Dynamic Motions of Beam-type Structural Element Subjected to a Harmonic Magnitude Partially Distributed Load at Uniform Velocity

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Abstract- This present paper examines the dynamic behaviour of a uniform simply supported beam resting on an elastic subgrade under a partially distributed load of harmonically moving magnitude at a uniform speed. The governing equation is a fourth-order partial differential equation with singular and variable coefficients. Fourier sine transform is employed to reduce the motion equation to a set of coupled second-order ordinary differential equations and finally solved by Laplace transforms. From the analytical and numerical results, it was observed that the amplitude of the deflection profile of the simply supported beam decreases with an increase in the value of effective torsional rigidity and load natural frequency. Furthermore, an increase in the value of other vital structural parameters such as damping coefficient, foundation modulus, and distance along the span of the beam decreases the deflection profile of the system. Also, the resonance phenomenon of the vibrating system is critically examined.

Index Terms- Damping, Effective torsional rigidity, Foundation modulus, Load natural frequency, Distributed load.

I. INTRODUCTION

Research works on the analysis of the dynamic vibration of structural members due to moving loads on elastic subgrades are of great importance in the field of transport engineering, Applied Mathematics and Mathematical Physics. Tremendous efforts have been achieved by researchers on the dynamic vibration of the structural member (beam or plate) resting on an elastic foundation and traversed by moving load using both analytical and numerical methods. Fryba [1] presented an excellent analysis of moving load problems in the dynamics of structures.

Moving loads are forces acting a structures and continually changing positions. Previous research effort on moving load includes [2-8]. Load is any force that is transmitted to a body from another body by means of direct contact over an area on the surface of the body. The common type of load that has received the attention of authors is the point-like (concentrated) load, where the area of contact is relatively small compared to the total area of the entire length of the structure. A number of investigations have been carried out in this area. Among them is the work of Ismail [9], who studied the dynamic behaviour of a beam carrying accelerating moving mass using finite element approximation. The dynamic response of the beam was obtained depending on the mass ratio and the acceleration of the mass. Ahmadian et al [10] investigated the analysis of a variable cross-section beam when subjected to a moving concentrated force and mass using the finite element method. The work of Samani and Pellicano [11] centred on the analysis of the effectiveness of the dynamic vibration absorbers applied to beam excited by moving load and the performances of dynamic vibration absorbers in suppressing the vibrations of a simply supported beam subjected to an infinite sequence of regularly spaced concentrated moving loads. Masoud [12] presented the problem of non-uniform cross-section beam with different boundary conditions subjected to moving loads, such as moving concentrated mass, a simple quarter-car (SQC) planner model, and a two-axle dynamic system with four DOF by developing the derivation of characteristic equations and including the effect of structural damping and moving load inertia on the beam. Amin [13] presented analytical-numerical examples to demonstrate the simplicity and efficiency of the dynamic Green function.

However, in structural design, loads are actually distributed over a small segment or over the entire length of the structural members they traversed. This is termed distributed load. The analysis of the dynamic behaviour Bernoulli-Euler beam carrying uniformly partially distributed moving masses was addressed by Emailzadeh and Ghorashi [14]. They solved the problem by means of conventional analytical technique, which is only suitable for the simple horizontal beam and will suffer many difficulties if the structures are

complicated. Gbadeyan and Dada [15] studied the dynamic response of plates on the Pasternak foundation to distributed moving load. In a recent development, Oni and Ogunyebi [16] investigated the dynamic response of uniform Rayleigh beam on variable Pasternak foundation when under the action of moving partially distributed load. Both the foundation stiffness and shear modulus are of the variable type for moving distributed forces and moving distributed masses respectively. It is noted that in the above works, numerical simulations were mainly used to address the governing partial differential equations for moving a distributed load of classical boundary conditions.

This present study therefore, concerns the dynamic response of harmonically varying numerically magnitude moving at a constant speed. It will among others explore the dynamic behaviour of vital structural parameters (load natural frequency, damping, effective torsional rigidity, foundation modulus) under moving distributed loads.

II. MATHEMATICAL MODEL

Consider the vibration of a one-dimensional structural member resting on an elastic foundation and traversed by harmonic magnitude moving partially distributed load. The mass M is assumed to strike the beam at the point $x = 0$ and at time $t = 0$ and travel across the beam at a constant speed c. The motion equation governing the system of the uniform cross-section is the fourth-order partial differential equation by

$$EI \frac{\partial^4}{\partial x^4} z(x,t) + \mu \frac{\partial^2 z(x,t)}{\partial t^2} - 2T_r \frac{\partial^2 z(x,t)}{\partial x^2} + \alpha \frac{\partial z(x,t)}{\partial t} + Kz(x,t) = P(x,t) \qquad (1)$$

where EI is the flexural rigidity, $z(x,t)$ is the deflection of the beam, x is the spatial coordinate, t is time, μ as the mass per unit length of the beam, T_r is the effective torsional rigidity, α is the damping coefficient, K as the foundation reaction, E is the modulus of elasticity, I is the moment of inertia, and $P(x,t)$ as the travelling distributed load.

The uniform beam under consideration has simple supports at both ends $x = 0$ and $x = L$. Thus the boundary conditions are

$$z(0,t) = 0 = z(L,t), \quad \frac{\partial^2 z(0,t)}{\partial x^2} = \frac{\partial^2 z(L,t)}{\partial x^2} \qquad (2)$$

and the initial condition is given as

$$z(x,0) = 0 = \frac{\partial z(x,0)}{\partial t} \qquad (3)$$

The harmonic magnitude force $P(x,t)$ acting on the slender member is given as [17]

$$P(x,t) = P_o \left(1 + \frac{1}{2} \cos \omega t \right) H(x - ct) \qquad (4)$$

where is ω the load natural frequency and $H(x - ct)$ is the Heaviside unit step function with the property

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases} \qquad (5)$$

The foundation reaction is taken to be

$$Kz(x,t) = K_f z(x,t) \qquad (6)$$

Substituting equations (4) and (6) into (1), one obtains

$$EI \frac{\partial^4}{\partial x^4} z(x,t) + \mu \frac{\partial^2 z(x,t)}{\partial t^2} - 2T_r \frac{\partial^2 z(x,t)}{\partial x^2} + \alpha \frac{\partial z(x,t)}{\partial t} + K_f z(x,t)$$
$$= P(x,t) = P_o \left(1 + \frac{1}{2} \cos \omega t \right) H(x - ct) \qquad (7)$$

Equation (7) is the non-homogeneous fourth-order partial differential equation of our dynamical system for the uniform non-prestressed beam traversed by harmonic magnitude moving partially distributed load on a constant elastic foundation.

III. SOLUTION

In this section, an analytical technique that is suitable to treat the above dynamical system problem will be discussed. To this end, use is made of the integral transform technique particularly, finite Fourier sine integral transform defined by

$$z(x,t) = \int_0^L U(m,t)V_m(x)dx \tag{8}$$

with the inverse transform defined as

$$U(m,t) = \frac{2}{L}\sum_{m=1}^{\infty} z(x,t)V_m(x) \tag{9}$$

and

$$V_m(x) = \frac{\sin m\pi x}{L} \tag{10}$$

Using equation (8), equation (7) on arrangements gives

$$\ddot{U}(m,t) + a_{m1}\dot{U}(m,t) + \left(a_{m2} + a_{m3} + a_{m4}\right)U(m,t) = \left(a_{m5} + a_{m6}\cos\omega t\right)H(x-ct) \tag{11}$$

where

$$a_{m1} = \frac{\alpha}{\mu}, \quad a_{m2} = \frac{EI}{\mu}\left(\frac{m^4\pi^4}{L^4}\right), \quad a_{m3} = \frac{2T_r}{\mu}\left(\frac{m^2\pi^2}{L^2}\right), \quad a_{m4} = \frac{K_f}{\mu}, \quad a_{m5} = \frac{P_o}{\mu}, \quad a_{m6} = \frac{P_o}{2\mu} \tag{12}$$

Equation (11) is the transformed equation governing the motion of a simply supported uniform beam on elastic foundation subjected to harmonic moving partially distributed load at constant speed.

Laplace Transform Solution of the Beam Equation

Equation (11) is a second-order ordinary differential equation. In what follows, we subject the system to a Laplace transformation defined as

$$(\tilde{\ }) = \int_0^\infty (\cdot)e^{-st}dt \tag{13}$$

where s is the Laplace parameter.
Applying the initial condition (3), one obtains the following algebraic equations

$$U(m,s) = \frac{1}{d_a - d_b}\left[\frac{P_1}{S^2 + \theta^2}\frac{P_1\Omega_1}{S^2 + \Omega_1^2} + \frac{P_2\Omega_2}{S^2 + \Omega_2^2}\right]\left(\frac{1}{S - d_a} - \frac{1}{S - d_b}\right) \tag{14}$$

where

$$d_a = \frac{-a_{m1} + \sqrt{a_{m1}^2 - 4a_m^*}}{2}, \quad d_a = \frac{-a_{m1} - \sqrt{a_{m1}^2 - 4a_m^*}}{2} \tag{15}$$

and

$$a_m^* = a_{m2} - a_{m3} + a_{m4} \tag{16}$$

Equation (14) after some simplification leads to

$$U(m,s) = P_s\left[\frac{1}{d_a}\left(\left(\frac{P_1 S}{S^2 + \theta^2}\cdot\frac{d_a}{S - d_a}\right) + \frac{P_2\Omega_1}{S^2 + \Omega_1^2}\cdot\frac{d_a}{S - d_a} + \frac{P_2\Omega_2}{S^2 + \Omega_2^2}\cdot\frac{d_a}{S - d_a}\right)\right.$$
$$\left. - \frac{1}{d_b}\left(\left(\frac{P_1 S}{S^2 + \theta^2}\cdot\frac{d_b}{S - d_b}\right) + \frac{P_2\Omega_1}{S^2 + \Omega_1^2}\cdot\frac{d_b}{S - d_b} + \frac{P_2\Omega_2}{S^2 + \Omega_2^2}\cdot\frac{d_b}{S - d_b}\right)\right] \tag{17}$$

where

$$P_s = \frac{1}{d_a - d_b} \tag{18}$$

To obtain the Laplace inversion of equation (16), the following representations are adopted

$$F_1(s) = \frac{d_a}{S - d_a}, F_2(s) = \frac{d_2}{S - d_b}, G_1(s) = \frac{S}{S^2 + \theta^2}, G_2(s) = \frac{\Omega_1}{S^2 + \Omega_1^2}, G_3(s) = \frac{\Omega_2}{S^2 + \Omega_2^2} \tag{19}$$

So that the Laplace inversion of each of the right hand side of equation (18) is the convolution of $f_i(t)$ & $g_i(t)$'s denoted by

$$f_i(t) * g_i(t) \tag{20}$$

where

$$f_i(t) = L^{-1}\{F_i(s)\}, \quad g_i(t) = L^{-1}\{G_i(s)\} \tag{21}$$

$f_i(t) * g_i(t)$ is represented by the integral

$$f_i(t) * g_i(t) = \int_0^t f(t-u)\,g\,udu, \quad i = 1, 2, 3 \cdots\cdots \tag{22}$$

To this end, one obtains the inversion of equation (18),

$$g_1(u) = \cos\theta u, \quad g_2(u) = \sin\Omega_1 u, \quad g_2(u) = \sin\Omega_1 u$$

$$f_1(t-u) = e^{da(t-u)}, \quad f_2(t-u) = e^{db(t-u)} \tag{23}$$

Therefore, equation (18) becomes

$$U(m,s) = P_s\left[\frac{e^{d_a t}}{d_a}\left(P_1 J_1^* + P_2(J_2^* + J_3^*)\right) - \frac{e^{d_b t}}{d_b}\left(P_1 J_4^* + P_2(J_5^* + J_6^*)\right)\right] \tag{24}$$

where

$$J_1^* = \int_0^t e^{-d_a u}\cos\theta u\,du, \quad J_2^* = \int_0^t e^{-d_a u}\sin\Omega_1 u\,du, \quad J_3^* = \int_0^t e^{-d_a u}\sin\Omega_2 u\,du$$

$$J_4^* = \int_0^t e^{-d_b u}\cos\theta u\,du, \quad J_5^* = \int_0^t e^{-d_b u}\sin\Omega_1 u\,du, \quad J_6^* = \int_0^t e^{-d_b u}\sin\Omega_2 u\,du \tag{25}$$

Evaluating the integrals above, one obtains

$$J_1^* = \frac{-\theta e^{-d_a t}\sin\theta t + d_a - d_a e^{-d_a t}\cos\theta t}{d_a^2 + \theta^2} \tag{26}$$

$$J_2^* = \frac{-\Omega_1 e^{-d_a t}\cos\Omega_1 t + \Omega_1 - d_a e^{-d_a t}\sin\Omega_1 t}{d_a^2 + \Omega_1^2} \tag{26}$$

$$J_3^* = \frac{-\Omega_2 e^{-d_a t}\cos\Omega_2 t + \Omega_2 - d_a e^{-d_a t}\sin\Omega_2 t}{d_a^2 + \Omega_2^2} \tag{27}$$

$$J_4^* = \frac{-\theta e^{-d_b t}\sin\theta t + d_b - d_b e^{-d_b t}\cos\theta t}{d_b^2 + \theta^2} \tag{28}$$

$$J_5^* = \frac{-\Omega_1 e^{-d_b t}\cos\Omega_1 t + \Omega_1 - d_b e^{-d_b t}\sin\Omega_1 t}{d_b^2 + \Omega_1^2} \tag{29}$$

$$J_6^* = \frac{-\Omega_2 e^{-d_b t}\cos\Omega_2 t + \Omega_2 - d_b e^{-d_a t}\sin\Omega_2 t}{d_b^2 + \Omega_2^2} \tag{30}$$

Substituting equations (26) – (30) into equation (24) and after some simplifications, one obtains

$$U(m,t) = P_s\Bigg\{\frac{e^{d_a t}}{d_a}\Big[-P_1\frac{e^{d_a t}}{d_a^2+\theta^2}\big(\theta\sin\theta t + d_a(1-\cos\theta t)\big)$$

$$-P_2\left(\frac{\Omega_1 e^{d_a t}}{d_a^2+\Omega_1^2}\left(\cos\Omega_1 t + 1 - \frac{d_a}{\Omega_1}\sin\Omega_1 t\right) - \frac{\Omega_2 e^{d_a t}}{d_a^2+\Omega_1^2}\left(\cos\Omega_2 t + 1 - \frac{d_a}{\Omega_2}\sin\Omega_2 t\right)\right)\Big]$$

$$-\frac{e^{d_b t}}{d_b}\Big[-P_1\frac{e^{d_b t}}{d_b^2+\theta^2}\big(\theta\sin\theta t + d_b(1-\cos\theta t)\big)$$

$$-P_2\left(\frac{\Omega_1 e^{d_b t}}{d_b^2+\Omega_1^2}\left(\cos\Omega_1 t + 1 - \frac{d_b}{\Omega_1}\sin\Omega_1 t\right) - \frac{\Omega_2 e^{d_b t}}{d_b^2+\Omega_1^2}\left(\cos\Omega_2 t + 1 - \frac{d_b}{\Omega_2}\sin\Omega_2 t\right)\right)\Big]\Bigg\} \tag{31}$$

which on inversion yields

$$z(x,t) = \frac{2}{L} \sum_{m=1}^{\infty} P_s \left\{ \frac{e^{d_a t}}{d_a} \left[-P_1 \frac{e^{d_a t}}{d_a^2 + \theta^2} \left(\theta \sin \theta t + d_a (1 - \cos \theta t) \right) \right. \right.$$

$$\left. -P_2 \left(\frac{\Omega_1 e^{d_a t}}{d_a^2 + \Omega_1^2} \left(\cos \Omega_1 t + 1 - \frac{d_a}{\Omega_1} \sin \Omega_1 t \right) - \frac{\Omega_2 e^{d_a t}}{d_a^2 + \Omega_1^2} \left(\cos \Omega_2 t + 1 - \frac{d_a}{\Omega_2} \sin \Omega_2 t \right) \right) \right]$$

$$-\frac{e^{d_b t}}{d_b} \left[-P_1 \frac{e^{d_b t}}{d_b^2 + \theta^2} \left(\theta \sin \theta t + d_b (1 - \cos \theta t) \right) \right.$$

$$\left. \left. -P_2 \left(\frac{\Omega_1 e^{d_b t}}{d_b^2 + \Omega_1^2} \left(\cos \Omega_1 t + 1 - \frac{d_b}{\Omega_1} \sin \Omega_1 t \right) - \frac{\Omega_2 e^{d_b t}}{d_b^2 + \Omega_1^2} \left(\cos \Omega_2 t + 1 - \frac{d_b}{\Omega_2} \sin \Omega_2 t \right) \right) \right] \right\} \cdot \frac{\sin m\pi x}{L} \quad (32)$$

which represents the transverse-displacement response to simply supported uniform beam under partially distributed load of harmonically moving magnitude at uniform velocity.

IV. NUMERICAL RESULTS AND DISCUSSION

This section presents the numerical results for the prismatic beam problem. A uniform beam of length 12.192 m and velocity 8.128 m/s is considered. Other values used are modulus of elasticity 3.34×10^{10} N/m², moment of inertia $1.042 \times 10^4 m^4$, mass per unit length of the beam $3401.563 kg/m$.
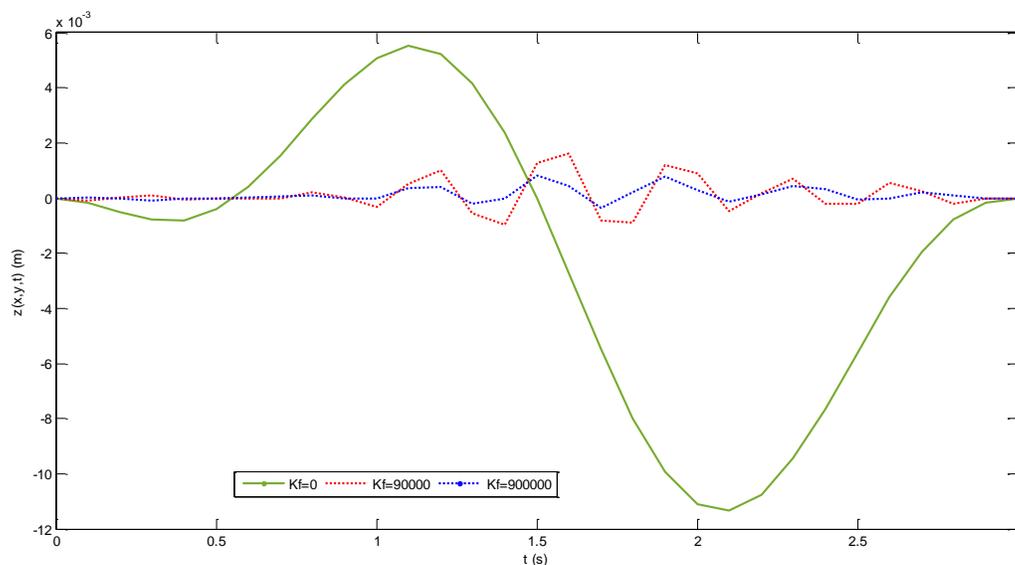

Figure 1: Deflection profile of Beam-type structural element on Winkler foundation subjected to moving load for various values of foundation modulus and fixed values of torsional rigidity, load natural frequency and damping.

Figure 1 shows the deflection profile of a uniform beam resting on the Winkler elastic foundation and traversed by moving distributed load at uniform velocity. From the figure, it is observed that for fixed values of damping, torsional rigidity and load natural frequency, the transverse displacement of a uniform beam resting on Winkler elastic foundation and traversed by moving distributed load moving at uniform velocity decreases as the values of foundation modulus K_f increases.
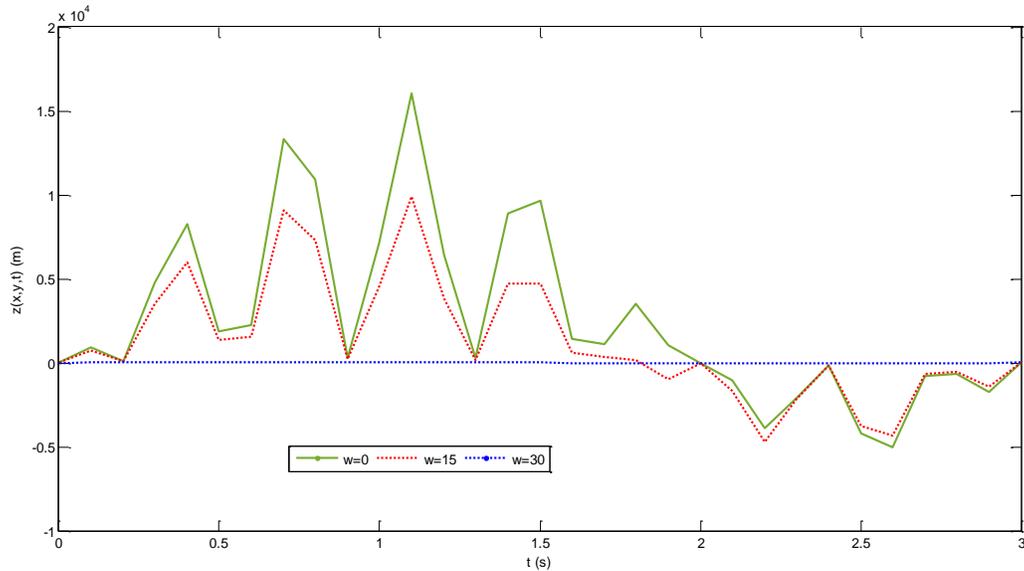
Figure 2: Displacement response of Beam-type structural element on Winkler foundation subjected to moving load for various values of load natural frequency and fixed values of torsional rigidity, foundation modulus and damping.

Figure 2 displays the displacement response of a uniform beam resting on the Winkler elastic foundation and traversed by moving distributed load at uniform velocity. Clearly, it is shown that the higher the value of the load natural frequency ω the lower the deflection of the uniform beam at uniform velocity.
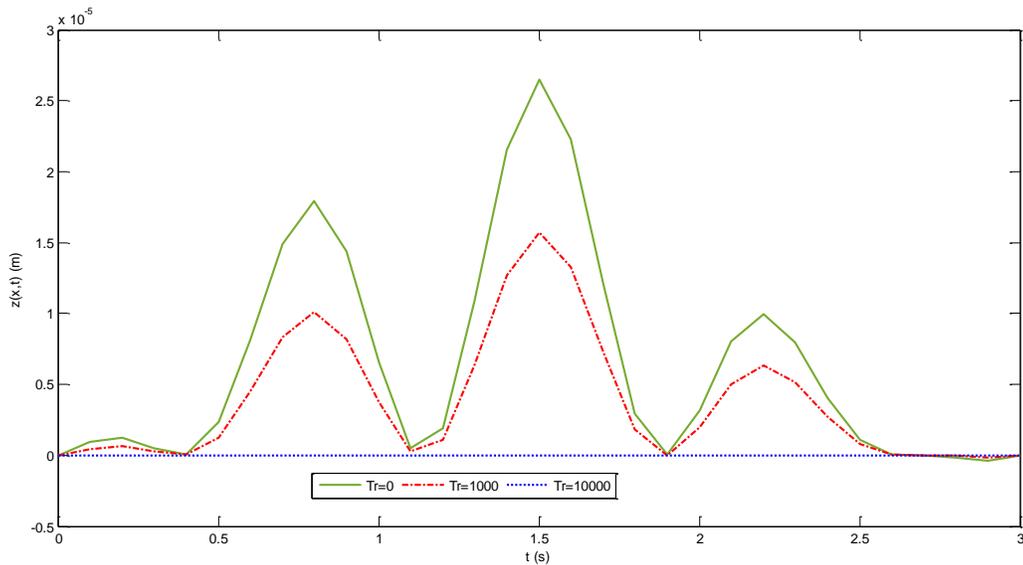

Figure 3: Displacement response of Beam-type structural element on Winkler foundation subjected to moving load for various values of torsional rigidity and fixed values of load natural frequency, foundation modulus and damping.

Figure 3 shows the dynamic response amplitude of the uniform beam resting on the Winkler elastic foundation and traversed by moving distributed load at constant velocity. It is clearly observed from the figure that higher values of effective torsional rigidity T_R reduce the deflection profile of the beam.
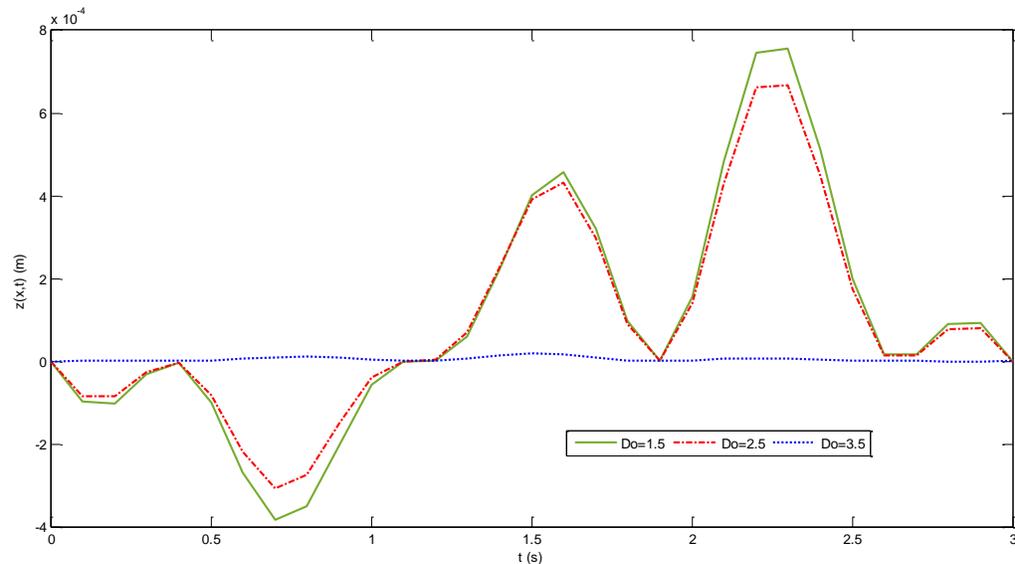
Figure 4: Deflection profile of Beam-type structural element on Winkler foundation subjected to moving load for various values of damping and fixed values of torsional rigidity, foundation modulus and load natural frequency.

Finally, figure 4 displays the transverse deflection of a uniform beam moving at uniform velocity for fixed values of other parameters and various values of the damping parameter α . It is observed that as the damping increases, the deflection of the beam decreases.

V. CONCLUSION

In this present paper, an analytical method of solution is presented for the problem of the dynamic behaviour to uniform simply-supported beam subjected to harmonic magnitude moving distributed load resting on elastic foundation at constant velocity. The approximation procedure is based on the finite Fourier sine transform and Laplace transformation. The closed-form solution of the governing fourth-order partial differential equation with variable and singular coefficients of the Uniform Bernoulli-Euler beam is presented. The dynamic influence of important parameters such as elastic foundation, effective torsional rigidity, load natural frequency and damping are closely examined. From the numerical analysis of results in plotted curves, it can be seen an increase in the structural parameters decreases the deflection of the uniform Bernoulli-Euler beam. Thus, the presence of the elastic foundation and damping clearly reduce the intensity of vibration and. also guarantee the safety of the traversing load.

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