Calculation of the Gravitational Constant

Abstract

$$G^4 = \frac{\pi^5 \alpha^{21} \omega^{11} \left(\frac{1}{R^2} + \frac{\alpha_1}{R_1^2} \tan \frac{\theta}{2} \right)^2}{\omega_1^{20} T^5 \cos^{20} \beta}$$
$$M = \frac{\pi^3 \alpha^{12} \omega^8}{\omega_1^{12} G^4 T^3 \cos^{12} \beta}$$

$'G'$ is the Gravitational constant and $'M'$ is the mass of Earth .

$\omega =$ Angular speed of revolution of Moon = 2. 668986971 $\times 10^{-6}$ rad/s

$\omega_1 =$ Angular speed of rotation of Earth = 7.292115856 $\times 10^{-5}$ rad /s

R^2= Square of the radius of Earth = 40.60770616 $\times 10^{12} m^2$

R_1 = Radius of Moon = 1.737844684 $\times$ 10^6 m

T= Time during 12 synodical period of Moon = 3.053371592$\times$ 10^7 seconds

α_1 = The ratio of mass of Moon to the mass of Earth= 1.229546595 $\times 10^{-2}$

$$\alpha = 0.9952702095$$
$$\beta = 2.442307055^0$$

$\theta =$ Angle between the plane of the orbit of Moon and the equatorial plane of Earth = 43.27570763^0

Hence we get

$$G = 6.672684704 \times 10^{-11}$$

and

$$M = 5.9767529499 \times 10^{24}$$

Calculation of the Gravitational Constant

1. <u>Angular Speed of Revolution of Moon</u> $:-$

The time period of revolution of Moon is

$$T = 27.3216615 \text{ days}$$

$$\begin{aligned} \text{One day} &= 23 \text{ hours } 56 \text{ minutes } 4.09053 \text{ seconds} \\ &= 23.93446959\ hours \end{aligned}$$

So , the angular speed of revolution of Moon is

$$\begin{aligned} w &= \frac{2\pi}{T} \\ &= 2.668986971 \times 10^{-6} rad/s \end{aligned}$$

2. <u>Angular Speed of Rotation of Earth : -</u>

The time period of rotation of Earth is

$$\begin{aligned} T &= 23\ hours\ 56\ minutes\ 4.09053\ seconds \\ &= 23.93446959\ hours \end{aligned}$$

So , the angular speed of rotation of Earth is

$$\begin{aligned} w &= \frac{2\pi}{T} \\ &= 7.292115856 \times 10^{-5} rad/s \end{aligned}$$

3. <u>Mass of Moon :-</u>

The mass of Earth is

$$M_1 = 5.977 \times 10^{24} kg$$

and the mass of Moon is

$$M_2 = 7.349 \times 10^{22} kg$$

The ratio of mass of Earth to the mass of Moon is

$$\begin{aligned} M &= \frac{M_1}{M_2} \\ &= 81.33079331 \end{aligned}$$

We determine the mass of Moon by using the formula

$$\sqrt{M} = \frac{w_1^2 R\ cos\ \theta}{w_2^2 a}$$

where $'M'$ is the ratio of mass of Earth to the mass of Moon , $'w_1'$ is the angular speed of rotation of Earth , $'w_2'$ is the angular speed of revolution of Moon , $'a'$ is the mean distance of Moon from Earth , $'R'$ is the equatorial radius of Earth and $'\theta'$ is the angle between the plane of the orbit of Moon and the equatorial plane of Earth.

$$\therefore cos\theta = \frac{\sqrt{M} w_2^2 a}{w_1^2 R} \text{............................}(1)$$

4. **<u>Radius of Earth :-</u>**

The equatorial and polar radii of Earth are respectively given by

$$a = 6.37814 \times 10^6 m$$

and

$$b = 6.356755 \times 10^6 m$$

So, the mean radius of Earth is

$$R_1 = \frac{2a+b}{3}$$
$$= 6.371011667 \times 10^6 m$$

Let $'\alpha'$ be the eccentric angle of the mean radius $'R_1'$ of Earth

$$\therefore\ R_1^2 = a^2 cos^2\alpha + b^2 sin^2\alpha$$
$$\therefore\ cos^2\alpha = \frac{R_1^2 - b^2}{a^2 - b^2}$$
$$= 0.6662935167$$
$$\therefore\ cos\alpha = 0.8162680422$$
$$\therefore\ \alpha = 35.28706334^o$$

is the eccentric angle of the mean radius of Earth.

The angle between the planes of the orbits of Moon and Sun

$$= 5.1453964^o$$
$$\therefore \frac{5.1453964^o}{2} = 2.5726982^o$$
$$= \alpha_1 (s\ a\ y)$$

The angle of inclination of Moon is

$$\alpha_2 = 1^o\ 32^{'}\ 32.7^{''}$$
$$= 1.542416667^o$$
$$\therefore \alpha - (\alpha_1 + \alpha_2) = 31.171948473^o$$
$$= \theta(s\ a\ y)$$

The radius $'R'$ of Earth for its ecentric angle $'\theta'$ is given by

$$R^2 = a^2 cos^2\theta + b^2 sin^2\theta$$
$$= 40.60770616 \times 10^{12} m^2$$

5. <u>Radius of Moon : -</u>

Let $'a'$ be the mean distance of Moon from Earth. The radius of Moon at its mean distance from Earth is

$$\theta = 15^{'}\ 32.58^{''}$$
$$= 0.25905^o$$

Hence , the radius of Moon is

$$R = a \times \theta \times \frac{\pi}{180^o}$$

6. <u>Mean Distance of Moon from Earth : -</u>

The length of the mean sidereal day is

$$t_1 = 23\ hours\ 56\ minutes\ 4.09053\ seconds$$
$$= 23.93446959\ hours$$

The length of the mean solar day is

$$t_2 = 24\ hours\ 3\ minutes\ 56.55537\ seconds$$
$$= 24.06570983\ hours$$

The length of the nodical month (node to node) is

$$T_1 = 27.2122207\ days$$

The length of the sidereal month (fixed star to fixed star) is

$$T_2 = 27.3216615\ days$$

The geometric mean of $\frac{t_1}{t_2}$ and $\frac{T_1}{T_2}$ is

$$\alpha = \sqrt{\frac{t_1}{t_2} \times \frac{T_1}{T_2}}$$
$$= 0.9952702095$$

The distance $'d'$ of Moon from Earth is given by

$$d^3 = \frac{GM}{w^2}$$

where $'M'$ is the mass of Earth , $'w'$ is the angular speed of revolution of Moon and $'G'$ being the Gravitational constant .

$$\therefore d = \frac{(GM)^{\frac{1}{3}}}{w^{\frac{2}{3}}}$$

Hence, the mean distance of Moon from Earth is

$$a = \frac{d}{\alpha}$$
$$= \frac{(GM)^{\frac{1}{3}}}{\alpha w^{\frac{2}{3}}} \text{..........................(2)}$$

7. Gravitational Constant:-

Now ,

$$w^2 a \ \alpha \ g^2$$

and

$$w \ \alpha \ g^2$$

where $'w'$ is the angular speed of revolution of Moon , $'a'$ is its mean distance from Earth and $'g'$ is the acceleration due to gravity of both Earth and Moon .

$$\therefore G = \frac{w^3 a}{g^2}$$

where $'G'$ is the Gravitational constant .

We write ,

$$\frac{g - g_0}{g_1} = tan\frac{\theta}{2}$$

where $'g'$ is the acceleration due to gravity of both Earth and Moon , $'g_0'$ is the acceleration due to gravity of Earth , $'g_1'$ is the acceleration due to gravity of Moon and $'\theta'$ is the angle between the plane of the orbit of Moon and the equational plane of Earth .

The mass of Earth is

$$M = 5.977 \times 10^{24} kg$$

and the mass of Moon is

$$M_1 = 7.349 \times 10^{22} kg$$

The ratio of mass of Moon to the mass of Earth is

$$\alpha_1 = \frac{M_1}{M}$$
$$= 1.229546595 \times 10^{-2}$$

We have

$$g = g_0 + g_1 tan\frac{\theta}{2}$$
$$= \frac{GM}{R^2} + \frac{GM_1}{R_1^2} tan\frac{\theta}{2}$$

where $'R'$ is the radius of Earth and $'R_1'$ is the radius of Moon

$$\therefore g = \frac{GM}{R^2} + \frac{G\alpha_1 M}{R_1^2} tan\frac{\theta}{2}$$

Hence ,

$$G = \frac{w^3 a}{g^2}$$
$$= \frac{w^3 a}{\left(\frac{GM}{R^2} + \frac{G\alpha_1 M}{R_1^2} tan\frac{\theta}{2}\right)^2}$$
$$= \frac{w^3 (GM)^{\frac{1}{3}}}{\alpha w^{\frac{2}{3}} G^2 M^2 \left(\frac{1}{R^2} + \frac{\alpha_1}{R_1^2} tan\frac{\theta}{2}\right)^2} \quad (using\ equation\ (2)\ )$$
$$\therefore G^{\frac{8}{3}} = \frac{w^{\frac{7}{3}}}{\alpha M^{\frac{5}{3}} \left(\frac{1}{R^2} + \frac{\alpha_1}{R_1^2} tan\frac{\theta}{2}\right)^2} \text{..............................(3)}$$

The unit of " $\frac{w^3 a}{g^2}$ " in the R.H.S. of the equation

$$G = \frac{w^3 a}{g^2}$$

is

$$\frac{\frac{1}{s^3}.m}{(m/s^2)^2} = \frac{s}{m}$$

But , the unit of the Gravitational constant $'G'$ is

$$\frac{N.m^2}{kg^2} = \frac{kg.m}{s^2}.\frac{m^2}{kg^2}$$
$$= \frac{m^3}{s^2.kg}$$

So, in order to transfer $'s/m'$ to $"\frac{m^3}{s^2.kg}"$; we should have to multiply $'s/m'$ with $"\frac{m^4}{s^3.kg}"$

We write ,

$$\frac{(w_1 a\ cos\ \beta)^4 T}{\pi M} = 1$$

where $'M'$ is the mass of Earth , $'w_1'$ is its angular speed of rotation , $'a'$ is the mean distance of Moon from Earth, $'T'$ being the time during 12 synodical period of Moon and $'\beta'$ is an angle of derivation . Thus , the unit of the L.H.S. of the above equation becomes $"\frac{m^4}{s^3.kg}"$. Hence , multiplying it with $"\frac{w^3 a}{g^2}"$ of the equation

$$G = \frac{w^3 a}{g^2}$$

we get the desired unit of $'G'$ without the change of its numerical value.
Henceforth , there should not be any confusion about the dimension of $'G'$.

As a result of it we get

$$M = \frac{(w_1 a\ cos\ \beta)^4 T}{\pi}$$
$$= \frac{w_1^4 (GM)^{\frac{4}{3}} T cos^4 \beta}{\pi \alpha^4 w^{\frac{8}{3}}} (using\ equation\ (2))$$

$$\therefore M^{-\frac{1}{3}} = \frac{w_1^4 G^{\frac{4}{3}} T cos^4 \beta}{\pi \alpha^4 w^{\frac{8}{3}}}$$
$$or,\ M^{\frac{1}{3}} = \frac{\pi \alpha^4 w^{\frac{8}{3}}}{w_1^4 G^{\frac{4}{3}} T cos^4 \beta} ----------(4)$$
$$\therefore\ M^{\frac{5}{3}} = \frac{\pi^5 \alpha^{20} w^{\frac{40}{3}}}{w_1^{20} G^{\frac{20}{3}} T^5 cos^{20} \beta} ---------(5)$$

Using equation (5) in equation (3) we have

$$G^{\frac{8}{3}} = \frac{w^{\frac{7}{3}} w_1^{20} G^{\frac{20}{3}} T^5 cos^{20}\beta}{\alpha\pi^5\alpha^{20} w^{\frac{40}{3}} \left(\frac{1}{R^2} + \frac{\alpha_1}{R_1^2} tan\frac{\theta}{2}\right)^2}$$

$$= \frac{w_1^{20} G^{\frac{20}{3}} T^5 cos^{20}\beta}{\pi^5\alpha^{21} w^{11} \left(\frac{1}{R^2} + \frac{\alpha_1}{R_1^2} tan\frac{\theta}{2}\right)^2}$$

$$\therefore G^4 = \frac{\pi^5\alpha^{21} w^{11} \left(\frac{1}{R^2} + \frac{\alpha_1}{R_1^2} tan\frac{\theta}{2}\right)^2}{w_1^{20} T^5 cos^{20}\beta} ----------(6)$$

Let us derive the angle β . Recall that we have the radius $'R'$ of Earth given by

$$R^2 = 40.60770616 \times 10^{12} m^2$$

and the eccentric angle of $'R'$ is

$$\beta_1 = 31.171948473^o$$

The angle of inclination of Moon is

$$\beta_2 = 1^o\ 32^{'}\ 32.7^{''}$$
$$= 1.542416667^o$$

The angle between the geographic and magnetic meridians of Earth is

$$\beta_3 = 17^o$$

The angle between the planes of the orbits of Moon and Sun is

$$\beta_4 = 5.1453964^o$$
$$\therefore \beta = \frac{\beta_4}{2} cos(\beta_3 + \beta_2\ cos\ \beta_1) = 2.442307055^o$$

The synodical period of Moon is

$$T_0 = 29.5305888\ days$$
$$One\ day = 23\ hours\ 56\ minutes\ 4.09053\ seconds$$
$$= 23.93446959\ hours$$
$$\therefore\ T = 12 \times T_0$$
$$= 3.053371592 \times 10^7\ seconds$$

The ratio of mass of Earth to the mass of Moon is

$$M_0 = 81.33079331$$

The equatorial radius of Earth is

$$\begin{aligned}
R_0 &= 6.37814 \times 10^6 m \\
w &= 2.668986971 \times 10^{-6} rad/s \\
w_1 &= 7.292115856 \times 10^{-5} rad/s \\
\alpha &= 0.9952702095 \\
\alpha_1 &= 1.229546595 \times 10^{-2} \\
R^2 &= 40.60770616 \times 10^{12} m^2
\end{aligned}$$

From equation (1) we have

$$\begin{aligned}
cos\theta &= \frac{\sqrt{M_0} w^2 a}{w_1^2 R_0} \\
&= \frac{\sqrt{M_0} w^2 G^{\frac{1}{3}} M^{\frac{1}{3}}}{w_1^2 R_0 \alpha w^{\frac{2}{3}}} \quad (using\ equation (2)) \\
&= \frac{\sqrt{M_0} w^{\frac{4}{3}} G^{\frac{1}{3}} M^{\frac{1}{3}}}{w_1^2 R_0 \alpha} \\
&= \frac{\sqrt{M_0} w^{4/3} G^{\frac{1}{3}} \pi \alpha^4 w^{\frac{8}{3}}}{w_1^2 R_0 \alpha w_1^4 G^{\frac{4}{3}} T cos^4\beta} \quad (using\ equation\ (4)) \\
&= \frac{\sqrt{M_0} w^4 \pi \alpha^3}{G w_1^6 R_0 T cos^4\beta} \\
&= \frac{\phi_1}{G} \quad (s\ a\ y)
\end{aligned}$$

where

$$\begin{aligned}
\phi_1 &= \frac{\sqrt{M_0} w^4 \pi \alpha^3}{w_1^6 R_0 T cos^4\beta} \\
&= 4.858137957 \times 10^{-11}
\end{aligned}$$

From equation (6) we have

$$\begin{aligned}
G^4 &= \frac{\pi^5 \alpha^{21} w^{11}}{w_1^{20} T^5 cos^{20}\beta} \left(\frac{1}{R^2} + \frac{\alpha_1}{R_1^2} tan\frac{\theta}{2} \right)^2 \\
&= \phi_2 \left(\frac{1}{R^2} + \frac{\alpha_1}{R_1^2} tan\frac{\theta}{2} \right)^2 \quad (s\ a\ y)
\end{aligned}$$

where

$$\phi_2 = \frac{\pi^5 \alpha^{21} w^{11}}{w_1^{20} T^5 cos^{20}\beta}$$
$$= 2.879025302 \times 10^{-14}$$

The radius of Moon at its mean distance $'a'$ from Earth is

$$\psi = 0.25905^o$$
$$\therefore R_1 = a \times \psi \times \frac{\pi}{180^o}$$
$$= \frac{G^{\frac{1}{3}} M^{\frac{1}{3}} \pi\psi}{\alpha w^{\frac{2}{3}} \times 180} \quad (using\ equation\ (2))$$
$$= \frac{G^{\frac{1}{3}} \pi \alpha^4 w^{\frac{8}{3}} \pi\psi}{\alpha w^{\frac{2}{3}} w_1^4 G^{\frac{4}{3}} T cos^4\beta \times 180} \quad (using\ equation\ (4))$$
$$= \frac{\pi^2 \alpha^3 w^2 \psi}{G \times 180 \times w_1^4 T cos^4\beta}$$
$$\therefore \frac{\alpha_1}{R_1^2} = \frac{G^2 \times (1.8)^2 \times 10^4 \times \alpha_1 w_1^8 T^2 cos^8\beta}{\pi^4 \alpha^6 w^4 \psi^2}$$
$$= \phi_3 G^2 \quad (s\ a\ y)$$

where

$$\phi_3 = \frac{(1.8)^2 \times 10^4 \times \alpha_1 w_1^8 T^2 cos^8\beta}{\pi^4 \alpha^6 w^4 \psi^2}$$
$$= 9.143697791 \times 10^5$$

Hence,

$$
\begin{aligned}
G^4 &= \phi_2 \left(\frac{1}{R^2} + \phi_3 G^2 tan \frac{\theta}{2} \right)^2 \\
&= \phi_2 \left(\frac{1}{R^2} + \phi_3 G^2 \sqrt{\frac{1 - cos\theta}{1 + cos\theta}} \right)^2 \\
&= \phi_2 \left(\frac{1}{R^2} + \phi_3 G^2 \sqrt{\frac{1 - \phi_1/G}{1 + \phi_1/G}} \right)^2 \left(\because cos\theta = \frac{\phi_1}{G} \right) \\
&= \phi_2 \left(\frac{1}{R^2} + \phi_3 G^2 \sqrt{\frac{G - \phi_1}{G + \phi_1}} \right)^2 \\
\therefore G^2 &= \sqrt{\phi_2} \left(\frac{1}{R^2} + \phi_3 G^2 \sqrt{\frac{G - \phi_1}{G + \phi_1}} \right) \\
or, \frac{G^2}{\sqrt{\phi_2}} - \frac{1}{R^2} &= \phi_3 G^2 \sqrt{\frac{G - \phi_1}{G + \phi_1}} \\
or, \frac{1}{\sqrt{\phi_2}} - \frac{1}{R^2 G^2} &= \phi_3 \sqrt{\frac{G - \phi_1}{G + \phi_1}}
\end{aligned}
$$

On iteration it yields the

$$L.H.S = 3.627235866 \times 10^5$$

and the

$$R.H.S = 3.627235872 \times 10^5$$

for the value of

$$G = 6.672684704 \times 10^{-11}$$

Hence, the Gravitational constant is

$$G = 6.672684704 \times 10^{-11} N.m^2/kg^2$$

The value of the Gravitational constant $'G'$ is correct provided for the given ratio of mass of Earth equal to 5.977×10^{24} kg to the mass of Moon equal to 7.349×10^{22} kg .

8. Mass of Earth :-

From equation (4) we have

$$
\begin{aligned}
M^{\frac{1}{3}} &= \frac{\pi \alpha^4 w^{\frac{8}{3}}}{w_1^4 G^{\frac{4}{3}} T cos^4\beta} \\
\therefore M &= \frac{\pi^3 \alpha^{12} w^8}{w_1^{12} G^4 T^3 cos^{12}\beta} \\
&= 5.976752949 \times 10^{24}
\end{aligned}
$$

is the mass of Earth in kg .

9. Mass of Moon : -

The mass of Moon is

$$M_1 = \frac{M}{M_0}$$
$$= 7.348696239 \times 10^{22} kg$$

10. Mean Distance of Moon from Earth :-

The mean distance of Moon from Earth is

$$a = \frac{1}{\alpha}\left(\frac{GM}{w^2}\right)^{\frac{1}{3}}$$
$$= 3.843704529 \times 10^8 m$$

11. Radius of Moon :-

The radius of Moon at its mean distance $'a'$ from Earth is

$$R_1 = a \times \psi \times \frac{\pi}{180^o}$$
$$= 1.737844684 \times 10^6 m$$

12. Acceleration due to Gravity of Moon :-

The acceleration due to gravity of Moon is

$$g_1 = \frac{GM_1}{R_1^2}$$
$$= 1.623637154 \quad m/s^2$$

13. Value of θ:-

We have

$$cos\theta = \frac{\sqrt{M_0} w^2 a}{w_1^2 R_0}$$
$$= 0.7280638422$$
$$\therefore \theta = 43.27570763^o$$

14. Value of g :- We have

$$G = \frac{w^3 a}{g^2}$$
$$\therefore g = \sqrt{\frac{w^3 a}{G}}$$
$$= 10.46512394 \ m/s^2$$

15. Value of g_0 :- We have

$$g_0 = \frac{GM}{R^2}$$
$$= 9.821039343 \quad m/s^2$$

16. Value of $g_1 tan\frac{\theta}{2}$:-

$$g_1 tan\frac{\theta}{2} = \frac{GM_1}{R_1^2} tan\frac{\theta}{2}$$
$$= 0.6440845991 \quad m/s^2$$

17. Conclusion -

The Gravitational constant is

$$G = 6.672684704 \times 10^{-11} s/m$$
$$or, G = 6.672684704 \times 10^{-11} N.m^2/kg^2$$

References

[1] College Physics by Weber, Manning & White

[2] Clark's Tables

[3] Siddhanta Darpana

[4] Indian Astronomical Ephemeris, 2006.

[5] Pears Cyclopaedia