ON STARLIKENESS AND COVEXITY PROPERTIES OF CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS

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Abstract- Let $A(w)$ be the class of functions analytic and univalent in the unit disk $U = \{z: |z| < 1\}$ and have the form:

$$f(z) = (z - w) + \sum_{k=2}^{\infty} a_k (z - w)^k$$

In this paper we provide the coefficient bounds for function in $A(w)$ that are starlike and convex.

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I. Introduction

Let A denote the class of functions which are analytic in the unit disk $U = \{z: |z| < 1\}$ and of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad 1.1$$

$$S = \{f \in A: f \text{ is univalent in } U\}$$

For function of class S, we recall the following definitions of the well known classes of starlike and convex functions S^* and S^c respectively see [1].

$$S^* = \left\{f \in A: Re\left(\frac{zf'(z)}{f(z)}\right) > 0,\ \ z \in U\right\} \qquad 1.2$$

$$S^c = \left\{f \in A: Re\left(1 + \frac{zf'(z)}{f(z)}\right) > 0,\ \ z \in U\right\} \qquad 1.3$$

S. Kanas and F. Ronning in [3] introduced the following classes of functions:

$$S(w) = \{f \in A(w): f \text{ is univalent in } U\} \qquad 1.4$$

$$S^*(w) = \left\{f \in S(w): Re\left(\frac{(z-w)f'(z)}{f(z)}\right) > 0,\ \ z \in U\right\} \qquad 1.5$$

$$S^c(w) = \left\{f \in S(w): Re\left(1 + \frac{(z-w)f''(z)}{f(z)}\right) > 0,\ \ z \in U\right\} \qquad 1.6$$

Let $P(w)$ denote the class of all functions:

$$P(z) = 1 + \sum_{n=1}^{\infty} B_n (z - w)^n \qquad 1.7$$

that are regular in U and satisfy $P(w) = 1$ and $Re\, P(z) > 0$ for $z \in U$, and w is fixed point in U.
We shall need the following results
Theorem 1[1, 7]: If $P \in P(w)$

$$P(z) = 1 + \sum_{n=1}^{\infty} B_n (z - w)^n$$

Then;

$$|B_n| \le \frac{2}{(1+d)(1-d)^n}$$

Where $d = |w|$ and $n \ge 1$.
Theorem 2 [2]: Let f be as in 1.1 and $f \in S^*$, then for each $n \ge 2$, $|a_n| \le n$

II. MAIN RESULTS

Theorem 3: Let $f(z)$ be of the form:

$$f(z) = (z-w) + \sum_{k=2}^{\infty} a_k (z-w)^k \qquad 2.1$$

And $f_2 \in S^*(w)$, then for each $k \geq 2$,

$$|a_k| \leq k$$

Proof:
Let $f \in S^*(w)$, then:

$$\frac{(z-w)f'(z)}{f(z)} = P(z) = 1 + \sum_{k=1}^{\infty} b_k (z-w)^k$$

Multiplying through by $f(z)$, we obtain:

$$(z-w)f'(z) = \left(1 + \sum_{k=1}^{\infty} b_k (z-w)^k\right)\left((z-w) + \sum_{k=2}^{\infty} a_k (z-w)^k\right) \qquad 2.2$$

Using 2.1 in 2.2 we obtain:

$$(z-w) + \sum_{k=2}^{\infty} k a_k (z-w)^k = \left(1 + \sum_{k=1}^{\infty} b_k (z-w)^k\right)\left((z-w) + \sum_{k=2}^{\infty} a_k (z-w)^k\right)$$

Which implies that comparing coefficients, we obtain;

$$k a_k (z-w)^k = a_k (z-w)^k + b_{k-1}(z-w)^k + \sum_{k=1}^{n-2} b_k a_{n-k} (z-w)$$

$$k a_k = a_k + b_{k-1} + \sum_{k=1}^{n-2} a_{n-k} b_k$$

$$|k a_k| = \left| a_k + b_{k-1} + \sum_{k=1}^{n-2} a_{n-k} b_k \right|$$

$$|a_k (k-1)| \leq |b_{k-1}| + \sum_{k=1}^{n-2} |a_{n-k}||b_k|$$

Using the theorem 2.1 and theorem 2.2, we have

$$|a_k|(k-1) \leq \frac{2}{(1+d)(1+d)^{k-1}} + \sum_{k=1}^{n-2} n-k \frac{2}{(1+d)(1+d)^{k-1}}, where\ d = |w|$$

And for $d > 1$, we obtain:

$$|a_k|(k-1) \leq 2\left(1 + \sum_{k=1}^{n=2} k\right)$$

$$= k(k-1), k \geq 2$$

$$|a_k| \leq k.$$

Theorem 4: Let $f(z)$ be as 3.1 if $f \in S^*(w)$ then for $k \geq 2$, $|a_k| \leq 1$
Proof:
Let

$$F(w) = (z-w)f'(z) \in S^*(w) \qquad 2.3$$

Then

$$\frac{(z-w)F'(z)}{F(z)} = P(z) = 1 + \sum_{k=1}^{\infty} b_k (z-w)^k \qquad 2.4$$

Using 2.3 in 2.4 we obtain;

$$1 + \frac{(z-w)f''(z)}{F(z)} = P(z) \qquad 2.5$$

Which shows that $f \in S^c(w)$; Now, multiply 3.4 by $F(z)$, gives

$$(z-w)F'(z) = \left(1 + \sum_{k=1}^{\infty} b_k (z-w)^k\right)\left((z-w) + \sum_{k=1}^{\infty} k a_k (z-w)^k\right) \qquad 2.6$$

Using 2.1 in 2.6, we obtain:

$$(z-w)+\sum_{k=2}^{\infty} k^2 a_k (z-w)^k = \left(1+\sum_{k=1}^{\infty} b_k (z-w)^k\right)\left((z-w)+\sum_{k=2}^{\infty} k a_k (z-w)^k\right)$$

Comparing coefficients of $(z-w)^k$, we obtain:

$$k^2 a_k = k a_k + b_{k-1} + \sum_{k=1}^{n-2} b_k a_{n-k}$$

$$|k^2 a_k| = \left| k^2 a_k = k a_k + b_{k-1} + \sum_{k=1}^{n-2} b_k a_{n-k} \right|$$

$$|k a_k (z-w)| \leq |b_{k-1}| + \sum_{k=1}^{n-1} |b_k|\,|a_{n-k}|$$

Using the theorem 1 and theorem 2, we have
Thus,

$$|k a_k (z-w)| \leq \frac{2}{(1+d)(1-d)^{k-1}} + \sum_{k=1}^{n-1} \frac{2}{(1+d)(1-d)^k}$$

$$|k a_k (z-w)| \leq 2 + \sum_{k=1}^{n-2} 2k$$

$$|k a_k (z-w)| \leq 2\left(1+\sum_{k=1}^{n-1} k\right) = k(k-1), k \geq 2$$

$$|k a_k (z-w)| \leq k(k-1)$$

Multiply through by $k(k-1), k \geq 2, we\ have$

$$|a_k| \leq 1$$

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