MINIMUM UNORTHODOX MEASURE OF ENTROPY FOR RESCRIBED ARITHMETIC MEAN AND SECOND ORDER MOMENT

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***Abstract*-**Minimum entropy probability distribution is necessary for complete information of probability distribution. But it has not been calculated so much whenever Maximum entropy probability distribution has been obtained by using different measures. Due to concave nature of entropy, minimization is complicated than maximization. In the present paper, We use Unorthodox measure of entropy to obtain minimum entropy for given Arithmetic mean and Second order moment.

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Index Terms- Unorthodox measure, switching point, consistent values of moments, feasible region.

I. Introduction

The word 'Uncertainty' is associated with entropy. Shannon introduced the concept of entropy [7] in 1948 to provide a quantitative measure of this uncertainty. After this measure many other measures of entropy came in existence. These are Renyi's [6], Havrda – Charvat [3] measure etc. Since Shannon entropy is concave function, a lot of work has been done on its maximization and its applications.

Kapur [4] introduced Unorthodox measure of entropy,

$$S = -\ln p_{max}$$

where $p_{max} = \max(p_1, p_{2.} ---- p_n)$

It is concave function of probability distribution. Entropy is maximum when probability distribution is as equal as possible . Here we have minimum information about system. As we increase information consistent with initial information in the form of moments, entropy decreases. This decreases until we obtain minimum entropy probability distribution. Now, we have complete information about system. . Maximum entropy probability distribution is most unbiased, most uniform and most random while minimum entropy probability distribution is most biased, least uniform and least random. Entropy is concave function so minimization of entropy is complicated than maximization.

Kapur [5] initiated the work to obtain minimum Shannon entropy. Anju Rani [2] obtained minimum entropy for Shannon measure and Havrda- Charvat measure when one moment is prescribed. In this paper, we have obtained analytical expressions for minimum Unorthodox measure of entropy for given Arithmetic mean and Second order moment. We have obtained also numerical values of entropy for given these two moments.

II MINIMUM VALUE OF AN UNORTHODOX MEASURE OF ENTROPY WHEN ARITHMETIC MEAN AND SECOND ORDER MOMENT ARE GIVEN: SPECIAL CASE:

Let x be a discrete variate which takes all values from 1 to n with probabilities, $p_1, p_{2.} ---- p_n$. The Arithmetic mean and Second order moment of this distribution are prescribed as M and $\left(\mu_2'\right)^{1/2}$. There will be many distributions having these particular values of M & $\left(\mu_2'\right)^{1/2}$ and each of these will have a particular value of entropy. Out of these entropies our aim is to find out minimum value of entropy i.e. S_{min}. Hence, we have to minimize

$$S = -\ln p_{max} \quad --(1)$$

subject to

$$\sum_{i=1}^{n} p_i = 1 \ , \ \sum_{i=1}^{n} i\, p_i = M \ , \ \sum_{i=1}^{n} i^2 p_i = \mu_2' \quad --(2)$$

Since there are three linear constraints, the minimum entropy probability distribution will have at most three non-zero components. Let these be p_h, p_k, p_l at points h, k and l respectively, where $1 \le h < k < l \le n$.

Then from equation (2)

$$p_h + p_k + p_l = 1\,, \quad hp_h + kp_k + lp_l = \mathrm{M}\,, \quad h^2 p_h + k^2 p_k + l^2 p_l = \mu_2' \qquad \text{- -(3)}$$

from equation (3)

$$p_h = \frac{\mu_2' + kl - \mathrm{M}(k+l)}{kl - h(k+l-h)}, \quad p_k = \frac{\mu_2' + hl - \mathrm{M}(h+l)}{hl - k(h+l-k)},$$

$$p_l = \frac{\mu_2' + hk - \mathrm{M}(h+k)}{hk - l(h+k-l)} \qquad \text{- -(4)}$$

Here we study the shifting behavior of p_{max} to calculate minimum entropy. Probability p_h increases with h & k and decreases with l ; p_k decreases with h and increases with k & l ; p_l increases with h, decreases with k, l. According to this we study the shifting of p_{max} from one set of (h, k, l) to another set of (h, k, l).

First we calculate feasible range of $\left(\mu_2'\right)^{1/2}$ for given value of M. For this we use following expressions by Anju Rani [2].

(i) If M is an integer , then

$$\left(\mu_2'\right)_{min}^{1/2} = \mathrm{M} \qquad \text{- -(5)}$$

If M is not an integer, $\mathrm{M} = [\mathrm{M}] + \mathrm{L}$, $0 < \mathrm{L} < 1$, where [M] represents integral part of M. Then

$$\left(\mu_2'\right)_{min} = [\mathrm{M}]^2 + \mathrm{L}(2[\mathrm{M}] + 1) \qquad \text{- -(6)}$$

(ii) The expression for maximum value of $\left(\mu_2'\right)_{max}$ is given as

$$\left(\mu_2'\right)_{max} = \mathrm{M}(\mathrm{n}+1) - \mathrm{n} \qquad \text{- -(7)}$$

For the given values of M and $\left(\mu_2'\right)_{min}$, probability p_h is zero at point $(1, a, a+1)$ or p_l is zero at point $(a, a+1, n)$ & for the given values of M and $\left(\mu_2'\right)_{max}$, probability $p_k = 0$ at point $(1, n-1, n)$. Here $p_h = 0$ for $\{1 \le h < k < \mathrm{H} \le l \le n\}$ or $\{1 \le h < k \le \mathrm{H} < l \le n\}$ and $p_l = 0$ for $\{1 \le h \le \mathrm{H} < k < l \le n\}$ or $\{1 \le h < \mathrm{H} \le k < l \le n\}$. For the given values of M and $\left(\mu_2'\right)_{min}$, the values of entropies are same at all existing points and similarly for the given values of M and $\left(\mu_2'\right)_{max}$, the values of entropies are same at all existing points. Here $(a, a+1]$ is interval in which Arithmetic mean lies.

Every interval is divided into many subintervals such that for given value of Arithmetic mean, the value of minimum entropy for any two subintervals is same. These values are called switching points. For these values, we switch over entropy from one set of values of (h, k, l) to another set of values of (h, k, l).

Let $\mathrm{M} \in (a, a+1]$, $1 \le a < \mathrm{n}$, where a is an integer. h can take values 1,2,….,a; k can take values h+1,….,n-1; l can take values a+1,……..,n. We calculate probability distribution in each possible interval for different values of M and $\left(\mu_2'\right)^{1/2}$.

For value $\left(\mu_2'\right)_{min}$, probability distribution is calculated at point $(a,\ a+1,\ n)$ for $a < n-1$ and at point $(1, a,\ a+1)$ for a = n-1. On the basis of probability distribution, we consider two cases.

CASE I: WHEN $a < n-1$

There are three possibilities:

(a) When p_h is maximum probability at point (a, a+1, a+2): In this case, minimum entropy occurs at point $p_h(a, a+1, a+2)$. From equations (1) &(4)

$$S_{min} = -ln\left[\frac{\mu_2' + (a+1)(a+2) - \mathrm{M}(2a+3)}{(a+1)(a+2) - a(a+3)}\right] \qquad \text{- -(8)}$$

(b) When p_k is maximum probability at point (a, a+1, a+2):

For value $\left(\mu_2'\right)_{min}$, $p_k(a, a+1, a+2) = p_l(a-1, a, a+1)$

then $p_l(a-1, a, a+1)$ is considered as maximum probability since p_k is decreasing and p_l is increasing with respect to Second order moment for given value of Arithmetic mean.

From equations (1) & (4)

$$S_{min} = -ln\left[\frac{\mu_2' + (a-1)a - \mathrm{M}(2a-1)}{(a-1)a - (a+1)(a-2)}\right] \qquad \text{- -(9)}$$

(c) When $p_h(a, a+1, a+2) = p_k(a, a+1, a+2)$:

In this situation, also $p_k(a\text{-}1, a, a\text{+}1) = p_l(a\text{-}1, a, a\text{+}1)$ for $(\mu_2^{'})_{min}$. Since p_k is decreasing, p_h and p_l are increasing with respect to s[th] order moment , then $p_h(a, a\text{+}1, a\text{+}2)$ and $p_l(a\text{-}1, a, a\text{+}1)$ are considered. Out of these values one may be maximum for $\mu_2^{'}$, where $\mu_2^{'} \gtrsim (\mu_2^{'})_{min}$ .Now we take greatest among

$$s_{min} = -ln\left[\frac{\mu_2^{'}+(a+1)(a+2)-\text{M}(2a+3)}{(a+1)(a+2)-a(a+3)}\right], \text{for } p_h > p_l \quad \text{- -(10)}$$

and, from equation (9)

$$s_{min} = -ln\left[\frac{\mu_2^{'}+(a-1)a-\text{M}(2a-1)}{(a-1)a-(a+1)(a-2)}\right], \text{for } p_l > p_h \quad \text{- -(11)}$$

CASE II: WHEN $a = n - 1$

There are two possibilities. These are

(a) If p_k is maximum probability for point (1, a, a+1):

In this case, $s_{min} = -lnp_k(1, a, a+1)$

$$s_{min} = -ln\left[\frac{\mu_2^{'}+(a+1)-\text{M}(a+2)}{(a+1)-2a}\right] \text{(from eq. 4)} \quad \text{- -(12)}$$

(b) If p_l is maximum probability at point (1, a, a+1):

In this case we consider point (a-1, a, a+1) for entropy to be minimum since p_l is maximum for maximum value of h and $p_l(1, a, a+1) = p_l(a-1, a, a+1)$, for $(\mu_2^{'})_{min}$

then, $s_{min} = -lnp_l(a-1, a, a+1)$

$$s_{min} = -ln\left[\frac{\mu_2^{'}+(a-1)a-\text{M}(2a-1)}{(a-1)a-(a+1)(a-2)}\right] \quad \text{- -(13)}$$

Now, we study three types of points to calculate minimum entropy

and its shifting behavior. These are-

(A) p_h at point $(a, a\text{+}\propto, a\text{+}\propto\text{+}1)$

(B) p_l at point $(a\text{+}\beta, a\text{+}\beta\text{+}1, a\text{+}\gamma)$

$(C) p_k$ at point $(1, a\text{+}\delta, n)$

where $1 \le \propto \le$ n-2, $1\text{-}a \le \beta \le$ n-3, $2 \le \gamma \le$ n-1, $1 \le \delta \le$ n-2 .

IF MAXIMUM PROBABILITY OCCURS AT $p_h(a, a\text{+}\propto, a\text{+}\propto\text{+}1)$:

There are two cases:

(a) $a > 1$ (b) a=1

(a) When $a > 1$: We study the shifting of minimum entropy from point (a, $a\text{+}\propto, a\text{+}\propto\text{+}1$) to another point. Maximum these probabilities. Hence,

From equation (8)

probability or minimum entropy may shifts from point $p_h(a, a\text{+}\propto, a\text{+}\propto\text{+}1)$ to point $p_l(a\text{-}1, a, a\text{+}1)$. Since p_l is maximum for maximum value of h & minimum value of k and l. Now, we are equating probabilities at two different points to find out switching point, at which maximum probability shifts from one point to another point. These probabilities are as:

$p_h(a, a\text{+}\propto, a\text{+}\propto +1) = p_l(a-1, a, a+1)$

$$\frac{\mu_2^{'}+(a+\propto)(a+\propto+1)-\text{M}(2a+2\propto+1)}{(a+\propto)(a+\propto+1)-a(a+2\propto+1)} = \frac{\mu_2^{'}+(a-1)a-\text{M}(2a-1)}{a(a-1)-(a+1)(a-2)} \quad \text{- -(14)}$$

by solving this equation, we get

$$(\mu_2^{'})_{\text{a}} = \frac{\text{A}\{(a-1)a-\text{M}(2a-1)\}-\text{B}\{(a+\propto)(a+\propto+1)-\text{M}(2a+2\propto+1)\}}{\text{B}-\text{A}} \quad \text{- -(15)}$$

Where A = $(a\text{+}\propto)(a\text{+}\propto +1) - a(a + 2 \propto +1)$

B = $a(a-1) - (a+1)(a-2)$

Now, two cases arise:

(A) $(\mu_2^{'})_{\text{a}}$ lies in the feasible region.

(B) $(\mu_2^{'})_{\text{a}}$ does not lie in the feasible region.

(A) $(\mu_2^{'})_{\text{a}}$ lies in the feasible region:

It means, probability distribution should exist for $(\mu_2^{'})_{\text{a}}$ at corresponding points. In this case minimum entropy shifts from $p_h(a, a\text{+}\propto, a\text{+}\propto\text{+}1)$ to $p_l(a\text{-}1, a, a\text{+}1)$. So,

$$s_{min} = -ln\left[\frac{\mu_2^{'}+(a+\propto)(a+\propto+1)-\text{M}(2a+2\propto+1)}{(a+\propto)(a+\propto+1)-a(a+2\propto+1)}\right],$$

$$\text{for } \mu_2^{'} \le (\mu_2^{'})_{\text{a}} \quad \text{- -(16)}$$

and $$s_{min} = -ln\left[\frac{\mu_2^{'}+(a-1)a-\text{M}(2a-1)}{a(a-1)-(a+1)(a-2)}\right],$$

$$\text{for } (\mu_2^{'})_{\text{a}} \le \mu_2^{'} \quad \text{- -(17)}$$

(B) If $\left(\mu_2'\right)_a$ does not lie in the feasible region: In this case maximum probability can not shift from $p_h(a, a{+}\propto, a{+}\propto{+}1)$ to $p_l(a\text{-}1, a, a{+}1)$. Depending on the value of a, there are two cases:

(B_1) $a = 2$ (B_2) $a > 2$

(B_1)When $a = 2$:

Maximum probability shifts to $p_l(a\text{-}1, a, a{+}2)$ since value of h can not be reduced further. Then

$$p_h(a, a{+}\propto, a{+}\propto{+}1) = p_l(a-1, a, a+2) \quad \text{- -(18)}$$

$$\frac{\mu_2' + (a{+}\propto)(a{+}\propto{+}1) - M(2a + 2\propto{+}1)}{(a{+}\propto)(a{+}\propto{+}1) - a(a + 2\propto{+}1)} = \frac{\mu_2' + (a-1)a - M(2a-1)}{a(a-1) - (a+2)(a-3)}$$

by solving this equation , we get

$$\left(\mu_2'\right)_b = \frac{A\{(a-1)a - M(2a-1)\} - C\{(a{+}\propto)(a{+}\propto{+}1) - M(2a{+}2\propto{+}1)\}}{C - A} \quad \text{- -(19)}$$

Where $C = a(a-1) - (a+2)(a-3)$

If $\left(\mu_2'\right)_b$ lies in the feasible region then maximum probability shifts from $p_h(a, a{+}\propto, a{+}\propto{+}1)$ to $p_l(a-1, a, a+2)$. Then,

$$s_{min} = -ln\left[\frac{\mu_2' + (a{+}\propto)(a{+}\propto{+}1) - M(2a{+}2\propto{+}1)}{(a{+}\propto)(a{+}\propto{+}1) - a(a{+}2\propto{+}1)}\right],$$

$$\text{for } \mu_2' \le \left(\mu_2'\right)_b \quad \text{- -(20)}$$

and $$s_{min} = -ln\left[\frac{\mu_2' + (a-1)a - M(2a-1)}{a(a-1) - (a+2)(a-3)}\right],$$

$$\text{for } \left(\mu_2'\right)_b \le \mu_2' \quad \text{- -(21)}$$

If $\left(\mu_2'\right)_b$ does not lie in the feasible region then value of l is increased gradually upto $a{+}\propto$ in $p_l(a-1, a, a+2)$ and probabilities will be equated with $p_h(a, a{+}\propto, a{+}\propto{+}1)$ as above. And if any switching point can not be obtained then probabilities are equated as

$$p_h(a, a{+}\propto, a{+}\propto{+}1) = p_h(a, a{+}\propto{+}1, a{+}\propto{+}2). \quad \text{- -(22)}$$

by solving this equation, we get

$$\left(\mu_2'\right)_c = (2a{+}\propto{+}1) - a(a{+}\propto{+}1) \quad \text{- -(23)}$$

Minimum entropy shifts from $p_h(a, a{+}\propto, a{+}\propto{+}1)$ to $p_h(a, a{+}\propto{+}1, a{+}\propto{+}2)$.

For $\mu_2' \le \left(\mu_2'\right)_c$

$$s_{min} = -ln\left[\frac{\mu_2' + (a{+}\propto)(a{+}\propto{+}1) - M(2a{+}2\propto{+}1)}{(a{+}\propto)(a{+}\propto{+}1) - a(a{+}2\propto{+}1)}\right] \quad \text{- -(24)}$$

And, for $\left(\mu_2'\right)_c \le \mu_2'$

$$s_{min} = -ln\left[\frac{\mu_2' + (a{+}\propto{+}1)(a{+}\propto{+}2) - M(2a{+}2\propto{+}3)}{(a{+}\propto{+}1)(a{+}\propto{+}2) - a(a{+}2\propto{+}3)}\right] \quad \text{- -(25)}$$

(B_2) When $a > 2$:

In this case, values of h and k are decreased by one from $p_l(a-1, a, a+1)$ simultaneously and probabilities are equated as:

$$p_h(a, a{+}\propto, a{+}\propto{+}1) = p_l(a-2, a-1, a+1) \quad \text{- -(26)}$$

$$\frac{\mu_2' + (a{+}\propto)(a{+}\propto{+}1) - M(2a{+}2\propto{+}1)}{(a{+}\propto)(a{+}\propto{+}1) - a(a{+}2\propto{+}1)} = \frac{\mu_2' + (a-2)(a-1) - M(2a-3)}{(a-2)(a-1) - (a+1)(a-4)}$$

by solving this equation, we get

$$\left(\mu_2'\right)_d = \frac{A\{(a-2)(a-1) - (2a-3)M\} - D\{(a{+}\propto)(a{+}\propto{+}1) - M(2a{+}2\propto{+}1)\}}{D - A} \quad \text{- -(27)}$$

where, $D = (a-2)(a-1) - (a+1)(a-4)$

If $\left(\mu_2'\right)_d$ lies in the feasible region then maximum probability shifts from $p_h(a, a{+}\propto, a{+}\propto{+}1)$ to $p_l(a-2, a-1, a+1)$ and if $\left(\mu_2'\right)_d$ does not lie in the feasible region then we proceed similarly as above by decreasing the values of h & k from $p_l(a-1, a, a+1)$ and equating probabilities with $p_h(a, a{+}\propto, a{+}\propto{+}1)$. If any value of switching point can not be obtained then probabilities are equated as:

$$p_h(a, a{+}\propto, a{+}\propto{+}1) = p_h(a, a{+}\propto{+}1, a{+}\propto{+}2)$$

by solving this equation, we get $\left(\mu_2'\right)_c$ (eq. 23)

For $a+\propto+1=n$, minimum entropy shifts from $p_h(a,a+\propto, a+\propto+1$ to $pk1,a, n$ and switching point for this shifting is $\left(\mu_2'\right)_c$.

(b)When $a=1$: In this situation, maximum probability shifts from $p_h(a,a+\propto,a+\propto+1)$ to $p_h(a,a+\propto+1,a+\propto+2$. So,

$$p_h(a,a+\propto,a+\propto+1)=p_h(a,a+\propto+1,a+\propto+2) \quad \text{- -(28)}$$

by solving this equation, we get $\left(\mu_2'\right)_c$ as switching point.

For $a+\propto+2=n$, $p_h(a,a+\propto+1,a+\propto+2)$ becomes $p_h(1,n-1,n)$ and maximum probability may shifts from $p_h(1,n-1,n)$ to $p_l(1,2,n)$. For getting switching point for this shifting, probabilities are equated as:

$$p_h(1,n-1,n)=p_l(1,2,n) \quad \text{- -(29)}$$

by solving this equation, we get $\left(\mu_2'\right)_e$ - -(30)

$$s_{min}=-ln p_h(1,n-1,n) \quad \text{for } \mu_2'\le\left(\mu_2'\right)_e$$

$$s_{min}=-ln p_l(1,2,n) \quad \text{for } \left(\mu_2'\right)_e\le\mu_2'.$$

IV IF MAXIMUM PROBABILITY OCCURS AT $p_l(a+\beta,a+\beta+1,a+\gamma)$: There are two cases:

(a) $a+\beta>1$ (b) $a+\beta=1$

(a) When $a+\beta>1$:

From calculation, we observe that maximum probability shifts from $p_l(a+\beta,a+\beta+1,a+\gamma)$ to $p_l(a+\beta-1,a+\beta,a+\gamma$. So, probabilities are equated as:

$$p_l(a+\beta,a+\beta+1,a+\gamma)=p_l(a+\beta-1,a+\beta,a+\gamma) \quad \text{- -(31)}$$

by solving this equation, we get

$$\left(\mu_2'\right)_f=\mathrm{M}(2a+\beta+\gamma)-(a+\beta)(a+\gamma) \quad \text{- -(32)}$$

Now, for $\mu_2'\le\left(\mu_2'\right)_f$,

$$s_{min}=-ln\left[\frac{\mu_2'+(a+\beta)(a+\beta+1)-\mathrm{M}(2a+2\beta+1)}{(a+\beta)(a+\beta+1)-(a+\gamma)(a+2\beta-\gamma+1)}\right] \quad \text{-(33)}$$

And, for $\left(\mu_2'\right)_f\le\mu_2'$,

$$s_{min}=-ln\left[\frac{\mu_2'+(a+\beta-1)(a+\beta)-\mathrm{M}(2a+2\beta-1)}{(a+\beta-1)(a+\beta)-(a+\gamma)(a+2\beta-\gamma-1)}\right] \quad \text{- -(34)}$$

Similarly, we reduce the values of h and k by one from $p_l(a+\beta-1,a+\beta,a+\gamma)$ simultaneously and find out switching points. We continue the process till $h=1,k=2$.

(b) When $a+\beta=1$:

From calculation, we observe that maximum probability shifts from $p_l(1,2,a+\gamma)$ to $p_k(1,a+\gamma,\mathrm{n})$. So

$$p_l(1,2,a+\gamma)=p_k(1,a+\gamma,n) \quad \text{- -(35)}$$

by solving this equation, we get

$$\left(\mu_2'\right)_g=\mathrm{M}(a+\gamma+1)-(a+\gamma) \quad \text{- -(36)}$$

Now, for $\mu_2'\le\left(\mu_2'\right)_g$

$$s_{min}=-ln\left[\frac{\mu_2'+2-3\mathrm{M})}{2-(a+\gamma)(3-a-\gamma)}\right] \quad \text{- -(37)}$$

and, for $\left(\mu_2'\right)_g\le\mu_2'$

$$s_{min}=-ln\left[\frac{\mu_2'+n-\mathrm{M}(n+1)}{n-(a+\gamma)(n+1-a-\gamma)}\right] \quad \text{- -(38)}$$

V IF MAXIMUM PROBABILITY OCCURS AT $p_k(1,a+\delta,n)$: To obtain switching point for minimum entropy, probabilities are equated as:

$$p_k(1,a+\delta,n)=p_h(a+\delta-1,a+\delta,a+\delta+1) \quad \text{- -(39)}$$

$$\frac{\mu_2'+n-\mathrm{M}(n+1)}{n-(a+\delta)(n+1-a-\delta)}=\frac{\mu_2'+(a+\delta)(a+\delta+1)-\mathrm{M}(2a+2\delta+1)}{(a+\delta)(a+\delta+1)-(a+\delta-1)(a+\delta+2)}$$

by solving this equation, we get

$$\left(\mu_2'\right)_h=\frac{\mathrm{E}\{(a+\delta)(a+\delta+1)-\mathrm{M}(2a+2\delta+1)\}-\mathrm{F}\{n-\mathrm{M}(n+1)\}}{\mathrm{F}-\mathrm{E}} \quad \text{- -(40)}$$

where, $\mathrm{E}=n-(a+\delta)(n+1-a-\delta)$

$\mathrm{F}=(a+\delta)(a+\delta+1)-(a+\delta-1)(a+\delta+2)$

And, $p_k(1,a+\delta,n)=p_l(a+\delta-1,a+\delta,a+\delta+1)$- -(41)

$$\frac{\mu_2'+n-\mathrm{M}(n+1)}{n-(a+\delta)(n+1-a-\delta)}=\frac{\mu_2'+(a+\delta-1)(a+\delta)-\mathrm{M}(2a+2\delta-1)}{(a+\delta-1)(a+\delta)-(a+\delta+1)(a+\delta-2)}$$

by solving this equation, we get

$$\left(\mu_2'\right)_i=\frac{\mathrm{E}\{(a+\delta-1)(a+\delta)-\mathrm{M}(2a+2\delta-1)\}-\mathrm{G}\{n-\mathrm{M}(n+1)\}}{\mathrm{G}-\mathrm{E}} \quad \text{- -(42)}$$

where, $\mathrm{G}=(a+\delta-1)(a+\delta)-(a+\delta+1)(a+\delta-2)$

First, we check the existence of probability distributions for values $\left(\mu_2'\right)_h$ and $\left(\mu_2'\right)_i$ at corresponding points. If both values lie in the feasible region then minimum value out of $\left(\mu_2'\right)_h$ and $\left(\mu_2'\right)_i$ is considered as switching point. If one of these values $\left(\mu_2'\right)_h$ and $\left(\mu_2'\right)_i$ lies in the feasible region then feasible value is considered as switching point whether this value is greater or smaller.

If $(\mu_2')_h$ lies in the feasible region and is minimum then minimum entropy shifts from $p_k(1, a+\delta, n)$ to $p_h(a+\delta-1, a+\delta, a+\delta+1)$. So, for $\mu_2' \leq \mu_2'h$

$$s_{min} = -ln\left[\frac{\mu_2'+n-M(n+1)}{n-(a+\delta)(n+1-a-\delta)}\right] \text{ (from eq. 1 \& 4)} \quad --(43)$$

and, for $(\mu_2')_h \leq \mu_2'$

$$s_{min} = -ln\left[\frac{\mu_2'+(a+\delta-1)(a+\delta)-M(2a+2\delta-1)}{(a+\delta-1)(a+\delta)-(a+\delta+1)(a+\delta-2)}\right] \quad --(44)$$

And, if $(\mu_2')_i$ lies in the feasible region and is minimum then minimum entropy shifts from $p_k(1, a+\delta, n)$ to $p_l(a+\delta-1, a+\delta, a+\delta+1)$. Hence, for $\mu_2' \leq \mu_2'i$

$$s_{min} = -lnp_k(1, a+\delta, n)$$

and, for $(\mu_2')_i \leq \mu_2'$

$$s_{min} = -ln\left[\frac{\mu_2'+(a+\delta-1)(a+\delta)-M(2a+2\delta-1)}{(a+\delta-1)(a+\delta)-(a+\delta+1)(a+\delta-2)}\right] \quad --(45)$$

If both $(\mu_2')_h$ and $(\mu_2')_i$ do not lie in the feasible region:

In this case, minimum entropy does not shift from p_k at point $(1, a+\delta, n)$ to p_h or p_l at point $(a+\delta-1, a+\delta, a+\delta+1)$.

So, three cases arise-

(a) If $+\delta-1 > 1$, $a+\delta+1 = n$

(b) If $+\delta-1 = 1$, $a+\delta+1 < n$

(c) If $+\delta-1 > 1$, $a+\delta+1 < n$

(a) If $+\delta-1 > 1$, $a+\delta+1 = n$

Since probability p_h is maximum for maximum values of h, k & p_l is maximum for maximum value of h and minimum value of k. Then for equating probabilities, value of h is reduced by one from $p_h(a+\delta-1, a+\delta, a+\delta+1)$ and values of h,k are reduced by one from $p_l(a+\delta-1, a+\delta, a+\delta+1)$. Hence probabilities are equated as:

$$p_k(1, a+\delta, n) = p_h(a+\delta-2, a+\delta, a+\delta+1)$$

$$\frac{\mu_2'+n-M(n+1)}{n-(a+\delta)(n+1-a-\delta)} = \frac{\mu_2'+(a+\delta)(a+\delta+1)-M(2a+2\delta+1)}{(a+\delta)(a+\delta+1)-(a+\delta-2)(a+\delta+3)} \quad --(46)$$

by solving this equation, we get

$$(\mu_2')_j = \frac{E\{(a+\delta)(a+\delta+1)-M(2a+2\delta+1)\}-H\{n-M(n+1)\}}{H-E} \quad --(47)$$

Again, we are equating probabilities as:

$$p_k(1, a+\delta, n) = p_l(a+\delta-2, a+\delta-1, a+\delta+1)$$

$$p_k(1, a+\delta, n) = \frac{\mu_2'+(a+\delta-2)(a+\delta-1)-M(2a+2\delta-3)}{(a+\delta-2)(a+\delta-1)-(a+\delta-4)(a+\delta+1)} \quad --(48)$$

by solving this equation, we get

$$(\mu_2')_k = \frac{E\{(a+\delta-2)(a+\delta-1)-M(2a+2\delta-3)\}-I\{n-M(n+1)\}}{I-E} \quad --(49)$$

Where, $I = (a+\delta-2)(a+\delta-1)-(a+\delta-4)(a+\delta+1$

If $(\mu_2')_j$ lies in the feasible region and is minimum then minimum entropy shifts from $p_k(1, a+\delta, n)$ to $p_h(a+\delta-2, a+\delta, a+\delta+1)$. For $\mu_2' \leq \mu_2'j$

$$s_{min} = \frac{\mu_2'+n-M(n+1)}{n-(a+\delta)(n+1-a-\delta)} \quad --(50)$$

and, for $(\mu_2')_j \leq \mu_2'$

$$s_{min} = -ln\left[\frac{\mu_2'+(a+\delta)(a+\delta+1)-M(2a+2\delta+1)}{(a+\delta)(a+\delta+1)-(a+\delta-2)(a+\delta+3)}\right] \quad --(51)$$

If $(\mu_2')_k$ lies in the feasible region and is minimum then minimum entropy shifts from $p_k(1, a+\delta, n)$ to $p_l(a+\delta-2, a+\delta-1, a+\delta+1)$.

$$s_{min} = -lnp_k(1, a+\delta, n) \text{ , for } \mu_2' \leq (\mu_2')_k$$

and, for $(\mu_2')_k \leq \mu_2'$

$$s_{min} = -ln\left[\frac{\mu_2'+(a+\delta-2)(a+\delta-1)-M(2a+2\delta-3)}{(a+\delta-2)(a+\delta-1)-(a+\delta-4)(a+\delta+1)}\right] \quad --(52)$$

If $(\mu_2')_j$ and $(\mu_2')_k$ do not lie in the feasible region then we proceed similarly and find out switching points by decreasing the values of h & k from p_l upto 1 & 2 respectively and only value of h from p_h up to 1.

(b) If $+\delta - 1 = 1$, $a + \delta + 1 < n$

In this case, $h = 1$ is fixed. So there will be no change in value of h. p_h is maximum for maximum value of k & minimum value of l and p_l is maximum for minimum values of k & l then value of k is increased by one and hence value of l is increased by one from $p_h(a+\delta-1, a+\delta, a+\delta+1$ and value of l is increased by one from $p_l(a+\delta-1, a+\delta, a+\delta+1)$. Hence, we equate probabilities as:

$$p_k(1, a+\delta, n) = p_h(1, a+\delta+1, a+\delta+2)$$

$$\frac{\mu_2' + n - M(n+1)}{n-(a+\delta)(n+1-a-\delta)} = \left[\frac{\mu_2' + (a+\delta+1)(a+\delta+2) - M(2a+2\delta+3)}{(a+\delta+1)(a+\delta+2)-(2a+2\delta+2)}\right] \quad --(53)$$

by solving this equation, we get

$$(\mu_2')_l = \frac{E\{(a+\delta+1)(a+\delta+2)-M(2a+2\delta+3)\}-J\{n-M(n+1)\}}{J-E} \quad --(54)$$

Again, we are equating probabilities as:

$$p_k(1, a+\delta, n) = p_l(1, a+\delta, a+\delta+2)$$

$$\frac{\mu_2' + n - M(n+1)}{n-(a+\delta)(n+1-a-\delta)} = \frac{\mu_2' + (a+\delta) - M(a+\delta+1)}{2(a+\delta+1)} \quad --(55)$$

by solving this equation, we get

$$(\mu_2')_m = \frac{E\{(a+\delta)-M(a+\delta+1)\}-K\{n-M(n+1)\}}{K-E} \quad --(56)$$

Where, $K = 2(a+\delta+1)$

If $(\mu_2')_l$ lies in the feasible region and is minimum then minimum entropy shifts from point $p_k(1, a+\delta, n)$ to $p_h(1, a+\delta+1, a+\delta+2)$. Then, for $\mu_2' \le (\mu_2')_l$

$$s_{min} = -ln p_k(1, a+\delta, n)$$

and, for $(\mu_2')_l \le \mu_2'$

$$s_{min} = -ln\left[\frac{\mu_2' + (a+\delta+1)(a+\delta+2) - M(2a+2\delta+3)}{(a+\delta+1)(a+\delta+2)-(2a+2\delta+2)}\right] \quad --(57)$$

And, if $(\mu_2')_m$ lies in the feasible region and is minimum then minimum entropy shifts from $p_k(1, a+\delta, n)$ to $p_l(1, a+\delta, a+\delta+2$, then

$$s_{min} = -ln p_k(1, a+\delta, n)\text{ , for } \mu_2' \le (\mu_2')_m$$

and, for $(\mu_2')_m \le \mu_2'$

$$s_{min} = -ln\left[\frac{\mu_2' + (a+\delta) - M(a+\delta+1)}{2(a+\delta+1)}\right] \quad --(58)$$

If both $(\mu_2')_l$ and $(\mu_2')_m$ do not lie in the feasible region then we proceed similarly and increase the value of l upto n and find out switching points.

(c) If $a + \delta - 1 > 1$ and $a + \delta + 1 < n$:

In this case probabilities are equated as follow:

$$p_k(1, a+\delta, n) = p_h(a+\delta-1, a+\delta+1, a+\delta+2)$$

and $\quad p_k(1, a+\delta, n) = p_l(a+\delta-2, a+\delta-1, a+\delta+1)$

Now,

$$p_k(1, a+\delta, n) = p_h(a+\delta-1, a+\delta+1, a+\delta+2)$$

$$\frac{\mu_2' + n - M(n+1)}{n-(a+\delta)(n+1-a-\delta)} = \frac{\mu_2' + (a+\delta+1)(a+\delta+2) - M(2a+2\delta+3)}{(a+\delta+1)(a+\delta+2)-(a+\delta-1)(a+\delta+4)} \quad --(59)$$

by solving this equation, we get

$$(\mu_2')_n = \frac{E\{(a+\delta+1)(a+\delta+2)-M(2a+2\delta+3)\}-L\{n-M(n+1)\}}{L-E} \quad --(60)$$

Where, $L = (a+\delta+1)(a+\delta+2) - (a+\delta-1)(a+\delta+4$

Again, we are equating probabilities as:

$$p_k(1, a+\delta, n) = p_l(a+\delta-2, a+\delta-1, a+\delta+1)$$

$$\frac{\mu_2' + n - M(n+1)}{n-(a+\delta)(n+1-a-\delta)} = \frac{\mu_2' + (a+\delta-2)(a+\delta-1) - M(2a+2\delta-3)}{(a+\delta-2)(a+\delta-1)-(a+\delta+1)(a+\delta-4)} \quad --(61)$$

by solving this equation, we get

$$(\mu_2')_o = \frac{E\{(a+\delta-2)(a+\delta-1)-M(2a+2\delta-3)\}-M\{n-M(n+1)\}}{M-E} \quad --(62)$$

Where, $M = (a+\delta-2)(a+\delta-1) - (a+\delta+1)(a+\delta-4)$

If $\left(\mu_2'\right)_n$ lies in the feasible region and is minimum then minimum entropy shifts from $p_k(1, a+\delta, n)$ to $p_h(a+\delta-1, a+\delta+1, a+\delta+2)$, then

$$s_{min} = -ln p_k(1, a+\delta, n)\ , \text{ for } \mu_2' \leq \left(\mu_2'\right)_n$$

And, for $\left(\mu_2'\right)_n \leq \mu_2'$

$$s_{min} = -ln\left[\frac{\mu_2' + (a+\delta+1)(a+\delta+2) - M(2a+2\delta+3)}{(a+\delta+1)(a+\delta+2)-(a+\delta-1)(a+\delta+4)}\right] \quad \text{- -(63)}$$

And if $\left(\mu_2'\right)_o$ lies in the feasible region and is minimum then minimum entropy shifts to $p_l(a+\delta-2, a+\delta-1, a+\delta+1$, then

$$s_{min} = -ln p_k(1, a+\delta, n)\ , \text{ for } \mu_2' \leq \left(\mu_2'\right)_o$$

And, for $\left(\mu_2'\right)_o \leq \mu_2'$

$$s_{min} = -ln\left[\frac{\mu_2' + (a+\delta-2)(a+\delta-1) - M(2a+2\delta-3)}{(a+\delta-2)(a+\delta-1)-(a+\delta+1)(a+\delta-4)}\right] \quad \text{- -(64)}$$

If $\left(\mu_2'\right)_n$ and $\left(\mu_2'\right)_o$ do not lie in the feasible region then we proceed similarly and increase the values of k & l up to $n-1$ & n in p_h and decrease the values of h & k up to 1 & 2 from p_l respectively.

VI MINIMUM VALUE OF AN UNORHTODOX MEASURE OF ENTROPY WHEN ARITHMETIC MEAN AND SECOND ORDER MOMENT ARE GIVEN: SIX FACED DICE:

Now we calculate minimum value of entropy of an unorthodox measure for six faced dice i.e. $n = 6$. We find out probability distributions for given moments belong to all possible intervals and observe how the maximum probability shifts from one set of points of non-zero probability to another set of points of non-zero probability.

Here we have to minimize

$$s = -ln p_{max} \quad \text{- -(65)}$$

subject to

$$\sum_{i=1}^{6} p_i = 1\ ,\quad \sum_{i=1}^{6} i^2 p_i = \mu_2'\ ,\quad \sum_{i=1}^{6} i\, p_i = M \quad \text{- -(66)}$$

CASE - 1 we consider the case when M ∈ (2,3]. In this interval we take values M=2.25, 2.5, 2.75, 3.0. But in the present paper we are considering only for M=2.25. Values of entropies are given in the table 1 for given M=2.25. In this case *h*=1,2; *k*=2,3,4,5; *l*=3,4,5,6.

(a) M=2.25

From equations (6) & (7), $\left(\mu_2'\right)_{min}^{1/2} = 2.2913$ and $\left(\mu_2'\right)_{max}^{1/2} = 3.1225$ [table 1].

For $\left(\mu_2'\right)_{min}^{1/2}$, p_k is maximum probability. From equation (22),

$$[s_{min}]_{p_h(a, a+\propto, a+\propto+1)} = [s_{min}]_{p_h(a, a+\propto+1, a+\propto+2)}$$

here *a*=2, ∝=1, M=2.25.

$$[s_{min}]_{p_h(2,3,4)} = [s_{min}]_{p_h(2,4,5)}$$

$$-ln\left[\frac{\mu_2' - 3.75}{2}\right] = -ln\left[\frac{\mu_2' - .25}{6}\right]$$

by solving this equation, we get $\left(\mu_2'\right)_1^{1/2} = 2.3452$ [table 1].

The value of $\left(\mu_2'\right)_1^{1/2}$ can be obtained from equation (23) for *a*=2, ∝=1, H=2.25.

$$s_{min} = -ln\left[\frac{\mu_2' - 3.75}{2}\right] \quad \text{for } \left(\mu_2'\right)^{1/2} \in [2.2913, 2.3452]$$

From equation (22),

$$[s_{min}]_{p_h(a, a+\propto, a+\propto+1)} = [s_{min}]_{p_h(a, a+\propto+1, a+\propto+2)}$$

here *a*=2, ∝=2, M=2.25.

$$[s_{min}]_{p_h(2,4,5)} = [s_{min}]_{p_h(2,5,6)}$$

$$-ln\left[\frac{\mu_2' - .25}{6}\right] = -ln\left[\frac{\mu_2' + 5.25}{12}\right]$$

by solving this equation, we get $\left(\mu_2'\right)_2^{1/2} = 2.3979$

The value of $\left(\mu_2'\right)_2^{1/2}$ can be obtained from equation (23) for *a*=2, ∝=2, M=2.25.

ISSN 2250-3153

$$s_{min} = -\ln\left[\frac{\mu_2^{'}-.25}{6}\right] \quad \text{for } \left(\mu_2^{'}\right)^{1/2} \in [2.3452, 2.3979]$$

Here $a+\propto+2=n$, then maximum probability shifts from point $p_h(a,a+\propto+1,a+\propto+2)$ to $p_k(1,a,n)$.

$$[s_{min}]_{p_h(a,a+\propto+1,a+\propto+2)} = [s_{min}]_{p_k(1,a,n)}$$

$$[s_{min}]_{p_h(2,5,6)} = [s_{min}]_{p_k(1,2,6)}$$

$$-\ln\left[\frac{\mu_2^{'}+5.25}{12}\right] = -\ln\left[\frac{9.75-\mu_2^{'}}{4}\right]$$

by solving this equation, we get $\left(\mu_2^{'}\right)_3^{1/2} = 2.4495$

$$s_{min} = -\ln\left[\frac{\mu_2^{'}+5.25}{12}\right], \text{ for } \left(\mu_2^{'}\right)^{1/2} \in [2.3979, 2.4495]$$

$$[s_{min}]_{p_k(1,2,6)} = [s_{min}]_{p_h(1,4,5)}$$

$$-\ln\left[\frac{9.75-\mu_2^{'}}{4}\right] = -\ln\left[\frac{\mu_2^{'}-.25}{3}\right]$$

by solving this equation, we get $\left(\mu_2^{'}\right)_4^{1/2} = 2.7157$

$$s_{min} = -\ln\left[\frac{9.75-\mu_2^{'}}{4}\right], \text{ for } \left(\mu_2^{'}\right)^{1/2} \in [2.4495, 2.7157]$$

Now, we equate entropies from equation (35) as:

$$[s_{min}]_{p_h(1,4,5)} = [s_{min}]_{p_h(1,5,6)}$$

$$-\ln\left[\frac{\mu_2^{'}-.25}{3}\right] = -\ln\left[\frac{\mu_2^{'}+5.25}{20}\right]$$

by solving this equation, we get $\left(\mu_2^{'}\right)_5^{1/2} = 2.9155$

$$s_{min} = -\ln\left[\frac{\mu_2^{'}-.25}{3}\right], \quad \text{for } \left(\mu_2^{'}\right)^{1/2} \in [2.7157, 2.9155]$$

$$s_{min} = -\ln\left[\frac{\mu_2^{'}+5.25}{20}\right], \quad \text{for } \left(\mu_2^{'}\right)^{1/2} \in [29155, 3.1225]$$

$\left(\mu_2^{'}\right)^{1/2}$ h,k,l	**2.2913**	**2.3**	**2.3452**	**2.3979**
1,2,3	(0,.75,.25)	(.02,.71,.27)	(.125,.50,.375)	(.25,.25,.50)
1,2,4			(0,.875,.125)	(.0833,.75,.1667)
1,2,5				(0,.9167,.0833)
2,3,4	(**.75**,.25,0)	(**.77**,.21,.02)	(**.875**, 0,.125)	
2,3,5	(.75,.25,0)	(.7633,.23,.0067)	(.8333,.125,.0417)	(.9167, 0, .08333)
2,3,6	(.75,.25,0)	(.76,.2367,.0033)	(.8125,.1667,.0208)	(.875,.0833,.0417)
2,4,5			(**.875**,.125,0)	(**.9167**, 0,.0833)
2,4,6			(.875,.125,0)	(.9063,.0625,.0313)
2.5,6				(**.9167**,.0833,0)

Contd….

$\left(\mu_2^{'}\right)$ h,k,l	**2.4**	**2.4495**	**2.5**	**2.6**
1,2,3	(.255,.24,.505)	(.375 ,0,.625)		
1,2,4	(.0867..745..1683)	(.1667,.625,.2083)	(.25,.50,.25)	(.42,.245,.335)
1,2,5	(.0025,.9133..0842)	(.0625,.8333,.1042)	(.125,.75,.125)	(.2525,.58,.1675)
1,2,6		(0,**.9375**,.0625)	(.05,**.875**,.075)	(.152,**.7475**,.1005)
1.3,4		(.375,.625,0)	(.4167,.50,.0833)	(.5017,.245,.2533)
1.3,5		(.375,.625,0)	(.4063,.5625,.0313)	(.47,.435,.095)
1,3,6		(.375,.625,0)	(.40,.5833,.0167)	(.451,.4983,.0507)
2,3,6	(.8775,.08,.0425)	(.8775,.08,.0425)		
2,4,6	(.9075,.06,.0325)	(.9075,.06,.0325)		
2.5,6	(**.9175**,.08,.0025)	(**.9175**,.08,.0025)		

Contd….

$\left(\mu_2^{'}\right)^{1}$ h,k,l	**2.6926**	**2.7**	**2.8**	**2.9**
1,2,4	(.5833,0,.4167)	(.385,.4033,.2117)	(.5225,.22,.2575)	(.665,.03,.305)
1,2,5	(.375,.4167,.2083)	(.258,**.615**,.127)	(.368,.4775,.1545)	(.482,.335,.183)
1,2,6	(.25,**.625**,.125)			
1.3,4	(.5833,0,.4167)	(.5363,.3025,.1613)	(.605,.165,.23)	(.6763,.0225,.3013)
1.3,5	(.5313,.3125,.1563)	(.504,.41,.086)	(.559,.3183,.1227)	(.616,.2233,.1607)
1,3,6	(.50,.4167,.0833)	(.5867,.4023,.01)	(**.6325**,.22,.1475)	(**.68**,.03,.29)
1,4,5	(.5833,.4167,0)	(.586,.41,.004)	(.6227,.3183,.059)	(.6607,.2233,.116)
1,4,6	(.5833,.4167,0)			

Contd…

$\left(\mu_2^{'}\right)^{1}$	**2.9155**	**3.0**	**3.1**	**3.1225**

h,k,l				
1,2,5	(**.6875**,0,.3125,)			
1,2,6	(.50,.3125,.1875)	(.60,.1875,.2125)	(.722,.035,.243)	(.75,0,.25)
1.3,5	(.6875,0,.3125)			
1,3,6	(.625,.2083,.1607)	(.675,.125,.20)	(.736,.0233,.2407)	(.75,0,.25)
1,4,5	(**.6875**,0,.3125)			
1,4,6	(.6667,.2083,.125)	(.70,.125,.175)	(.7407,.0233,.236)	(.75,0,.25)
1,5,6	(**.6875**,.3125,0)	(**.7125**,.1875,.10)	(**.743**,.035,.222)	(.75,0,.25)

Table [1]

Similarly, we can obtain minimum entropy for all values of Arithmetic mean and Second order moment.

VII CONCLUDING REMARKS:

We have obtained the expressions of minimum unorthodox measure of entropy for the given values of Arithmetic mean and Second order moment. So, we observe that

1. For the given values of M and $(\mu_2')_{min}$, probability distribution is same at all existing points and similarly for the given values of M and $(\mu_2')_{max}$, probability distribution is same at all existing points.
2. s_{min} is piecewise concave function of $(\mu_2')^{1/2}$, for the given value of M.
3. In a given interval, if p_k is maximum then minimum entropy increases with Second order moment for a given value of Arithmetic mean.
4. If p_h and p_l are maximum probabilities then minimum entropy decreases with Second order moment for a given value of Arithmetic mean.

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