Estimating the Performance Measure of Exponential-Gamma Distribution with Application to Failure Rate Data

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Abstract- Statistical distributions pose importance usefulness in predicting and analyzing real life occurrences. Several researchers have applied different statistical distributions in analyzing real life data. Therefore, this study focused on applying the newly developed Exponential-Gamma distribution in modeling data on the number of successive failures of air conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes. The result of the performance and adequacy of the Exponential-Gamma distribution was compared with other existing statistical distribution. The result showed that Exponential-Gamma distribution is adequate and fits data better than other exiting distribution compared.

Index Terms- Exponential-Gamma distribution, Failure Rates, AIC, BIC

I. Introduction

The call for statistical inferences and modeling of data sets demands the prior knowledge of suitable distribution and assumptions for the data sets. Consequently, virtually all the existing statistical distributions have been applied to data of various types in many areas of life to makes inferences about a particular subject matter which includes: modeling environmental pollution in environmental science, modeling duration without claims in actuarial science, modeling machine life span in engineering, modeling survival times of patients after surgery and modeling failure rate of software in computer science. Moreover, the process of generating data in many of these areas is typically based on varied degrees of skewness and kurtosis. Also, the data may exhibit non-monotonic hazard rates such as the bathtub, unimodal or modified unimodal hazard rates. Hence, modeling the data with the existing classical distributions does not provide a reasonable parametric fit and is often an approximation rather than reality.

In view of these, the development of new statistical distributions in the recent decades have continuously received great interest from researchers who study and develop new methods for modifying existing distributions to make them more flexible and robust or developing new statistical distributions for modeling data sets from different areas of study these methods are aimed towards yielding distributions with heavy tails, monotonic and non-monotonic hazard rates, tractable cumulative distribution function for efficiency in simulation and analyzing data with varieties of degrees of skewness and kurtosis.

In light of their adequacies, variety in usage and performance, statistical distributions have received numerous attentions from various researchers such as; [1],[2],[3],[4],[5]and[6] Therefore, this study aim to examine the adequacy and performance of the new Exponential-Gamma distribution to other exiting probability distributions using the data on the number of successive failures for the air conditioning system on each member in a fleet of 13 Boeing 720 jet airplanes, using the model selection criteria like the Akaike information criterion (AIC), Bayesian information criterion (BIC) and the log likelihood function (l).

II. METHODS

The Exponential-Gamma distribution was developed by [7] and its pdf is defined as

$$f(x;\alpha,\lambda)=\frac{\lambda^{\alpha+1}x^{\alpha-1}e^{-2\lambda x}}{\Gamma(\alpha)},x,\lambda,\alpha>0 \qquad (1)$$

With the mean and variance;

$$\mu=\frac{\alpha}{2^{\alpha+1}} \qquad (2)$$

and $$V(x)=\frac{\alpha\left(\alpha 2^{\alpha}-\lambda\alpha+2^{\alpha}\right)}{\lambda\left(2^{2(\alpha+1)}\right)} \qquad (3)$$

The cumulative distribution function is defined as

$$F(x)=\frac{\lambda\gamma(\alpha,x)}{2^{\alpha}\Gamma(\alpha)} \qquad (4)$$

The survival function for the distribution defined by $S(x)=1-F(x)$ was obtained as;

$$S(x)=1-\frac{\lambda\gamma(\alpha,x)}{2^{\alpha}\Gamma(\alpha)} \qquad (5)$$

While the corresponding hazard function defined by $h(x)=\dfrac{f(x)}{S(x)}$ was obtained as;

$$h(x)=\frac{\lambda^{\alpha+1}x^{\alpha-1}e^{-2\lambda x}2^{\alpha}}{2^{\alpha}\Gamma(\alpha)-\lambda\gamma(\alpha,x)} \qquad (6)$$

The cumulative hazard function for distribution defined by $H(x)=W(F(x))=-\log(1-F(x))\equiv\int_0^x h(x)dx$ and was obtained as;

$$H(x)=\frac{\lambda\gamma(\alpha,x)}{2^{\alpha}\Gamma(\alpha)-\lambda\gamma(\alpha,x)} \qquad (7)$$

A. Maximum Likelihood Estimator

Let $X_1, X_2, ..., X_n$ be a random sample of size n from Exponential-Gamma distribution. Then the likelihood function is given by;

$$L(\alpha,\lambda;x)=\left(\frac{\lambda^{\alpha+1}}{\Gamma(\alpha)}\right)^n \prod_{i=1}^{n} x_i^{\alpha-1}\exp\left(-2\lambda\sum_{i=1}^{n}x_i\right) \qquad (8)$$

by taking logarithm of (9), we find the log likelihood function as;

$$\log(L)=\alpha n\log\lambda+n\log\lambda-n\log\Gamma(\alpha)+(\alpha-1)\sum_{i=1}^{n}\log x_i-2\lambda\sum_{i=1}^{n}x_i \qquad (9)$$

Therefore, the MLE which maximizes (9) must satisfy the following normal equations;

$$\frac{\partial\log L}{\partial\alpha}=n\log\lambda-\frac{n\Gamma'(\alpha)}{\Gamma(\alpha)}+\sum_{i=1}^{n}\log x_i \qquad (10)$$

$$\frac{\partial\log L}{\partial\lambda}=\frac{\alpha n}{\lambda}+\frac{n}{\lambda}-2\sum_{i=1}^{n}x_i \qquad (11)$$

The solution of the non-linear system of equations is obtained by differentiating (9) with respect to (α,λ) gives the maximum likelihood estimates of the model parameters. The estimates of the parameters can be obtained by solving (10) and (11) numerically as it cannot be done analytically. The numerical solution can also be obtained directly by using python software using the data sets.

In this study, we applied the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and the log likelihood function (l) to compare the new developed Exponential-Gamma distribution with the existing probability distributions such as the Exponential and the Gamma distributions.

The AIC is defined by;

$$AIC=2k-2In(\hat{\ell}) \qquad (12)$$

where k is the number of the estimated parameter in the model

$\hat{\ell}$ is the maximized value of the model.

The BIC is defined by;

$$In(n)k-2In(\hat{\ell}) \qquad (13)$$

where k is the number of the estimated parameter in the model

n is the number of observations

$\hat{\ell}$ is the maximized value of the model.

The approach in (12) and (13) above is used when comparing the performance of different distributions to determine the best fit model. To select the appropriate model by considering the number parameters and maximum likelihood function; the AIC, BIC and likelihood function are examined; consequently an acceptable model has smaller AIC and BIC value while the log likelihood value is expected to be greater. The Python software was used for the comparison of the performance of the Exponential-Gamma, Exponential and Gamma distributions.

III. RESULTS

The analysis of the data set was carried out by using python software

This data has been previously used by [8],[9] and [10] consist of the number of successive failures for the air conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes. The data consist 213 observations as follows;

Table 1

194, 413, 90, 74, 55, 23, 97, 50, 359, 50, 130, 487, 102, 15, 14, 10, 57,320, 261, 51, 44, 9, 254, 493, 18, 209, 41, 58, 60, 48, 56, 87, 11, 102,12, 5, 100, 14, 29, 37, 186, 29, 104, 7, 4, 72, 270, 283, 7, 57, 33,100, 61, 502, 220, 120, 141, 22, 603, 35, 98, 54, 181, 65, 49, 12, 239, 14,18, 39, 3,12, 5, 32, 9, 14, 70, 47, 62, 142, 3, 104, 85, 67, 169,24, 21, 246, 47, 68, 15, 2, 91, 59, 447, 56, 29, 176, 225, 77, 197, 438,43, 134, 184, 20, 386, 182, 71, 80, 188, 230, 152, 36, 79, 59, 33, 246, 1,79, 3, 27, 201, 84, 27, 21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46,230, 59, 153, 104, 20, 206, 5, 66, 34, 29, 26, 35, 5, 82, 5, 61, 31,118, 326, 12, 54, 36, 34, 18, 25, 120, 31, 22, 18, 156, 11, 216, 139, 67,310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 26, 71, 39, 30, 7, 44, 11,63, 23, 22, 23, 14, 18, 13, 34, 62, 11, 191, 14, 16, 18, 130, 90, 163,208, 1, 24, 70, 16, 101, 52, 208, 95.

Table 2: Summary of data (Boeing jet)

Parameters	Values
n	213
Min	1.0
Max	603
Mean	93.14085
Variance	11398.47
Skewness	2.108908
Kurtosis	4.906632

The results from the table 2 indicated that the distribution of the data is skewed to the right with skewness 2.108908. This shows that Exponential-Gamma has the ability to fit a right skewed data. Also, it was observed that the kurtosis is 4.906632 which is greater than 3. This implies that the distribution has longer and fatter tails with a heavy peakedness when compared to that of the Normal distribution.

Table 3: Estimates and performance of the distributions (Boeing jet)

Distribution	Parameters	log likelihood (l)	AIC	BIC
Exponential-Gamma	$\hat{\alpha} = \mathbf{0.82}$ $\hat{\lambda} = \mathbf{1.0}$	**-8.09**	**22.30**	**32.26**
Exponential	$\hat{\lambda} = 1.0$	-1171.94	2347.94	2354.60
Gamma	$\hat{\alpha} = 0.82$ $\hat{\lambda} = 1.0$	-1165.17	2336.46	2346.41

The estimates of the parameters, log-likelihood, Akaike information criterion (AIC) and Bayesian information criterion (BIC) for the data on the number of successive failures for the air conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes is presented in Table 3. It was observed that Exponential-Gamma provides a better fit as compared to Exponential and Gamma distributions since it has highest value of log-likelihood (l) and the lowest value of Akaike information criterion (AIC) and Bayesian information criterion (BIC). Hence, the Exponential-Gamma distribution performed better than other distributions compared.

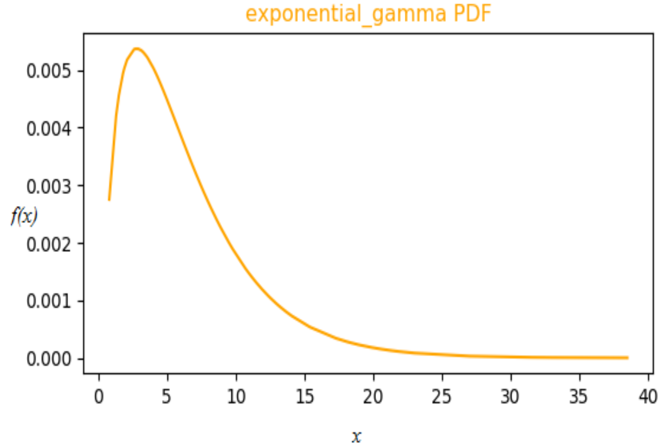

Figure:1 Exponential-Gamma distribution pdf plot for the data sets

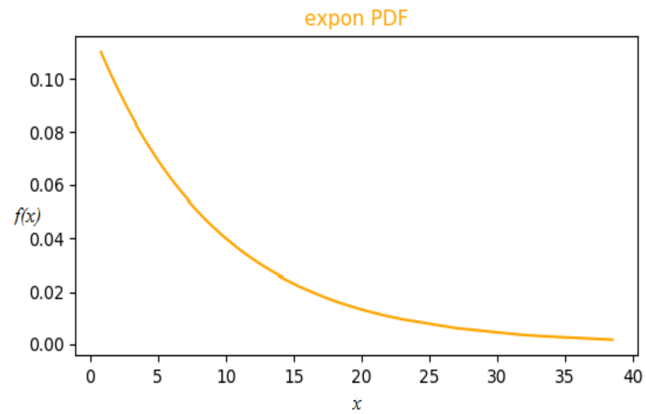

Figure:2 Exponential distribution pdf plot for the data sets

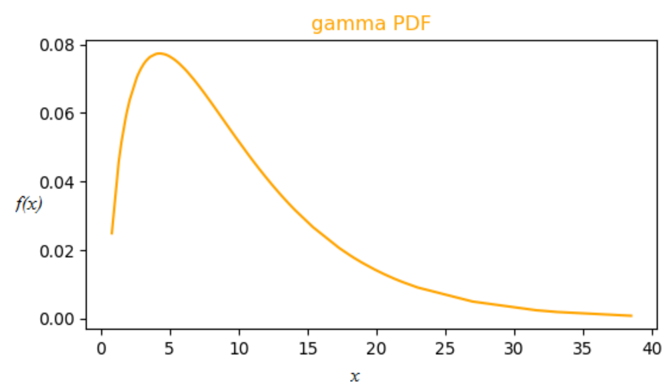

Figure 3: Gamma distribution pdf plot for the data sets

IV. CONCLUSION

Statistical distributions play a crucial role in describing and predicting real life occurrence, many distributions have been developed and there are always rooms for developing distributions which are more flexible and capable of handling real world application. In this study we apply the new Exponential-Gamma in modeling data on the number of successive failures for the air conditioning system on each member in a fleet of 13 Boeing 720 jet airplanes. The result of the performance and adequacy of the Exponential-Gamma distribution was compared with other existing statistical distribution. The result showed that Exponential-Gamma distribution is adequate and fits data better than other exiting distribution compared.

Therefore, for higher precision in analyzing data on failure rates, the use of Exponential-Gamma distribution is highly recommended in various fields where analysis of data failure rate is crucial.

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