On m-Derivation of BCI-Algebras with Special Ideals in BCK-Algebras

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Abstract- We collect some important concepts on BCK,BCI and BCL-algebras which are useful to develop the main results in subsequent topics. The left-right m-derivation of a BCI-algebra is introduced, and some related properties are investigated. Using the idea of regular m-derivation , we give characterizations of a p-semi-simple BCI-algebra .We introduce the useful properties of m-derivation in BCI-algebras and commutative and maximal ideals in BCK-algebras .

Index Terms- BCI- algebra, BCK-algebra, BCL-algebra, m-derivation in BCK-algebra, ideals in BCK-algebra.

I. Introduction and Preliminaries

To develop the main results in the following topics we need the following notions

1.1. Defintion.[1].

Let X be a set with binary operation $*$ and a constant 0. Then $(X,*,0)$ a BCI-algebra if it satisfies the following properties:

BCI-1. $\big((x*y)*(x*z)\big)*(z*y)=0$;
BCI-2. $(x*(x*y))*y=0$;
BCI-3. $x*y=0$;
BCI-4. $x*y=0$ and $y*x=0$ implies that $x=y$;
BCI-5. $0*x=0$, for all $x,y,z\in X$.

For brevity we also call X a BCK-algebra. In X we can define a binary relation$\le$ by putting $x\le y$ if and only if $x*y=0$. Then $(X,*,0)$ is a BCK-algebra if and only if satisfies that

BCK-1 $(x*y)*(x*z)\le(z*y)$,
BCK-2 $x*(x*y)\le y$,
BCK-3 $x\le x$,
BCK-4 $x\le y$ *and* $y\le x\Rightarrow x=y$,
BCK-5 $0\le x$.

1.2. Definition.[1].

A subset S of a BCI-algebra X is called an ideal of X if it satisfies

(i) $0\in I$;
(ii) $x*y\in S$ and $y\in I$ imply that $x\in I$ for all $x,y\in I$.

A subset S of a BCI-algebra X is called sub algebra of X if $x*y\in S$ for all $x,y\in S$.

1.3. Definition.[1].

A mapping f of a BCI-algebra X into itself is called an endomorphism of X if $f(x*y)=f(x)*f(y)$ for all $x,y\in X$.

1.4. Remark.

A BCI-algebra X has the following properties:

(1) $(x*y)*z=(x*z)*y$;
(2) $x\le y\ \Rightarrow\ x*z\le y*z$ and $z*y\le z*x$;
(3) $x*\big(x*(x*y)\big)=x*y$;
(4) $(x*z)*(y*z)\le x*y$;
(5) $0*(x*y)=(0*x)*(0*y)$;
(6) $x*0=0\ \Rightarrow\ x=0$.

For a BCI-algebra X, denote by $X_+\big(resp.,G(X)\big)$ the BCK-part $(resp.,the\ BCI-G\ part)$ of X, that is $X_+=\{x\in X|0\le x\}$ $(resp.,G(X)=\{x\in X|0*x=x\})$.
Note that: $G(X)\cap X_+=\{0\}$.

If $X_+=\{0\}$ then X is called a p-semi-simple BCI-algebra.

In a p-semi-simple BCI-algebra X, the following hold:

(7) $(x*y)*(y*z)=x*y$;
(8) $0*(0*x)=x$;
(9) $x*(0*y)=y*(0*x)$;
(10) $x*y=0\ \Rightarrow x=y$;
(11) $x*a=x*b\ \Rightarrow\ a=b$;
(12) $a*x=b*x\ \Rightarrow\ a=b$;
(13) $a*(a*x)=x$.

Remark. For any $x,y\in X$ we denote $x\wedge y=y*(y*x)$.
And $x\wedge x=x$, $x\wedge 0=0\wedge x=0$. But in general $x\wedge y\ne y\wedge x$.

1.5. Definition.[1]

Let X be a p-semi-simple BCI-algebra. We define addition $+$ as
$x+y=x+(0+y)\ \forall\, x,y\in X$. Then $(X,+)$ is an a belian group with identity 0 and $x-y=x*y$.

1.6. Example.

Let $X=\{0,1,2,\dots\}$ and the operation $*$ be defined as follows:

$$x*y=\begin{cases}0 & \text{if } x\le y\\ 1 & \text{otherwise}\end{cases}$$

Then $(X,*,0)$ BCI-1, BCI-3, BCI-4 and BCI-5, but it does not satisfy BCI-2.

1.7. Theorem. [1].

In any BCK-algebra, we have $x*(y\wedge x)=x*y$.

Proof.

Since $y\wedge x\leq y$, by properties of BCI-algebra we get $x*y\leq x*(y\wedge x)$.

On the other hand, by BCI-2 we have $x*(y\wedge x)=x*\big(x*(x*y)\big)\leq x*y$.

This means $x*y=x*(y\wedge x)$. #

1.8. definition.[1].

An algebra $(X,*,0)$ of type (2,0) is said to be a BCL-algebra if and only if for any $x,y,z\ in\ X$, the following conditions:

(1) BCL-1 : $x*x=0$;

(2) BCL-2 : $x*y=0$ and $y*x=0$ imply $x=y$;

(3) BCL-3 : $\big(((x*y)*z)*(x*z)*y)\big)*((z*y)*x)=0$.

1.9. Definition.[1].

Let $(X,*,0)$ is a BCL-algebra. A binary relation $\leq$ on X by which $x\leq y$ if and only if $x*y=0$ for any $x,y,z\in X$, we call the BCL-ordering $\leq$ is partial ordering on X.

1.10. Definition.[1].

Let $x\leq y$ if and only if $x*y=0$, the definition (1.8) can be written as

(1) BCL-1* $x\leq x$;

(2) BCL-2* $x\leq y$ and $y\leq x$ imply $x=y$;

(3) BCL-2* $\big((x*y)*z\big)*((x*z)*y\leq(z*y)*x$.

1.11. Theorem.[1].

(1) Any a BCK-algebra is a BCL algebra .

(2) $x*y=0$ if and only if $x\leq y$.

Proof.

Assume that (X,∗,0) is a BCL-algebra, then the BCL-ordering $\leq$ is a partial ordering on X. By definition of $\leq$, (2) is valid. Also, BCL-3 and (2) imply (1). Conversely, assume that $\leq$ a partial ordering on X, and satisfying (1) and (2) Also, by reflexing of $\leq$, we see that $x\leq x$, then (2) $\Rightarrow x*x=0$. Moreover, if

$x*y=0$ and $y*x=0$, then $x\leq y$ and $x\leq y$ and $y\leq x$ by (2) , and so the anti-symmetry of $\leq$ gives $x=y$. Therefore $(X,*,0)$ is a BCL-algebra.

II. M-DERIVATIONS.

Derivation is a very interesting and important area of research in the theory of algebraic structures in mathematics. Over the last some decals an interest for this topic has increased, many well known algebraists like K.I.Besdar, J.Bergen, M.Bresar, I.N.Hersteim.

In what follows, let m be an endomorphism of X unless otherwise specified. The derivations of BCI-algebras in different aspects as follows: in 2005 [8] , Zhen and Liu have given the notion of m-derivation of BCI-algebras and studied p-semi-simple BCI-algebras by using the idea of regular m-derivation in BCL-algebras.

2.1.Definition.[2].

Let X be a BCI-algebra. Then for any $x\in X$, we define a self map $d_m:X\longrightarrow X$ by: $d_m(x)=x*m\ for\ all\ x\in X$.

2.2. Definition.[2].

Let X be a BCI-algebra. By a left-right m-derivation (briefly , $(l,r)-m-$derivation of X, a self –map d_m of X satisfying the identity

$d_m(x*y)=\big(m(x)*d_m(y)\big)\wedge\ \big(d_m(x)*m(y)\big)$ for all $\in X$, then it is said that d_m is a right-left-m-derivation . Moreover, if d_m is both an (r,l) and an (l,r)-m-derivation, it is said that d_m is an $m-$ derivation.

2.3. Example.

Let $X=\{0,1,2,3,4,5\}$ be a BCI-algebra with the following Cayley table (1):

*	0	1	2	3	4	5
0	0	0	2	2	2	2
1	1	0	2	2	2	2
2	2	2	0	0	0	0
3	3	2	1	0	0	0
4	4	2	1	1	0	1
5	5	2	1	1	1	0

Define a map $d_m:X\longrightarrow X$ by

$$d_m(x)=\begin{cases}2 & \text{if } x=0,1\\ 0 & \text{otherwise}\end{cases},$$

And define an endomorphism of X by

$$m(x)=\begin{cases}0 & \text{if } x=0,1\\ 2 & \text{otherwise}\end{cases}.$$

Then it is easily checked that d_m is both derivation and $m-$ derivation of X.

2.4. Theorem.[2].

d_m be a self-map of a BCI- algebra X defined by $d_m(x)=m_x\ \forall\ x\in X$. Then d_m is an $(l,r)-\ m-$derivation of X. Moreover, if X is commutative, then d_m is an $(r,l)-m-$ derivation of X .

Proof.

Let X be an associative BCI -algebra, then we have

$d_m(x*y)=(x*y)*m=\{x*(y*m)\}*0$ by property (6) and (2)

$=\{x*(y*m)\}*[\{x*(y*m)\}*\{x*(y*m)\}]$ by property (iii)

$=\{x*(y*m)\}*[\{x*(y*m)\}*\{(x*y)*m\}]$ by property (6)

$=\{x*(y*m)\}*[\{x*(y*m)\}*\{(x*m)*y\}]$ by property (1)

$=((x*m)*y)\wedge\big(x*(y*m)\big)$ $=(d_m(x)*y)\wedge\big(x*d_m(y)\big)$.

2.5. Theorem.[2].

Let d_m be a self map of an associative BCI-algebra X. Then, d_m is a $m-$derivation of X.

2.6. Definition.[2].

A self map d_m of a BCI-algebra X is said to be $m-$regular if $d_m(0) = 0$.

2.7. Proposition.[2].

Let d_m be a self map of a BCI-algebra . Then :

(i) if d_m is a (l,r)-m- derivation of X, then $d_m(x) = d_m(x) \wedge x$ for all $x \in X$.

(ii) d_m is a (r,l)- m - derivation of X if and only if d_m is $m-$regular.

Proof.

(i) Let d_m be (l,r)-m- derivation of X , then

$$\begin{aligned} d_m(x) &= d_m(x*0) = (d_m(x)*0) \wedge (x*d_m(0)) \\ &= d_m(x) \wedge \{x*d_m(0)\} \\ &= \{x*d_m(0)\} * \{x*d_m(0)\} * d_m(x) \\ &= \{x*d_m(0)\} * [\{x*d_m(0)\} * d_m(0)] \\ &\le x*\{x*d_m(x)\} \quad \text{by property (3)} \\ &= d_m(x) \wedge x. \end{aligned}$$

But $d_m(x) \wedge x \le d_m(x)$ is trivial (i) holds.

(ii) Let d_m be a $(r,l)-m-$ derivation of X $d_m(x) = x \wedge d_m(x)$, then

$d_m(0) = 0 \wedge d_m(0) = d_m(0) * \{d_m(0)*0\} = d_m(0) * d_m(0) = 0$ there by implying d_m is $m-$regular. Conversely, suppose that d_m is $m-$regular, that is $d_m(0) = 0$, then we have

$$\begin{aligned} d_m(x) &= d_m(x*0) = (x*d_m(0)) \wedge (d_m(x)*0) \\ &= (x*0) \wedge d_m(x) = x \wedge d_m(x) \ \#. \end{aligned}$$

This is complete the proof.

2.8. Example.

Let $X = \{0, a, b\}$ be a BCI-algebra with the following Cayley table (2):

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

For any $m \in X$, we define a self map $d_m: X \longrightarrow X$ by

$$d_m(x) = x*m = \begin{cases} b & \text{if } x = 0, a \\ 0 & \text{if } x = b \end{cases}.$$

(i) Then it is easily checked that d_m *is* (l,r) and $(r,l)-m-$derivation of X, which is not $m-$ regular.

(ii) For any $m \in X$, define a self map $d'_m = x*m = \begin{cases} 0 & \text{if } x = 0, a \\ b & \text{if } x = b \end{cases}$.

Then it is easily checked that d'_m *is* (l,r) and $(r,l)-m-$derivation of X, which is $m-$ regular.

2.9. Definition.[2].

Let X be a BCI-algebra and let d_m , d'_m be two self maps of X. Then we define

$d_m \circ d'_m : X \longrightarrow X$ by: $(d_m \circ d'_m) = d_m(d'_m(x)) \ \forall x \in X$.

2.10. Proposition.[2].

Let X be a p-semi-simple BCI-algebra and let d_m , d'_m be (l,r) -m-derivations of X. Then , $d_m \circ d'_m$ of X. Then, $d_m \circ d'_m$ is also a (l,r)- m derivation of X.

Proof.

Let X be a p-semi-simple BCK-algebra. d_m and d'_m are $(l,r)-$m-derivations of X. Then $\forall x,y \in X$, we get

$$\begin{aligned} (d_m \circ d'_m) &= d_m(d'_m(x*y)) = d_m[(d'_m(x)*y) \wedge (x*d'_m(y))] \\ &= d_m[(x*d'_m(y)) * \{(x*d'_m(y)) * (d'_m(x)*y)\}] \\ &= d_m(d'_m(x)*y) \quad \text{by property (7)} \\ &= \{x*d_m(d'_m(y))\} * [\{x*d_m(d'_m(y))\} * \{d_m(d'_m(x)*y)\}] \\ &= \{d_m(d'_m(x)*y)\} \wedge \{x*d_m(d'_m(y))\} \\ &= ((d_m \circ d'_m)(x)*y) \wedge (x*(d_t \circ d'_m)(y)) . \end{aligned}$$

Therefore, $(d_m \circ d'_m)$ is a $(l,r)-m-$derivaion of X.

2.11. Proposition.[2].

Let X be a p-semi-simple BCI-algebra and let d_m, d'_m be $(l,r)-m$-derivation of X. Then, $d_m \circ d'_m$ is also a $(r,l)-m-$derivation of X.

III. IDEALS IN A BCK-ALGEBRA.

We deal with the study of some ideals in BCK-algebras.

3.1. Definition.[3].

Anon empty subset A of a BCI-algebra X is called a left (resp. right) 1-ideal of X if :

(1) $x.a \in A$ (resp. $a.x \in A$) whenever $x \in X$ and $a \in A$.

(2) for any $x,y \in X$ $x*y \in A$ and $y \in A$ imply $y\,x \in A$.

Both a left and a right 1-ideal is called 1-ideal.

3.2 Definition.[3].

Anon-empty subset A of a BCI-algebra X is called a left (resp. right) associative of X if :

(1) $x.a \in A$ (resp. $a.x \in A$) whenever $x \in A$ and $a \in A$.

(2) for any $x,y,z \in X, (x*y)*z \in A$ and $y*z \in A$ imply that $x \in A$.

3.3. Example.

Let $X = \{0,1,2\}$ in which $*$ is given by the table (3)

$*$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Then $(X,*,0)$ is an implicative BCK-algebra, and $\{0\}, X$, $\{0,1\}$ and $\{0,2\}$ are all ideals of X.

3.4. Definition.[3].

Given a BCK-algebra $(X.*,0)$, a nonempty subset I of X is said to be a positive implicative ideal if it satisfies, for all x,y,z in X,

(i) $0 \in I$.

(ii)$(x*y)*z \in I$ and $y*z \in I$ imply $x*z \in I$.

3.5. Example.
Let $X = \{0,1,2,3,4\}$ in which $*$ is given by table (4)

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

Then $(X,*,0)$ is a BCK-algebra. Clearly, $\{0,1,3\}$ and $\{0,1,2,3\}$ are positive implicative ideals of X. $\{0\}, \{0,2\}$ and $\{0,2,4\}$ are ideals of X, but not positive implicative.

3.6. Theorem.[3].

If we are given an I of a BCK-algebra, then I is positive implicative if and only if any $n \in X$, then

$$A_n = \{x \in X : x * n \in I\}$$

is and ideal of X.

Proof.

Suppose that I is positive implicative and $x * y \in A_n$ and $y \in A_n$. Then
$(x * y) * n \in I$ *and* $y * n \in I$. By (ii) we obtain $x * n \in I, i.e., x \in A_n$. This says A_n is an ideal.
$(\Leftarrow)$ Suppose that for any n in X, A_n is an ideal of X. If $(x * y) * z \in I$ and
$y * z \in I$, then $x * y \in A_z$ and $y \in A_z$. Since A_z is an ideal of X, $x \in A_z$, and so $x * z \in I$. This means that I is positive implicative #.

If I is a positive implicative ideal, then A_n is an ideal, as well as the least ideal containing I and n. In fact, if B is any ideal containing I and n, then $\forall\, x \in A_n$, we have $x * n \in I$. If follows that $x * n \in B$ an $n \in B$, hence $x \in B$. This shows that $A_n \subseteq B$. The assertion holds. Thus we have

3.7. Corollary.[3].

If I is appositive implicative ideal of X then for any $n \in X$, $A_n = \{x \in X : x * n \in I\}$ is the least ideal containing I and n .

3.8.Definition.[3].

Given a BCK-algebra $(X,*,0)$, an ideal I of X is called a maximal ideal if I is a proper ideal of X and not a proper subset of any proper ideal of X.

3.9.Example.
Let $X = \{0,1,2,3,4\}$, in which $*$ is given by the table (5)

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	1
3	3	3	3	0	3
4	4	4	4	4	0

Then $(X,*,0)$ is a BCK-algebra, $\{0,1,2,3\}$ and $\{0,1,2,4\}$ are two maximal ideal of X.

3.10. Theorem.[3].

Suppose $(X,*,0)$ is a bounded BCK-algebra and $|X| \geq 2$. Then X at least have one maximal ideal.

Proof.

First we prove that an ideal I of X is proper if and only if $1 \notin I$ .In fact, if $1 \notin I$ then $I \neq X$, and so I is a proper ideal. Conversely, assume that I is proper . If $1 \in I$ then $x \leq 1$ for all x in X, hence $x \in I$. This means that $I = X$, which contradicts to the hypothesis . Therefore $1 \notin I$. The second step. We prove that every ideal A is contained in a maximal ideal. The set of all proper ideals containing A is denoted by S. Obviously, $(S, \subseteq)$ is partially ordered set and

$S \neq \phi$. Let S_0 be a totally ordered subset of S and denote by $B = \cup \{I : I \in S_0\}$. Noticing that A is the least element of $(S, \subseteq)$ we have $A \subseteq B$. Hence $0 \in B$.

Let $x * y \in B$ and $y \in B$. Then there are $I_1, I_2 \in S_0$ such that $x * y \in I_1$ and $y \in I_2$. We can suppose $I_2 \subseteq I_1$ without loss of any generality. Thus $x * y \in I_1, y \in I_1$ and $y \in I_1$. It follows that $x \in B$. This means that B is an ideal. Since every ideal of S_0 does not contain the element 1, we have $1 \notin B$. By the first step B is a proper ideal, hence $B \in S$. This proves that every totally ordered subset of S have an upper bound in S. By Zorn's Lemma S have a maximal element M. Clearly, $A \subseteq M$. Therefore M is indeed a maximal ideal. The proof is completed. As an immediate consequence of above theorem we have.

3.11. Theorem.[3].

Suppose X is a bound BCK-algebra and let I be a proper commutative (resp. implicative, positive implicative) ideal, then there is a maximal commutative (resp. implicative, positive implicative) ideal containing 1. In the next theorem some of equivalent conditions of maximal ideals are given.

IV. Conclusion .

In this paper, we have considered the definition of BCK , BCI, and BCL- algebra , we get that any BCK-algebra is a BCL-algebra, we introduced the notation of m-derivations in BCI-algebras and investigated the useful properties of m-derivations in BCI-algebras. We discuss commutative and maximal ideals in BCK-algebras, the notion of maximal ideals in BCK-algebras is also given.

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