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IA-Automorphisms of a Solvable Product of Abelian Lie Algebras

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Abstract- Let S be a solvable product of class $n \geq 1$ of free abelian Lie algebras of finite rank. In this study we prove that every normal automorphism of S which acts identically on S' is an IA-automorphism.

Index Terms- Free Abelian Lie algebras, solvable product, automorphisms

I. INTRODUCTION

Let L be a Lie algebra over a field K and denote by Aut(L) the group of automorphisms of L. The kernel of a natural homomorphism

$$\pi: Aut(L) \to Aut(L/L')$$

consist of automorphisms which act identically modulo L', where L' derived subalgebra. It is called the group of IA-automorphisms and denoted by IA(L). In [2] it was studied IA-automorphisms of 2-generator metabelian Lie algebras. In [4] it is given the defining relations of the subgroup of IA-automorphisms of a free metabelian Lie algebra of rank 3. In [3] it is investigated generating sets of IA-automorphisms of a free metabelian Lie algebra of rank 3. An automorphism of L is called normal if it fixes every ideal of L. The set of normal automorphisms of L is a normal subgroup of the group Aut(L). Let F be a free Lie algebra of finite rank and R be an ideal of F. In [1] it is shown that every automorphism of F/R' which acts identically on R/R' is an IA-automorphism.

For an arbitrary variety of Lie algebras, the solvable product of Lie algebras F_i, $i = 1, \ldots, m$ of class $n \geq 1$ is defined as

$$\left(\prod * F_i\right)/\left(D \cap F^{(n+1)}\right),$$

where $F = \prod * F_i$ is the free product of the Lie algebras F_i and D is the cartesian subalgebra of $\prod * F_i$. If the algebras F_i are non-trivial free abelian Lie algebras then the solvable product of them is isomorphic *to* $F/F^{(n+1)}$, where $F^{(1)} = [F, F] = F'$ and $F^{(n+1)} = [F^{(n)}, F^{(n)}]$.

Let S be the solvable product of free abelian Lie algebras of finite rank. In this study it is shown that every normal automorphism of S which acts identically on S' is an IA-automorphism.

Let L be a Lie algebra and E any subset of L. We show that by $< E >$ the ideal of L generated by the set E.

II. IA-AUTOMORPHISMS OF A SOLVABLE PRODUCT

Let A_i, $i = 1, \ldots, m$, be free Abelian Lie algebras of finite rank and $F = \prod * A_i$ is the free product of the Abelian Lie algebras A_i, $i = 1, \ldots, m$. If S is the solvable product of class $n \geq 1$ of the algebras A_i, S is isomorphic to $F/F^{(n+1)}$.

Definition 1: Let L be a Lie algebra. An automorphism ϕ of L is called a normal automorphism If $\phi(I) = I$ for evey ideal I of L.

Definition 2: Let L be a Lie algebra. The kernel of a natural homomorphism

$$\pi: Aut(L) \to Aut(L/L')$$

consist of automorphisms which act identically modulo L', where L' derived subalgebra. It is called the group of IA-automorphisms and denoted by IA(L).

Theorem : Every normal automorphism of S which acts identically on S' is an IA-automorphism.

Proof : Let ϕ is a normal automorphism of S acting identically on S'. Now consider the automorphism $\bar{\phi}: S/S' \to S/S'$ defined as $\bar{\phi}(a + S') = \phi(a) + S'$. $\bar{\phi}$ is well defined since ϕ is a normal automorphism of S. Let $P = I/S'$ be an ideal of S/S', where I is an ideal of S. Since $\phi(I) = I$ we get

$$\begin{aligned}\bar{\phi}(P) &= \bar{\phi}(I/S') \\ &= \{\bar{\phi}(a + S'): a \in I\} \\ &= \{\phi(a) + S': a \in I\} \\ &= I/S' \\ &= P.\end{aligned}$$

Hence $\bar{\phi}$ is a normal automorphism of S/S'. Since S/S' is finitely generated there is a basis $\{v_1, \dots, v_l\}$ of S/S'. The normal automorphisms of S/S' are linear maps which preserve the vector subspaces of S/S'. We now consider the vector subspace Kv_i. Since $\bar{\phi}(Kv_i) = Kv_i$ we get that $\bar{\phi}(v_i) = \alpha_i v_i$, $0 \neq \alpha_i \in K$, $1 \leq i \leq l$. Similarly, $\bar{\phi}(v_i + v_j) = \alpha_i v_i + \alpha_j v_j = \alpha(v_i + v_j)$, $0 \neq \alpha \in K$, $i \neq j$. Thus, $\alpha_i = \alpha_j = \alpha$ and we have $\bar{\phi}(v) = \alpha v$. The algebra S can be considered as $S = F/F^{(n+1)}$. Let denote by $\hat{v} = v + F^{(n+1)}$, where $v \in F$. Then there exist $u_i \in F'$, $1 \leq i \leq m$, such that

$$\phi(\hat{a_i}) = \alpha\hat{a_i} + \hat{u_i}$$

where $a_i \in A_i$. Let $\hat{w}$ be an element of S'. Since ϕ acts identically on S' then we have

$$\phi([\hat{w}, \widehat{a_1}]) = [\hat{w}, \widehat{a_1}]$$

and

$$[\hat{w}, \alpha\widehat{a_1} + \widehat{u_1}] = [\hat{w}, \widehat{a_1}]$$

and

$$(\alpha - 1)[\hat{w}, \widehat{a_1}] + [\hat{w}, \widehat{u_1}] = \hat{0}.$$

Then we get $\alpha = 1$ and ϕ is an IA-automorphism.

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