Dihedral And Reflexive Homology Theory Of Polynomial Algebra

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Abstract- In this paper we study the excision property of short sequence to get the long sequence of the dihedral homology and reflexive homology of polynomial algebra. We give a new application of these theorems if we take a new category of graded lie algebras. Another application is relative homology.

***Index Terms*-** polynomial algebra, Excision, Dihedral cohomology, Reflexive, Mayer–Vietoris.

I. Introduction

Discrete and indiscrete are two types of (co)homology theory. In the discrete field, the hochschild (co)homology in mathematics denotes the homology theory of algebras over a field. Hochschild invented simplicial cohomology for algebras in [1] with id, and Henri and Samuel extended it to more rings [2]. Simplicial homology for polynomial algebra is proved for module differentials and the differential commutative graded algebra introduced by Loday. Particular (co)homology theories for associative algebras in non-commutative geometry and mathematics fields that generalize de Rham homology and cohomology of manifolds are called cyclic (co)homology. Boris Tsygan [3] and Alain Connes [4] independently introduced those notions for homology and cohomology. Many branches as de Rham's speculation, simplicial (co)homology, and $\mathcal{KK}$-theory, have interesting relationships with these invariants. The dihedral (co)homology, independently suggested in [3] and [4], and proven process in a variety of algebras, is the hermitian equivalent of a cyclic (co)homology. In 1985 and 1987, cyclic homology of algebra when the characteristic is 0 is calculated. For algebra, $\mathcal{A} = \mathcal{K}[X]/(f)$, the cyclic homology is calculated if f is polynomial and $\mathcal{K}$ is the ring with unital in 1991. In [11], the dimensions of simplicial (co)homology of periodic infinite algebras of polynomial growth are defined in the spaces of their dimensions. To apply this, he obtained a nonstandard periodic representation not subject to analysis the infinite algebra of the polynomial does not derive from scalar algebra. Dihedral homology of algebras over the field is defined in the report as the dihedral group homology in algebra ([5], [6]).

In indiscrete field, the analog simplicial cohomology of operator algebras was reported by Johnson [7] and others. The Banach cyclic (co)homology reported in [8], [9], and [10]. In 1987, he studied involutive unital algebra's dihedral and reflexive (co)homology, and in 1989, he studied the remaining (co)homology group. The relation between the dihedral cohomology and cyclic cohomology of Banach algebras were investigated in [12].

The Mayer–Vietoris sequence generally holds for theories satisfying the Eilenberg–Steenrod axioms but alters for theories with relative and reduced (co)homology. Since this (co)homology of the majority of spaces could not be computed immediately from their definitions, partial information is obtained via methods like the Mayer–Vietoris sequence. Numerous spaces in topology are generated by stitching together extremely simple patches. By carefully choosing the two covering subspaces such that they have a simpler (co)homology compared to entire space, a comprehensive derivation of the space's (co)homology may be possible. In this aspect, the Mayer–Vietoris sequence is equivalent to the Seifert–van Kampen theorem for the basic group, and a clear relation exists for dimension one homology. By the excision theorem in conjunction with the long-exact sequence, the Mayer–Vietoris sequence could be produced.

First section, we will go through the definitions of simplicial and cyclic homology of pure algebra.

Second part, we will review the definition of simplicial and cyclic homology. We will present some relations sequences of simplicial and cyclic (co)homology theory.

In part three, we will first study polynomial algebra's dihedral homology and reflexive homology. We will study the excision property. Second, in polynomial algebra, we will study Mayer-Vietoris for dihedral and reflexive homology. Finally, we will provide some examples of the theories that have been studied.

The following section contains definitions for the homology theorem. A hochschild homology description of pure algebra will be presented. The concept of cyclic homology will be expanded, defining both reflexive and dihedral algebraic homology.

II. Mathematical Review of pure algebra

We define and illustrate some (co)homological invariants for associative algebras. We begin by properly defining the central objects of non-commutative geometry, notably the associative algebras. Let (A, μ, η) is an associative algebra over k, where A is vector space, $\mu: A \otimes A \to A$ and $\eta: k \to A$ are linear maps satisfying associativity and unitality. The linear map μ is called the algebra product, whereas the element $1 = \eta(1)$ is called its unit. For all $a, b \in A$, (A, μ, η) is commutative if $ab = ba$. We will provide an example of commutative algebra to help us in this article. The commutative algebras are examples of the field k itself, and Polynomial in a single indeterminate x can always be written (or rewritten) in the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_0, \dots\dots, a_n$ are constants and x is the indeterminate.

We now give examples of non-commutative algebras, which are algebras that are not commutative [13]. The triple $A^{op} = (A, \mu^{op}, \eta)$ is an algebra known as the opposite algebra of A.

If A and B are algebra, then the tensor product $A \otimes B$ becomes an algebra with product

$$(a_0 \otimes b_0)(a_1 \otimes b_1) = a_0 a_1 \otimes b_0 b_1, \quad \forall a_0, a_1 \in A, \quad b_0, b_1 \in)$$

And unit $1 \otimes 1$. Let V be a vector space over a k. For any nonnegative integer k, we define the kth tensor power of V to be the tensor product of V with itself k times: $T^k V = V^{\otimes k} = V \otimes V \otimes \ldots \otimes V$. The Weyl algebra is a ring of differential operators with polynomial coefficients (in a single variable) [14], more precisely, equations of the type

$$f_m(X)\partial_X^m + f_{m-1}(X)\partial_X^{m-1} + \cdots + f_1(X)\partial_X + f_0(X).$$

We can name the Weyl algebra $W(V)$ as

$$W(V) := T(V)/((v \otimes u - u \otimes u - w(v,u), \quad \forall v, u \in V))$$

where $T(V)$ is a tensor algebra on V, and the notation (( ))means "the ideal generated by".
Let $\mathcal{A}$ is algebra over $\mathcal{K}$ and $\mathcal{M}$ bimodule with involution $*: \mathcal{A} \to \mathcal{A}; a \to a^* \ \forall a \in \mathcal{A}$. let

$$\mathsf{C}(\mathcal{A}) := \cdots \leftarrow \mathcal{A}_0 \overset{d_0}{\leftarrow} \mathcal{A}_1 \overset{d_1}{\leftarrow} \mathcal{A}_2 \overset{d_2}{\leftarrow} \cdots$$

is chain complex with the operator

$$d_n(a_0, a_1, \ldots, a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_i . a_{i+1}, \ldots, a_n) + (-1)^n (a_n . a_0, a_1, \ldots, a_{n-1}),$$

where $d^2 = d_n \, d_{n-1} = 0$.
If we take the (co)homology of the upper complex, then its (co)homology is named as simplicial (co)homology and taken the form $\mathcal{H}_n(\mathsf{C}_*(\mathcal{A})) = \frac{\mathcal{Z}(\mathsf{C}_*(\mathcal{A}))}{\mathcal{B}(\mathsf{C}_*(\mathcal{A}))} = \frac{ker(d_n)}{Im(d_{n-1})}$.
It is denoted by $\mathcal{HH}_n(\mathcal{A})$ [1]. Another definition of Hochschild (co)homology was reported [2]. If $\mathcal{A}$ be a tensor product $\mathcal{A}^e = \mathcal{A} \otimes \mathcal{A}^{op}$, it can be defined by $(\mathcal{Tor})$ and $(\mathcal{Ext})$ as following:

$$\mathcal{HH}_n(\mathcal{A}, \mathcal{M}) = \mathcal{Tor}_n^{\mathcal{A}}(\mathcal{A}, \mathcal{M}), \quad \mathcal{HH}^n(\mathcal{A}, \mathcal{M}) = \mathcal{Ext}_{\mathcal{A}}^n(\mathcal{A}, \mathcal{M}).$$

Before defining periodic homology, the cyclic operator $t_n: \mathsf{C}_n(\mathcal{A}) \to \mathsf{C}_n(\mathcal{A})$ must be defined, where

$$t_n(a_0, \ldots, a_{n-1}, a_n) = (-1)^n (a_n, a_0, \ldots, a_{n-1}). \tag{1}$$

If the next complex is named the subcomplex of $\mathsf{C}_n(\mathcal{A})$,

$$\begin{array}{lccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & \mathsf{C}_2 & \overset{1-t}{\longleftarrow} & \mathsf{C}_2 & \overset{N}{\leftarrow} & \mathsf{C}_2 & \overset{1-t}{\longleftarrow} & \mathsf{C}_2 & \overset{N}{\leftarrow} & \cdots \\
\mathsf{CC}_n(\mathcal{A}): & b \downarrow & & -b' \downarrow & & b \downarrow & & -b' \downarrow & & \\
 & \mathsf{C}_1 & \overset{1-t}{\longleftarrow} & \mathsf{C}_1 & \overset{N}{\leftarrow} & \mathsf{C}_1 & \overset{1-t}{\longleftarrow} & \mathsf{C}_1 & \overset{N}{\leftarrow} & \cdots \\
 & b \downarrow & & -b' \downarrow & & b \downarrow & & -b' \downarrow & & \\
 & \mathsf{C}_0 & \overset{1-t}{\longleftarrow} & \mathsf{C}_0 & \overset{N}{\leftarrow} & \mathsf{C}_0 & \overset{1-t}{\longleftarrow} & \mathsf{C}_0 & \overset{N}{\leftarrow} & \cdots
\end{array}$$

where $b = \sum_{i=0}^{n} (-1)^i d_i$, $b' = \sum_{i=0}^{n-1} (-1)^i d_i$, and $N := 1 + t + \cdots t^n$, then its homology is called cyclic homology and given as follows:

$$\mathcal{H}\mathsf{C}_n(\mathcal{A}) = \mathcal{H}_n(\mathsf{CC}_*(\mathcal{A}), b_*) = \mathcal{H}_n\left(\frac{\mathsf{C}_n(\mathcal{A})}{Im(1 - t_n)}, b_*\right).$$

Alternatively, if Λ is a category, the cyclic homology and cohomology are as follows:

$$\mathcal{H}\mathsf{C}_n(\mathcal{M}) = \mathcal{Tor}_n^{\mathcal{K}[\Lambda]^{op}}(\mathcal{K}^{\mathsf{C}}, \mathcal{M}), \mathcal{H}\mathsf{C}^n(\mathcal{M}) = \mathcal{Ext}_{\mathcal{K}[\Lambda]^{op}}^n(\mathcal{M}, \mathcal{K}^{\mathsf{C}}), n \geq 0.$$

The reflexive homology was defined by identifying a subcomplex of $\mathsf{C}_n(\mathcal{A})$ as following:

$$^{\alpha}\mathsf{C}\mathcal{R}_n(\mathcal{A}) = \frac{\mathsf{C}_n(\mathcal{A})}{Im(1 - r_n)},$$

with reflexive operator $r_n: \mathcal{C}_n(\mathcal{A}) \to \mathcal{C}_n(\mathcal{A})$, since

$$r_n(a_0, \ldots,, a_{n-1}, a_n) = a(-1)^{n(n+1)/2} (a_0^*, a_n^*, \ldots, a_1^*). \tag{2}$$

Its (co)homology is given by
$^{\alpha}\mathcal{HR}_n(\mathcal{A}) = \mathcal{H}_n(\mathsf{C}\mathcal{R}_*(\mathcal{A}), d_*) = \mathcal{H}_n(\frac{\mathsf{C}_*(\mathcal{A})}{Im(1-r_n)}, d_*)$, and it is called hyperhomology (hypercohomolog) of $\mathcal{A}$.

The ${}^{\alpha}\mathrm{C}\mathcal{D}_n(\mathcal{A}) = (\frac{\mathrm{C}_n(\mathcal{A})}{Im(1-t_n)+Im(1-r_n)})$ was obtained by using the (1) and (2) on $\mathrm{C}(\mathcal{A})$, which is a sub-complex of $\mathrm{C}_n(\mathcal{A})$. If the homology of this complex ${}^{\alpha}\mathrm{C}\mathcal{D}_*(\mathcal{A})$ is taken, the dihedral homology of $\mathcal{A}$ can be defined [15]. It is given by

$${}^{\alpha}\mathcal{HD}_n(\mathcal{A}) = \mathcal{H}_n(\mathrm{C}\mathcal{D}_*(\mathcal{A}), b_*) = \mathcal{H}_n\left(\frac{\mathrm{C}_n(\mathcal{A})}{Im(1-t_n)+Im(1-r_n)}, b_*\right).$$

An alternative perspective on dihedral (co)homology is as follows:

$$Tor_n^{\mathcal{K}[\xi]^{op}}(\mathcal{K}^{\mathcal{D}}, \mathcal{M}) = \mathcal{HD}_n(\mathcal{M}), \quad Ext^n_{\mathcal{K}[\xi]^{op}}(\mathcal{M}, \mathcal{K}^{\mathcal{D}}) = \mathcal{HD}^n(\mathcal{M}), \quad n \geq 0.$$

Theorem (1-1):

Suppose that $P' \subseteq P$ for k- associative algebra. Then we obtain the long exact sequence;

$$\cdots \to \mathcal{HH}_n(P') \to \mathcal{HH}_n(P) \to \mathcal{HH}_n(P/P') \to \mathcal{HH}_{n-1}(P') \to \cdots.$$

Prove: see [16].

Theory (1-2):

Let $P' \subseteq P$. Then long exact sequence for cyclic homology of associative algebra is;

$$\cdots \to \mathcal{HC}_n(P') \to \mathcal{HC}_n(P) \to \mathcal{HC}_n(P/P') \to \mathcal{HC}_{n-1}(P') \to \cdots.$$

Prove: see [16].

Theory (1-3):[15]

Long exact sequences are known as Connes' exact periodicity sequences;

$$\cdots \to \mathcal{H}\mathrm{C}_{n-1}(P) \xrightarrow{\mathcal{B}} \mathcal{HH}_n(P) \xrightarrow{I} \mathcal{H}\mathrm{C}_n(P) \xrightarrow{S} \mathcal{H}\mathrm{C}_{n-2}(P) \xrightarrow{\mathcal{B}} \mathcal{HH}_{n-1}(P) \to \cdots,$$

Where $\mathcal{I}$ is inclusion map, $\mathcal{B}$ is conn`s map, S is periodic map.

In this part, we study the property of excision theorem of the dihedral (co)homology of algebras. We study and explaine the Mayer-Vietortis for the dihedral homology of pure algebra.

III. Main Results

Suppose that P is a polynomial algebra over k with $(\sum_i p_i x^i)^* = \sum_i p_i^* x^i \quad , p_i \in P$. Let EP_α^D the dihedral submodule of the dihedral module P_α^D produced by Polynomial $p_1, \cdots, p_2 \in P$. We named the dihedral (co)homology of the group as;

$${}^{\alpha}\mathcal{H}D_n(P) = Tor_n^{k[E]^{op}}(k^D, EP_\alpha^D), \quad {}_{\alpha}\mathcal{H}D^n(P) = Ext^n_{k[E]}(EP_\alpha^D, k^D).$$

By using [18], we can name the dihedral homology of polynomial algebra as; $\mathcal{H}D_n(P) = \mathcal{H}_n(Tot\ CC^+(P))$ and skew-dihedral homology of P is $\mathcal{H}D_n'(P) = \mathcal{H}_n(Tot\ CC^-(P))$.

We can define the dihedral (co)homology of Laurent polynomial as the following by using [16];

${}^{\alpha}\mathcal{H}D_n(P[p,p^{-1}]) = {}^{\alpha}\mathcal{H}D_n(P) \otimes {}^{\alpha}\mathcal{H}D_{n-1}(P) \otimes \mathcal{H}R_n^{\infty}(P)$, if $P[p,p^{-1}]$ is the Laurent polynomial algebra over k with involution;

$$(\sum_i a_i y^i + \sum_j b_j y^{-j})^* = \sum_i a_i^* y^i + \sum_j b_j^* y^{-j} \quad \forall\ a_i, b_j \in P.$$

If M is a polynomial algebra, and U, V are polynomials algebra in M so that $M = U \cup V$. Then we have inclusion maps;

$$i_1 : U \to M, \quad i_2 : V \to M$$

and inclusion maps;

$$j_1 : U \cap V \to U, \quad j_2 : U \cap V \to V.$$

Let $C_*(M)$ is chain complex, and we have inclusion maps as a form:

$$\gamma : C_*(M) \to C_*(U) \oplus C_*(V)$$

and

$$\beta : C_*(U) \oplus C_*(V) \to C_*(U \cap V).$$

Then we have a short exact sequence:

$$0 \to C_*(M) \to C_*(U) \oplus C_*(V) \to C_*(U \cap V) \to 0$$

This is Mayer-Vietoris; for more, see [16].

In the next theorem, we will study and prove the sequence which relate between the dihedral and cyclic (co)homology of algebra.

Theorem (2-1):

Let P is a polynomial algebra, and then the following long exact sequence can be put in a commutative diagram;

$$\cdots \to {}^{-\alpha}\mathcal{H}D_n(P) \xrightarrow{j} \mathcal{H}C_n(P) \xrightarrow{i} {}^{\alpha}\mathcal{H}D_n(P) \to {}^{-\alpha}\mathcal{H}D_{n-1}(P) \to \cdots$$

Prove:

Let P is a polynomial algebra. Suppose that ${}^{\alpha}\mathfrak{C}(P) = \mathfrak{C}(P_{\alpha}^{D})$ is a bicomplex, let hochschild complex $\mathcal{C}(P)$ is isomorphic to the reflexive complex ${}^{\alpha}\mathfrak{R}(P)$.

If $(Tot\ {}^{\alpha}\mathfrak{C}(P)[-2])_n = (Tot\ {}^{\alpha}\mathfrak{C}(P))_{n-2}$, ${}^{\alpha}\mathfrak{R}(P) = \ker j$, where j is a natural map, and $i : {}^{\alpha}\mathfrak{R}(P) \to Tot\ {}^{\alpha}\mathfrak{C}(P)$. Then we obtaine the short sequence [13];

$$0 \to {}^{\alpha}\mathfrak{R}(P) \xrightarrow{i} Tot\ {}^{\alpha}\mathfrak{C}(P) \xrightarrow{j} Tot\ {}^{\alpha}\mathfrak{C}(P)[-2] \to 0 \qquad (3).$$

The sequence

$$0 \to \mathcal{K}[\mathbb{Z}/2] \to \mathcal{W}^{\alpha} \to \mathcal{W}^{\alpha}[-1] \to 0\,, \qquad (4)$$

is exact where

$$\mathcal{W}^{\alpha} = \left\{\mathcal{K}[\mathbb{Z}/2] \xleftarrow{1+\alpha R} \mathcal{K}[\mathbb{Z}/2] \xleftarrow{1-\alpha R} \mathcal{K}[\mathbb{Z}/2] \xleftarrow{1+\alpha R} \cdots\right\}, \quad \alpha = \pm 1.$$

If

$${}^{\alpha}\mathfrak{C}(P) = Tot\left({}^{\alpha}\mathfrak{C}(P) \otimes_{\mathcal{K}[\mathbb{Z}/2]} \mathcal{W}^{\alpha}\right), \quad {}^{\alpha}\mathcal{S}(P) = Tot\left({}^{\alpha}\mathfrak{C}(P) \otimes_{\mathcal{K}[\mathbb{Z}/2]} \mathcal{W}^{\alpha}\right),$$

and take the exact polynomial sequence of (3) and (4) over $\mathcal{K}[\mathbb{Z}/2]$. We get the commutative diagram of complexes with the columns and rows are exact;

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & {}^{\alpha}\mathfrak{R}(P) & \to & Tot\ {}^{\alpha}\mathfrak{C}(P) & \to & Tot\ {}^{\alpha}\mathfrak{C}(P)[-2] & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & {}^{\alpha}\mathcal{S}(P) & \to & {}^{\alpha}\mathfrak{C}(P) & \to & {}^{-\alpha}\mathfrak{C}(P)[-2] & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & {}^{-\alpha}\mathcal{S}(P)[-1] & \to & {}^{-\alpha}\mathfrak{C}(P)[-1] & \to & {}^{\alpha}\mathfrak{C}(P)[-3] & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \quad (5).$$

Where

$$\mathcal{H}_*\big(\mathcal{C}(P)\big) = \mathcal{HH}_*(P), \quad \mathcal{H}_*\big(Tot\ {}^{\alpha}\mathfrak{C}(P)\big) = \mathcal{H}\mathsf{C}_*(P),$$

$$\mathcal{H}_*(\mathbb{Z}/2\,;{}^{\alpha}\mathfrak{R}(P)) = \mathcal{H}_*\left({}^{\alpha}\mathcal{S}(P)\right) = {}^{\alpha}\mathcal{H}R_*(P),$$

$$\mathcal{H}_*\big({}^{\alpha}\mathfrak{C}(P)\big) = \mathcal{H}_*\big(\mathbb{Z}/2\,;{}^{\alpha}\mathfrak{C}(P)\big) = {}^{\alpha}\mathcal{H}D_*(P)\,.$$

From the equations previously, then we get an infinite commutative diagram for (5);

$$\begin{array}{ccccccccccc} & & \cdots & & \cdots & & \cdots & & \cdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & \mathcal{HH}_n(P) & \to & \mathcal{H}\mathsf{C}_n(P) & \to & \mathcal{H}\mathsf{C}_{n-2}(P) & \to & \mathcal{HH}_{n-1}(P) & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & {}^{\alpha}\mathcal{H}R_n(P) & \to & {}^{\alpha}\mathcal{H}D_n(P) & \to & {}^{\alpha}\mathcal{H}D_{n-2}(P) & \to & {}^{\alpha}\mathcal{H}R_{n-1}(P) & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & {}^{\alpha}\mathcal{H}R_{n-1}(P) & \to & {}^{\alpha}\mathcal{H}D_{n-1}(P) & \to & {}^{\alpha}\mathcal{H}D_{n-3}(P) & \to & {}^{\alpha}\mathcal{H}R_{n-2}(P) & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \to & \mathcal{HH}_{n-1}(P) & \to & \mathcal{H}\mathsf{C}_{n-1}(P) & \to & \mathcal{H}\mathsf{C}_{n-3}(P) & \to & \mathcal{HH}_{n-2}(P) & \to & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & \cdots & & \cdots & & \cdots & & \cdots & & \end{array}.$$

The long exact sequence of a sequence $0 \to B \to P \to P/B \to 0$ for reflexive (co)homology in polynomial algebra given in the following theorem. By using [14] ,[15] and [26].

Theorem (2-2):

Let $B \subset P$ of P polynomial algebra. Suppose that $0 \to B \to P \to P/B \to 0$ an extension of polynomial algebra over $\mathcal{K}$. The sequence of reflexive (co)homology is

$$\cdots \to \mathcal{H}R^n(P/B) \to \mathcal{H}R^n(P) \to \mathcal{H}R^n(B) \to \mathcal{H}R^{n+1}(P/B) \to \cdots.$$

Proof:

Let

$$\mathcal{F}\colon \mathcal{H}R_n(B) \to \mathcal{H}R_n(P,B), \ \ \mathcal{F}\colon \mathcal{H}R^n(B) \to \mathcal{H}R^n(P,B)$$

is a functor.

The association between cyclic and reflexive (co)homology of polynomial P given as a long exact sequence;

$$\cdots \to \mathcal{H}R^n(P) \xrightarrow{I} \mathcal{H}\mathsf{C}^n(P) \xrightarrow{S} \mathcal{H}\mathsf{C}^{n+2}(P) \xrightarrow{\mathcal{B}} \mathcal{H}R^{n+1}(P) \to \cdots.$$

From [12], the construction of $\mathcal{H}\mathsf{C}_n(B)$ & $\mathcal{H}\mathsf{C}^n(B)$, there is the long exact sequence of Connes. Consider a commutative diagrams;

$$\begin{array}{ccccccccc} \cdots & \to & \mathcal{H}R^n(P,B) & \to & \mathcal{H}\mathsf{C}^n(P,B) & \to & \mathcal{H}\mathsf{C}^{n+2}(P,B) & \to & \mathcal{H}R^{n+1}(P,B) \to \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \to & \mathcal{H}R^n(B) & \to & \mathcal{H}\mathsf{C}^n(B) & \to & \mathcal{H}\mathsf{C}^{n+2}(B) & \to & \mathcal{H}R^{n+1}(B) \to \cdots \end{array}.$$

In the next theorem, we will study and prove from short exact sequence $0 \to B \to P \to P/B$ of polynomial algebra, the commutative diagram of dihedral (co)homology.

Theorem (2-3):

The short sequence $0 \to B \to P \to P/B$ is exact of polynomial algebra where $B \subset P$. Then we get the commutative diagram of dihedral (co)homology

$$\begin{array}{ccc} \mathcal{H}D^0(B) & \leftarrow \mathcal{H}D^0(P) \leftarrow & \mathcal{H}D^0(P/B) \\ \downarrow & \uparrow & \\ \mathcal{H}D^1(P/B) \to & \mathcal{H}D^1(P) \to & \mathcal{H}D^1(B) \end{array}$$

Proof:

Let the sequence for the algebra P [16],

$$0 \to BP \to \mathcal{T}P \to P \to 0$$

where $\mathcal{T}P$ is the non-unital involution polynomial algebra over P, BP is unital polynomial algebra $\mathcal{T}P$, then the long exact sequence is

$$\cdots \to \mathcal{H}D^{n-1}(BP) \to \mathcal{H}D^{n-1}(\mathcal{T}P) \to \mathcal{H}D^{n-1}(P) \to \mathcal{H}D^n(BP) \to \mathcal{H}D^n(\mathcal{T}P) \to \mathcal{H}D^n(P) \to \cdots (6)$$

Let $\mathcal{K}$ be the kernel in the exact sequence

$$0 \to \mathcal{K} \to \mathcal{T}P \to P/B \to 0$$

$$\cdots \to \mathcal{H}D^{n-1}(\mathcal{K}) \to \mathcal{H}D^{n-1}(\mathcal{T}P) \to H\mathcal{H}^{n-1}(P/B) \to \mathcal{H}D^n(\mathcal{K}) \to \mathcal{H}D^n(\mathcal{T}P) \to \mathcal{H}D^n(P/B) \to \cdots (7)$$

where $\mathcal{K}$ is unital tensor algebra $\mathcal{T}P$ which is free, then BP and $\mathcal{K}$ are about H-unital from the long exact sequences for (6) and (7) we have

$$\mathcal{H}D^{n-1}(BP) \cong \mathcal{H}D^n(P) \quad \& \quad \mathcal{H}D^{n-1}(\mathcal{K}) \cong \mathcal{H}D^n(P/B)$$

For the short exact sequence

$$0 \to BP \to \mathcal{K} \to B \to 0$$

$$\cdots \to \mathcal{H}D^{n-1}(BP) \to \mathcal{H}D^{n-1}(\mathcal{K}) \to \mathcal{H}D^{n-1}(B) \to \mathcal{H}D^n(BP) \to \mathcal{H}D^n(\mathcal{K}) \to \mathcal{H}D^n(B) \to \cdots \qquad (8)$$

From the long exact sequence for this sequence(8), we have proof of our theorem.

From the sequence $0 \to B \to P \to P/B$ of polynomial, we obtaine the commutative diagram of reflexive (co)homology.

Lemma (2-4):

Where $B \subset P$, the short sequence $0 \to B \to P \to P/B$ is exact of polynomial algebra. Then we get the commutative diagram of reflexive cohomology

$$\begin{array}{ccc} \mathcal{H}R^0(B) & \leftarrow \mathcal{H}R^0(P) \leftarrow & \mathcal{H}R^0(P/B) \\ \downarrow & \uparrow & \\ \mathcal{H}R^1(P/B) \to & \mathcal{H}R^1(P) \to & \mathcal{H}R^1(B) \end{array}$$

Proof: Using [14] and using the same method of theory (2-3).

In the next theorem, we will study and prove the long exact sequence for Mayer-Vietoris of dihedral (co)homology.

Theorem (2-5):

If $M = U \cup V$ where U and V are polynomials algebra. Then the following long exact sequence is exact:

$$\cdots \to {}^{\alpha}\mathcal{H}D_n(M) \to {}^{\alpha}\mathcal{H}D_n(U) \oplus {}^{\alpha}\mathcal{H}D_n(V) \to {}^{\alpha}\mathcal{H}D_n(U \cap V) \to {}^{\alpha}\mathcal{H}D_{n-1}(M) \to \cdots.$$

Prove:

Let $M = U \cup V$ since U, V polynomial algebra. the diagram

$$\begin{array}{ccc} M & \to & U \\ \downarrow & & \downarrow \\ V & \to & U \cap V \end{array},$$

is commutative where $i_1: M \to U, \quad i_2: M \to V \quad , \quad j_1: U \to U \cap V, \quad j_2: V \to V \cap U.$

If we take the homology of the next short exact sequence:

$$0 \to {}^{\alpha}\mathrm{C}\mathcal{D}_*(M) \to {}^{\alpha}\mathrm{C}\mathcal{D}_*(U) \oplus {}^{\alpha}\mathrm{C}\mathcal{D}_*(V) \to {}^{\alpha}\mathrm{C}\mathcal{D}_*(U \cap V) \to 0, a = \pm 1.$$

And using [6], then we get a long exact sequence of dihedral homology

$$\cdots \to {}^{\alpha}\mathcal{H}D_n(M) \to {}^{\alpha}\mathcal{H}D_n(U) \oplus {}^{\alpha}\mathcal{H}D_n(V) \to {}^{\alpha}\mathcal{H}D_n(U \cap V) \to {}^{\alpha}\mathcal{H}D_{n-1}(M) \to \cdots.$$

In the next theorem, we will study and prove the long exact sequence for Mayer-Vietoris of reflexive (co)Homology.

Theorem (2-6):

If $M = U \cup V$ is a polynomial algebra. Then we got a long exact sequence of the hyperhomology

$$\cdots \to {}^{\alpha}\mathcal{H}R_n(M) \to {}^{\alpha}\mathcal{H}R_n(U) \oplus {}^{\alpha}\mathcal{H}R_n(V) \to {}^{\alpha}\mathcal{H}R_n(U \cap V) \to {}^{\alpha}\mathcal{H}R_{n-1}(M) \to \cdots.$$

Prove:

If ${}^{\alpha}\mathrm{C}\mathcal{R}_n(M)$ is a sub-complex of polynomial algebra, then the following short sequence is exact:

$$0 \to {}^{\alpha}\mathrm{C}\mathcal{R}_*(U \cup V) \xrightarrow{\gamma} {}^{\alpha}\mathrm{C}\mathcal{R}_*(U) \oplus {}^{\alpha}\mathrm{C}\mathcal{R}_*(V) \xrightarrow{\beta} {}^{\alpha}\mathrm{C}\mathcal{R}_*(U \cap V) \to 0, \quad a = \pm 1,$$

where $i_1: U \cup V \to U, \quad i_2: U \cup V \to V \quad , \quad j_1: U \to U \cap V, \quad j_2: V \to V \cap U.$

If we take the homology of the upper sequence and using [16], we get the long exact sequence

$$\cdots \to {}^{\alpha}\mathcal{H}R_n(U \cup V) \xrightarrow{\gamma} {}^{\alpha}\mathcal{H}R_n(U) \oplus {}^{\alpha}\mathcal{H}R_n(V) \xrightarrow{\beta} {}^{\alpha}\mathcal{H}R_n(U \cap V) \xrightarrow{\theta} {}^{\alpha}\mathcal{H}R_{n-1}(M) \to \cdots$$

Since $\theta \circ \gamma = 0$, this ends the proof. In the next section, we will present some new applications of the theories we have studied.

IV. Applications

In the first example, we give the long exact sequence of short exact sequence $0 \to M \to E \to A \to 0$ under some conditions as the example in the polynomial and Laurent polynomial algebras for dihedral and reflexive homology. In the second example, we give a long exact sequence of dihedral and reflexive for a short exact sequence of graded Lie algebras:

$$0 \to \mathbb{L}(W) \to B \to A/(a) \to 0.$$

In the third example, we give the long exact sequence of relative homology for dihedral and reflexive homology.

Example (3-1):

Let $0 \to M \to E \to A \to 0$ is a short exact sequence of associative k- polynomial algebras and Laurent polynomial algebras such that the sequence is split over k, where $E \cong M \oplus A$ and $M^2 = 0$. We can obtain and prove the long exact sequence of dihedral homology by using Theorem (2-3) and [15]; this long exact sequence takes the form of polynomial algebras;

$$\cdots \to {}^{\alpha}\mathcal{H}D_n(M) \to {}^{\alpha}\mathcal{H}D_n(E) \to {}^{\alpha}\mathcal{H}D_n(A) \to {}^{\alpha}\mathcal{H}D_{n-1}(M) \to \cdots, \quad \alpha = \pm 1.$$

and Laurent polynomial algebras $\alpha = \pm 1$.:

$$\cdots \to {}^{\alpha}\mathcal{H}D_n(M[k,k^{-1}]) \to {}^{\alpha}\mathcal{H}D_n(E[k,k^{-1}]) \to {}^{\alpha}\mathcal{H}D_n(A[k,k^{-1}]) \to {}^{\alpha}\mathcal{H}D_{n-1}(M[k,k^{-1}]) \to \cdots .$$

We can prove and get the long exact sequence for reflexive homology by using Result (2-4). This long exact sequence takes the form;

$$\cdots \to {}^{\alpha}\mathcal{H}R_n(M) \to {}^{\alpha}\mathcal{H}R_n(E) \to {}^{\alpha}\mathcal{H}R_n(A) \to {}^{\alpha}\mathcal{H}R_{n-1}(M) \to \cdots, \quad \alpha = \pm 1.$$

and Laurent polynomial algebras $\alpha = \pm 1$.:

$$\cdots \to {}^{\alpha}\mathcal{H}R_n(M[k,k^{-1}]) \to {}^{\alpha}\mathcal{H}R_n(E[k,k^{-1}]) \to {}^{\alpha}\mathcal{H}R_n(A[k,k^{-1}]) \to {}^{\alpha}\mathcal{H}R_{n-1}(M[k,k^{-1}]) \to \cdots.$$

Example (3-2):

A graded Lie algebra is a Lie algebra endowed with a gradation fit with the Lie bracket. Let $(\mathbb{L}(V), d)$ be a graded Lie algebra, and a be an element in A_n. If n is even, and a is in the center, then:

(1) a is nice element,

(2) The sequence of graded Lie algebras:

$$0 \to \mathbb{L}(W) \to B \to A/(a) \to 0,$$

Is split with W a graded space isomorphic to $s^{n+1}\ A/(a)_+$.

If n is odd, the bracket *[a, a]* equals zero, and so the triple Whitehead bracket $< a, a, a >$ is clearly defined [16]. The sequence of the dihedral (co)homology and reflexive (co)homology of graded lie algebras as the form, respectively;

$$\cdots \to {}^{\alpha}\mathcal{H}D_n(\mathbb{L}(W)) \to {}^{\alpha}\mathcal{H}D_n(B) \to {}^{\alpha}\mathcal{H}D_n(A/(a)) \to {}^{\alpha}\mathcal{H}D_{n-1}(\mathbb{L}(W)) \to \cdots.$$

$$\cdots \to {}^{\alpha}\mathcal{H}R_n(\mathbb{L}(W)) \to {}^{\alpha}\mathcal{H}R_n(B) \to {}^{\alpha}\mathcal{H}R_n(A/(a)) \to {}^{\alpha}\mathcal{H}R_{n-1}(\mathbb{L}(W)) \to \cdots.$$

We give another recent application of these theorems if we take a new category of relative homology.

Example (3-3):

The (singular) homology of a topological space concerning a subspace is a construct in singular homology for pairs of spaces in algebraic topology, a branch of mathematics. Relative homology is beneficial and significant in a variety of ways. Innately, it aids in determining which subspace contains what part of an absolute homology group.

Let a subspace $A \subseteq X$, the short exact sequence $0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0$ where $C_*(X)$ denotes the singular chains on the space X [15]. We can get the sequence of relative dihedral (co)homology as the following:

$$\cdots \to {}^{\alpha}\mathcal{H}D_n(A) \to {}^{\alpha}\mathcal{H}D_n(X) \to {}^{\alpha}\mathcal{H}D_n(X, A) \to {}^{\alpha}\mathcal{H}D_{n-1}(A) \to \cdots, \alpha = \pm 1$$

and the sequence of relative reflexive (co)homology as the form;

$$\cdots \to {}^{\alpha}\mathcal{H}R_n(A) \to {}^{\alpha}\mathcal{H}R_n(X) \to {}^{\alpha}\mathcal{H}R_n(X, A) \to {}^{\alpha}\mathcal{H}R_{n-1}(A) \to \cdots, \alpha = \pm 1.$$

V. Conclusion

We demonstrate the dihedral and reflexive homology of polynomial algebra. We give a commutative diagram of polynomial algebra as the following;

$$\begin{array}{ccccccccccc}
 & & \cdots & & \cdots & & \cdots & & \cdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \mathcal{H}\mathcal{H}_n(P) & \to & \mathcal{H}C_n(P) & \to & \mathcal{H}C_{n-2}(P) & \to & \mathcal{H}\mathcal{H}_{n-1}(P) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & {}^{\alpha}\mathcal{H}R_n(P) & \to & {}^{\alpha}\mathcal{H}D_n(P) & \to & {}^{\alpha}\mathcal{H}D_{n-2}(P) & \to & {}^{\alpha}\mathcal{H}R_{n-1}(P) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & {}^{\alpha}\mathcal{H}R_{n-1}(P) & \to & {}^{\alpha}\mathcal{H}D_{n-1}(P) & \to & {}^{\alpha}\mathcal{H}D_{n-3}(P) & \to & {}^{\alpha}\mathcal{H}R_{n-2}(P) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \mathcal{H}\mathcal{H}_{n-1}(P) & \to & \mathcal{H}C_{n-1}(P) & \to & \mathcal{H}C_{n-3}(P) & \to & \mathcal{H}\mathcal{H}_{n-2}(P) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \cdots & & \cdots & & \cdots & & \cdots & &
\end{array}.$$

We give the long exact sequence for reflexive homology and cohomology, respectively,

$$\cdots \to \mathcal{H}R_n(B) \to \mathcal{H}R_n(P) \to \mathcal{H}R_n(P/B) \to \mathcal{H}R_{n-1}(B) \to \cdots,$$

$$\cdots \to \mathcal{H}R^n(P/B) \to \mathcal{H}R^n(P) \to \mathcal{H}R^n(B) \to \mathcal{H}R^{n+1}(P/B) \to \cdots.$$

We give the commutative diagram of dihedral homology and cohomology;

$$\begin{array}{ccc} \mathcal{H}D^0(B) & \leftarrow \mathcal{H}D^0(P) \leftarrow & \mathcal{H}D^0(P/B) \\ \downarrow & & \uparrow \\ \mathcal{H}D^1(P/B) & \rightarrow \mathcal{H}D^1(P) \rightarrow & \mathcal{H}D^1(B) \end{array} \qquad \begin{array}{ccc} \mathcal{H}D_1(B) & \rightarrow \mathcal{H}D_1(P) \rightarrow & \mathcal{H}D_1(P/B) \\ \downarrow & & \uparrow \\ \mathcal{H}D_0(P/B) & \leftarrow \mathcal{H}D_0(P) \leftarrow & \mathcal{H}D_0(B) \end{array}.$$

We give the long exact sequence for Mayer -Vietoris of dihedral homology and hyperhomology, respectively;

$$\cdots \to {}^{\alpha}\mathcal{H}D_n(M) \to {}^{\alpha}\mathcal{H}D_n(U) \oplus {}^{\alpha}\mathcal{H}D_n(V) \to {}^{\alpha}\mathcal{H}D_n(U \cap V) \to {}^{\alpha}\mathcal{H}D_{n-1}(M) \to \cdots,$$

$$\cdots \to {}^{\alpha}\mathcal{H}R_n(M) \to {}^{\alpha}\mathcal{H}R_n(U) \oplus {}^{\alpha}\mathcal{H}R_n(V) \to {}^{\alpha}\mathcal{H}R_n(U \cap V) \to {}^{\alpha}\mathcal{H}R_{n-1}(M) \to \cdots.$$

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