On application of algebra of quaternions I

Emran Sasi Althabit [*], Ismail.M.Mohamed [**]

[*] Higher Institute for water Affairs
[**] Academic of Engineering and Medical Sciences

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Abstract- We introduced in this paper the quaternions concepts and quaternions algebra, Representation of quaternions, transformation of quaternion in space exactly rotation and as application : Quaternions in space.

Key words: Quaternion, Complex quaternions , Interpolation of quaternions .

I. Introduction

The important of quaternions, appeared in a physics and a Computer programs.

* In physic s , quaternions are correlated to nature of the universe at the level of quantum mechanics. They lead to elegant expression of the Lorentz transformations, which form the basis of the modern theory of relativity.[6]

* In a computer programs transformation of quaternions use for Animation and Simulation, The Quaternion Fourier Transform (QFT) is a powerful tool , its application include face recognition and voice recognition in and robot kinematics as a simulation of nature of the universe.

1. Hamilton's Quaternions

Definition1. 1:A quaternion is a four-dimensional complex number that can be used to represent the orientation of a rigid body or coordinate frame in three-dimensional space. The Hamilton's quaternions $\mathbb{H}$ can be generalized to allow coefficients in $\mathbb{R}$ as

$$\mathbb{H} = \{Q = w + xi + yj + zk : w, x, y, z \in \mathbb{R}\}$$

Or $\mathbb{H} = \{(w, x, y, z) : w, x, y, z \in \mathbb{R}\}$ (1.1)

So that quaternion is the sum of a scalar part w and a vectorial part $(x, y, z) \in \mathbb{R}^3$ i.e.,if $\vec{r} = xi + yj + zk$ is position vector in space quaternions is define as

$w + xi + yj + zk \leftrightarrow w + \vec{r} = [w, \vec{r}]$ (1.2)

in simply where w real part $Re(q)$, and $\vec{r}$ represent imaginary part= $Im(q)$-vector in

3-dimession space, If $= 0$, the quaternion

$q = xi + yj + zk$ (1.3)

is called pure quaternion i.e., A 3D vector is a pure quaternion whose real part is zero.

where i , j and k are the standard orthonormal basis in $\mathbb{R}^3$.

The multiplication for the primitive elements i, j and k is defined by

$$ij = -ji = k$$
$$jk = -kj = i$$
$$ki = -ki = j \quad (1.4)$$
$$i^2 = j^2 = k^2 = ijk = -1$$

	1	**−1**	**i**	**−i**	**j**	**−j**	**k**	**−k**
1	1	−1	i	−i	j	−j	k	−k
−1	−1	1	−i	i	−j	j	−k	k
i	i	−i	−1	1	k	−k	−j	j
−i	−i	i	1	−1	−k	k	j	−j
j	j	−j	−k	k	−1	1	i	−i
−j	−j	j	k	−k	1	−1	−i	i
k	k	−k	j	−j	−i	i	−1	1
−k	−k	k	−j	j	i	−i	1	−1

Table 1: Multiplication Table for the Quaternion Group. Also called a Cayley Group Table. Conversion order: Row entry first followed by column entry.(Adapted from Weisstein, 1999-2014)

and

$i \times j = k, j \times k = i, k \times i = j$ (1.5)

Which are very similar to the cross product of two unit Cartesian vectors.

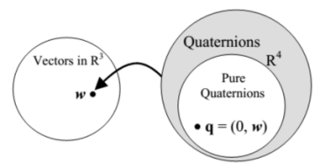

Fig.2 A one-to-one correspondence between pure quaternions and vectors in $\mathbb{R}^3$ [3]

Octonions

In the same year in which Hamilton discover the quaternions wrote to his friend John Graves about his discovery Three months later on December 26th Graves wrote to Hamilton about his discovery of a kind of ‘double-quaternion’ that he called ‘octives’, today are they are known as octonions. Like quaternions octonions form a division algebra, but unlike quaternions they are not associative. They are related to geometries in 7 and 8 dimensions.[6]

$$O = s + e_1x_1 + e_2x_2 + e_3x_3 + e_4x_4 + e_5x_5 + e_6x_6 + e_7x_7 \quad (1.6)$$

where $(x_1, x_2, x_3, x_4, x_5x_6, x_7) \in \mathbb{R}^7$

Mathematically, octonions are the largest normed division algebras. This means that they satisfy $|ab|^2 = |a|^2|b|^2$.

When Hamilton discovered quaternions had to give up commutative property in order to preserve the norm; now the associative property has to be given up in order to preserve the norm for the octonions.

A multiplication table for octonions can be made like quaternions as follows:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	$-e_7$	$-e_5$	-1	e_6	e_7	$-e_4$	e_1
e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

Table 2: Multiplication table for Octonions. (Baez, 2001c)

Some of the properties of octonions as they relate to quaternions are the multiplication table for octonions[6]

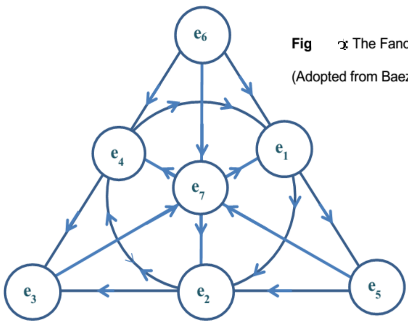

Fig 2: The Fano plane for octonions (Adopted from Baez, 2001, p.152)

For example: e_i, e_j, and e_k can be represented as follows: $e_i\, e_j = e_k;\; e_j e_i = -\, e_k$. (1.7)
Let 1 be the multiplicative identity.
Let each circle in the diagram be represented by the relationship $e_i^2 = -1$[6]

2. <u>Quaternions Algebras</u>

The Operations on quaternions 2.1:
Let Q_1 and Q_2 be two quaternions , given as

$$\begin{matrix} Q_1 = w_1 + x_1 i + y_1 j + z_1 k \\ Q_2 = w_2 + x_2 i + y_2 j + z_2 k \end{matrix} \text{, then}$$

1- The addition and subtraction are define as

$$Q_1 \pm Q_2 = (w_1 + x_1 i + y_1 j + z_1 k) \pm (w_2 + x_2 i + y_2 j + z_2 k)$$
$$= (w_1 \pm w_2) + (x_1 \pm x_2)i + (y_1 \pm y_2)j + (z_1 \pm z_2)k \quad (2.1)$$

2- The multiplication

From multiplication of the primitive elements i, j and k is defined multiplication of two quaternions Q_1 and Q_1as

$$Q_1 Q_2 = (w_1 + x_1 i + y_1 j + z_1 k)(wt_2 + x_2 i + y_2 j + z_2 k)$$
$$= \begin{matrix} (w_1 w_2 - x_1 x_2 - y_1 y_2 - z_1 z_2) \\ +(wt_1 x_2 + x_1 w_2 + y_1 z_2 - z_1 y_2)i \\ +(w_1 y_2 - x_1 z_2 + y_1 w_2 + z_1 x_2)j \\ +(wt_1 z_2 + x_1 y_2 - y_1 x_2 - z_1 wt_2)k \end{matrix} \quad (2.2)$$

We have $Q_1 Q_2 \neq Q_2 Q_1$ i.e., the multiplication of quaternions is not commutative in general . the quaternions is also viewed as $Q = w + \vec{r}$, where $\vec{r} = xi + yj + zk$, we identify with 3D vector (x, y, z) so the dot product $(\cdot)$ and cross product $(\times)$ of vectors introduced the familiar notation and so we define the canonical scalar product on $\mathbb{H}$ by

$$Q_1 Q_2 = (w_1 + \vec{r}_1)(wt_2 + \vec{r}_2)$$
$$= (w_1 w_2 - \vec{r_1}.\vec{r_2}) + w_1 \vec{r_2} + w_2 \vec{r_1} + \vec{r_1} \times \vec{r_2} \; (2.3)$$

The multiplication of quaternions is not a commutative over a real numbers so that it is called is a non-commutative algebra.

The square of a quaternion is given by
$Q = [w, \vec{r}]$, then

$$Q^2 = [w, \vec{r}][w, \vec{r}] = [w^2 - \vec{r}.\vec{r}, 2w\vec{r} + \vec{r} \times \vec{r}]$$
$$= [w^2 - \vec{r}.\vec{r}, 2w\vec{r}]$$
$$Q^2 = [w^2 - x^2 - y^2 - z^2, 2w(xi + yj + zk)]$$

(2.4)

The square of a pure quaternion is

$$Q^2 = [0, \vec{r}][0, \vec{r}] = [0 - \vec{r}.\vec{r}, \vec{r} \times \vec{r}]$$
$$= [-\vec{r}.\vec{r}, 0] = [-(x^2 + y^2 + z^2), 0]$$

3- The quotient

We represent the quotient Q_1/Q_2 as

$$Q_3 = Q_1 Q_2^{-1} = \frac{Q_1 Q_2^*}{|Q_2|^2} \quad (2.5)$$

Algebraic properties 2.2:

(1) Norm:

$$|Q| = \sqrt{\boldsymbol{w^2 + x^2 + y^2 + z^2}} \quad (2.6)$$

and for two quaternions p , q satisfies the norm of product $|\mathbf{Q}\boldsymbol{P}| = |\mathbf{Q}||\boldsymbol{P}|$

(2) Conjugate:

$$Q^* = (\boldsymbol{w + xi + yj + zk})^*$$
$$= \boldsymbol{w - xi - yj - zk} \quad (2.7)$$

and for two quaternions p , q satisfies the conjugate of product$(\boldsymbol{QP})^* = \boldsymbol{P}^*\boldsymbol{Q}^*$

(3) Multiplicative inverse:

$$\boldsymbol{Q^{-1}} = \frac{Q^*}{|Q|} \quad (2.8)$$

and for two quaternions P , Q satisfies the inverse of product

$$(\mathbf{P}\boldsymbol{Q})^{-\mathbf{1}} = \boldsymbol{Q}^{-\mathbf{1}}\mathbf{P}^{-\mathbf{1}}$$

(4) Real part of Q

$$\boldsymbol{w} = \frac{\boldsymbol{Q + Q^*}}{\mathbf{2}} \quad (2.9)$$

(5) The dot product

$$Q_1 \cdot Q_2 = \boldsymbol{w_1 w_2 + x_1 x_2 + y_1 y_2 + z_1 z_2}$$
$$= Q_1 \cdot Q_2^* \quad (2.10)$$

(6) The unit quaternion: has norm 1, i.e., $|Q| = \boldsymbol{w^2 + x^2 + y^2 + y^2 = 1}$ (2.11)
and inverse of unit quaternion $Q^{-\mathbf{1}} = Q^*$
and it can be represented by

$$Q = \mathbf{cos}\,\boldsymbol{\theta} + \hat{\mathbf{u}}\,\mathbf{sin}\,\boldsymbol{\theta} \quad (2.12)$$

Where $\hat{\mathbf{u}}$ is the vector has length 1. However, observe that the quaternion product $\hat{\mathbf{u}}\hat{\mathbf{u}} = -\mathbf{1}$
Note in general : a pure unit quaternion is a square root of $-\mathbf{1}$, such as imaginary units and Euler's identity of quaternion
The unit Quaternion is normalised with respect to the Hermitian inner product , where I_2 is the 2x2 identity matrix.
A quaternion with a unit norm is called a normalised quaternion

$$Q' = \frac{Q}{|Q|} = \frac{Q}{\sqrt{w^2 + |\vec{\boldsymbol{r}}|^2}} \quad (2.13)$$

(7) The unit length Euler's identity for quaternions

$$\mathbf{exp}(\hat{\mathbf{u}}\boldsymbol{\theta}) = \mathbf{cos}\,\boldsymbol{\theta} + \hat{\mathbf{u}}\,\mathbf{sin}\,\boldsymbol{\theta}\,. \quad (2.14)$$

where the exponential on the left-hand side is evaluated by

$$Q^{\boldsymbol{n}} = (\mathbf{cos}\,\boldsymbol{\theta} + \hat{\mathbf{u}}\,\mathbf{sin}\,\boldsymbol{\theta})^{\boldsymbol{n}}$$
$$= \mathbf{cos}(\boldsymbol{n\theta}) + \hat{\mathbf{u}}\,\mathbf{sin}(\boldsymbol{n\theta})$$
$$= \mathbf{exp}(\hat{\mathbf{u}}\boldsymbol{n\theta}) \quad (2.15)$$

(8) It is also possible to define the logarithm of a unit quaternion,
$\mathbf{ln}\, Q = \mathbf{ln}(\mathbf{cos}\,\boldsymbol{\theta} + \hat{\mathbf{u}}\,\mathbf{sin}\,\boldsymbol{\theta})$

$= \mathbf{ln\,exp}(\hat{\mathbf{u}}\boldsymbol{\theta}) = \hat{\mathbf{u}}\boldsymbol{\theta}$ (2.16)

The quaternions can be thought of as a choice of a group structure

(9) The quaternion polar coordinate representation is

$Q = |Q|(\mathbf{cos}\,\boldsymbol{\theta} + \hat{\mathbf{u}}\,\mathbf{sin}\,\boldsymbol{\theta})$

$= |Q|\,\mathbf{exp}(\hat{u}\boldsymbol{\theta})$ (2.17)

(10) The multiplication of two vectors

If we multiply together two imaginary quaternions $\boldsymbol{P}, Q \in \mathbb{H}_{\mathbf{p}}$, where $\mathbb{H}_{\mathbf{p}}$ is set of pure quaternions we obtain a quaternionic version of the scalar product and vector product on $\mathbb{R}^3$ as follows[10]

$$\boldsymbol{P}Q = -\boldsymbol{P} \cdot Q + \boldsymbol{P} \wedge Q \in \mathbb{R} \oplus \mathbb{H}_{\mathbf{p}} \cong \mathbb{H}$$

(2.18)

Others algebraic properties

The set of quaternions satisfy some other algerbic properties that worth mentioning

* The set of quaternions is an abelian group $(\mathbb{H}, +)$under quaternion addition .

* The set of quaternions is not abelian rig $(\mathbb{H}, +, \cdot)$where $+$ and $\cdot$ are quaternion addition and multiplication .[14]

3. Matrix representation of quaternion

The are at least two ways for representation of quaternions as matrices in such way that quaternions addition and multiplication correspond to matrix addition and multiplication .One is use 2×2 complex matrices ,and the other is to use 4×4 real matrices. In each case , the representation given is one of a family of lineally related representations .[13]

The 4×4 real matrix representation 3.1

The quaternion is written in 4×4 real matrix notation as follows

$$[w, x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}] \mapsto \boldsymbol{L}_Q = \begin{bmatrix} \boldsymbol{w} & -\boldsymbol{x} & -\boldsymbol{y} & -\boldsymbol{z} \\ \boldsymbol{x} & \boldsymbol{w} & \boldsymbol{z} & -\boldsymbol{y} \\ \boldsymbol{y} & -\boldsymbol{z} & \boldsymbol{w} & \boldsymbol{x} \\ \boldsymbol{z} & \boldsymbol{y} & -\boldsymbol{x} & \boldsymbol{w} \end{bmatrix} (3.1)$$

Orthogonal Matrix

We can demonstrate that the unit-norm quaternion matrix is orthogonal the product of $\mathrm{L_Q}$ with its transpose $\mathrm{L}_{Q^*} = \mathrm{L_Q^T}$ equals the identity matrix I(where I is 4×4 marix).

The 2×2 complex matrix representation 3.2

We switch from thinking of $\mathbb{H}$ as 4-D with real scalars and basis $, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, to being 2-D with complex scalars and basis $\mathbf{1}, \boldsymbol{j}$ by writing

$Q = \boldsymbol{w} + \boldsymbol{xi} + \boldsymbol{yj} + \boldsymbol{zk}$

$= (\boldsymbol{w} + \boldsymbol{xi}) + (\boldsymbol{y} + \boldsymbol{zi})\boldsymbol{j}$

$= \boldsymbol{\alpha} + \boldsymbol{\beta j}$

Where the map $j\colon \alpha + \beta j \mapsto -\bar{\beta} + \bar{\alpha} j$ is a conjugate-linear involution of $\mathbb{C}^2$ with $j^2 = -1$This identification $\mathbb{H} \cong \mathbb{C}^2$. [10]

Having written this down, it is easy to form the map: $\mathbb{H} \to \mathcal{H} \subset Mat(2, \mathbb{C})$

$$Q = \boldsymbol{\alpha} + \boldsymbol{\beta}\mathbf{j} \mapsto \boldsymbol{M}_Q = \begin{bmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} \\ -\bar{\boldsymbol{\beta}} & \bar{\boldsymbol{\alpha}} \end{bmatrix} \quad (3.2)$$

$$\mathbf{1} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}, \boldsymbol{i} = \begin{bmatrix} \boldsymbol{i} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{i} \end{bmatrix},$$

$$\boldsymbol{j} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix}, \boldsymbol{k} = \begin{bmatrix} \mathbf{0} & \boldsymbol{i} \\ \boldsymbol{i} & \mathbf{0} \end{bmatrix}$$

The quaternion algebra can thus be realised as a real subalgebra of Mat(2, $\mathbb{C}$), using the icidentifations

$$\boldsymbol{i} = \begin{bmatrix} \boldsymbol{i} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{i} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{bmatrix} \boldsymbol{i},$$

$$\boldsymbol{j} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \boldsymbol{i} \\ -\boldsymbol{i} & \mathbf{0} \end{bmatrix} \boldsymbol{i},$$

$$\boldsymbol{k} = \begin{bmatrix} \mathbf{0} & \boldsymbol{i} \\ \boldsymbol{i} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \boldsymbol{i}$$

so that if we define the matrices$\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}, \boldsymbol{\sigma_3}$such that

$$\boldsymbol{\sigma_1} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{bmatrix}, \boldsymbol{\sigma_2} = \begin{bmatrix} \mathbf{0} & -\boldsymbol{i} \\ \boldsymbol{i} & \mathbf{0} \end{bmatrix}, \boldsymbol{\sigma_3} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} (3.3)$$

we can write

$Q = \boldsymbol{w1} + \boldsymbol{xi} + \boldsymbol{yj} + \boldsymbol{zk}$

$= w\mathbf{1} + \boldsymbol{i}(\boldsymbol{z\sigma_1} + \boldsymbol{y\sigma_2} + \boldsymbol{x\sigma_3})$ (3.4)

The matrices$\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}, \boldsymbol{\sigma_3}$are called the Pauli spin matrices.

Note that the irtraces are null and that they are Hermitian (recall that a complex matrix is Hermitian if it is equal to the transpose of its conjugate, i.e.,$\boldsymbol{A}^* = \boldsymbol{A}$). If we let $\boldsymbol{e_0} = \boldsymbol{t}$, $\boldsymbol{e_1} = \boldsymbol{z}, \boldsymbol{e_2} = \boldsymbol{y}and\boldsymbol{e_3} = \boldsymbol{x}$, then $\boldsymbol{q}$ can be written as

$Q = \boldsymbol{e_0}\mathbf{1} + \boldsymbol{i}(\boldsymbol{e_1\sigma_1} + \boldsymbol{e_2\sigma_2} + \boldsymbol{e_3\sigma_3})$ (3.5)

And$\boldsymbol{e_0}, \boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$ are called theEuler parametersof the rotation specified by q

$|Q| = \mathbf{1}$, *then we can write also*

$Q = \mathbf{cos}\,½\theta + \boldsymbol{i}\,\mathbf{sin}\,½\theta\,(\boldsymbol{\beta\sigma_3} + \boldsymbol{\gamma\sigma_2} + \boldsymbol{\delta\sigma_1})$

(3.6)

Where

$$(\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}) = \frac{\mathbf{1}}{\mathbf{sin}\,½\theta}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$$

Letting $\boldsymbol{A} = \boldsymbol{\beta\sigma_3} + \boldsymbol{\gamma\sigma_2} + \boldsymbol{\delta\sigma_1}$, *it can be shown that*

$\boldsymbol{exp}(\boldsymbol{i\theta A}) = \mathbf{cos}\,\boldsymbol{\theta}\,\mathbf{1} + \boldsymbol{i}\,\mathbf{sin}\,\boldsymbol{\theta A}$ (3.7)

Note that : The exponential is the usual exponential of matrices, i.e., for a square $\boldsymbol{n} \times \boldsymbol{n}$ *matrix M*

$$\boldsymbol{exp}(\boldsymbol{A\theta}) = \boldsymbol{I_n} + \sum_{k\geq 1} \frac{(\boldsymbol{A\theta})^{\boldsymbol{k}}}{\boldsymbol{k}!}$$

Note that the rows (and columns) of such matrices are vectors in$\mathbb{C}^2$that are orthogonal with respect to the Hermitian inner product of $\mathbb{C}^2$given by

$\mathcal{H} \subset$*Furthermore, their norm is*

$$\sqrt{\alpha\bar{\alpha} + +\beta\bar{\boldsymbol{\beta}}} = \sqrt{\boldsymbol{w^2} + \boldsymbol{x^2} + \boldsymbol{y^2} + \boldsymbol{z^2}}$$

and the determinant of $\boldsymbol{A}$ *is*

$$\boldsymbol{w^2} + \boldsymbol{x^2} + \boldsymbol{y^2} + \boldsymbol{z^2}$$

The vector space $\mathbb{R}^3$*is a Lie algebra* 3.3

If we define the Lie bracket on$\mathbb{R}^3$*as the usual cross product* $\boldsymbol{u} \times \mathbf{v}$ *of vectors. Then the Lie algebra of* $\boldsymbol{SU}(\mathbf{2})$ *is isomorphic to* $(\mathbb{R}^3, \times)$ *and the exponential map can be viewed as a map* $\boldsymbol{exp}\colon (\mathbb{R}^3, \times) \to \boldsymbol{SU}(\mathbf{2})$ *given bythe formula*

$$\boldsymbol{exp}(\boldsymbol{\theta}\hat{\mathbf{u}}) = [\mathbf{cos}\,½\theta\,, \mathbf{sin}\,½\theta\,\hat{\mathbf{u}}] \quad (3.8)$$

a vector is rotated about the axis $\hat{\mathbf{u}}$ by an angle $\boldsymbol{\theta}$. This half-angle representation was discovered by Rodrigues .

Note that the squared norm $Q\,Q^*$ of a quaternion q is the same as the determinant of the matrix $\mathbf{M}_Q \in \mathbb{H}$. The isomorphism $\mathbf{M_Q}$ gives an easy way to deduce that $\mathbb{H}$ is an associative division algebra; the inverse of any nonzero matrix $\mathbf{A} \in \mathbb{H}$ is also in $\mathbb{H}$, and the only matrix in $\mathbb{H}$ whose determinant is zero is the zero matrix i.e., the quaternion algebra can be viewed as a union of complex planes, so the hyperbolic quaternion algebra is a union of split-complex number planes sharing the same real line .

4. complex quaternion

Definition.4.1: A complex quaternion is an element of the form

$$\boldsymbol{Q} = \boldsymbol{A_0} + \boldsymbol{A_1 e_1} + \boldsymbol{A_2 e_2} + \boldsymbol{A_3 e_3} \quad (4.1)$$

where

$\boldsymbol{A_0}, \boldsymbol{A_1}, \boldsymbol{A_2}$ and $\boldsymbol{A_3}$ complex numbers and $\boldsymbol{e_1}, \boldsymbol{e_2 e_3}$ are quaternionic units which satisfy the equalities

$$\begin{aligned} &\boldsymbol{e_1^2} = \boldsymbol{e_2^2} = \boldsymbol{e_3^2} = \boldsymbol{e_1 e_2 e_3} = -\boldsymbol{1}, \\ &\boldsymbol{e_1 e_2} = -\boldsymbol{e_2 e_1} = \boldsymbol{e_3}, \\ &\boldsymbol{e_2 e_3} = -\boldsymbol{e_3 e_2} = \boldsymbol{e_1}, \\ &\boldsymbol{e_3 e_1} = -\boldsymbol{e_1 e_3} = \boldsymbol{e_2}. \end{aligned} \quad (4.2)$$

As a consequence of this dentition, a complex quaternion Q can also be written

$\boldsymbol{Q} + \boldsymbol{iQ}^*$ (4.3)

Where $\boldsymbol{Q}$, $\boldsymbol{Q}^*$ *are real quaternions* and $\boldsymbol{i^2} = -\boldsymbol{1}$.

The set of all complex quaternions is denoted by $\mathbb{H}_{\mathbb{C}}$. Also, a complex quaternion $\mathbf{Q}$ is a sum of a scalar and a vector , called scalar part $\boldsymbol{S_Q} = \boldsymbol{A_0}$ and vector part

$\mathbf{V_Q} = \boldsymbol{A_1 e_1} + \boldsymbol{A_2 e_2} + \boldsymbol{A_3 e_3}$ (4.4)

If $\boldsymbol{S_Q} = \mathbf{0}$*;* then Q is called pure complex quaternion, we may call it a complex vector. The algebra of biquaternions can be considered as a tensor product $\mathbb{C} \otimes \mathbb{H}$ (taken over the reals)where $\mathbb{C}$ is the field of complex numbers and $\mathbb{H}$ is the algebra of real quaternions . In other words, the bi-quaternions are just the complexication of the real quaternions.

The product of two complex quaternions $\boldsymbol{Q} = \boldsymbol{S_Q} + \vec{\mathbf{V}}_{\mathbf{Q}}$ and $\boldsymbol{P} = \boldsymbol{S_P} + \vec{\mathbf{V}}_{\mathbf{P}}$ and If a complex quaternion is looked at as a four-dimensional vector, the complex quaternion product can be describe by a matrix-vector products as:

$$\boldsymbol{QP} = \begin{bmatrix} \boldsymbol{A_0} & -\boldsymbol{A_1} & -\boldsymbol{A_2} & -\boldsymbol{A_3} \\ \boldsymbol{A_1} & \boldsymbol{A_0} & -\boldsymbol{A_3} & \boldsymbol{A_2} \\ \boldsymbol{A_2} & \boldsymbol{A_3} & \boldsymbol{A_0} & -\boldsymbol{A_1} \\ \boldsymbol{A_3} & -\boldsymbol{A_2} & \boldsymbol{A_1} & \boldsymbol{A_1} \end{bmatrix} \begin{bmatrix} \boldsymbol{B_0} \\ \boldsymbol{B_1} \\ \boldsymbol{B_2} \\ \boldsymbol{B_3} \end{bmatrix} \quad (4.5)$$

From now one, we will identify the quaternion with the “complex” quaternion .[4]

5. Quaternions in Space

In 1898 the German mathematician, Adolf Hurwitz (1859–1919), proved that the product of the sum of $\boldsymbol{n}$ squares by the sum of $\boldsymbol{n}$ squares is the sum of n squares only when $\boldsymbol{n}$ is equal to 1, 2, 4 and 8, which are represented by the reals, complex numbers, quaternions and octonions. This is known as ‘Hurwitz’s Theorem’ or the ‘1, 2, 4, 8 Theorem’. No other system is possible, which shows how important quaternions are within the realm of mathematics. Consequently, Hamilton’s search for a system of triples was futile, because there is no three-square identity.

Aunit-norm quaternion $\boldsymbol{Q}$ used to rotate a vector $\vec{\mathbf{v}}$ stored as a pure quaternion p as the product $\boldsymbol{QP}$, where

$$\begin{aligned} &\boldsymbol{Q} = [\boldsymbol{w}, \boldsymbol{\lambda}\hat{\mathbf{v}}], \boldsymbol{w}, \boldsymbol{\lambda} \in \mathbb{R}, \hat{\mathbf{v}} \in \mathbb{R}^3, \\ &|\hat{\mathbf{v}}| = \mathbf{1}, \boldsymbol{w^2} + \boldsymbol{\lambda^2} = \mathbf{1}. \end{aligned} \quad (\mathbf{4.1})$$

and the pure quaternion $\boldsymbol{P}$ stores the vector $\vec{\boldsymbol{r}}$ to be rotated:

$\boldsymbol{P} = [\mathbf{0}, \vec{\boldsymbol{r}}]$, $\boldsymbol{p} \in \mathbb{R}^3$

We compute the product $\boldsymbol{P}' = \boldsymbol{QP}$
and examine the vector part of p
to see if $\boldsymbol{P}$ is rotated:

$$\begin{aligned} \boldsymbol{P}' = \boldsymbol{QP} &= [\boldsymbol{w}, \boldsymbol{\lambda}\hat{\mathbf{v}}][\mathbf{0}, \vec{\boldsymbol{r}}] \\ &= [-\boldsymbol{\lambda}\hat{\mathbf{v}}.\vec{\boldsymbol{r}}, \boldsymbol{w}\vec{\boldsymbol{r}} + \boldsymbol{\lambda}\hat{\mathbf{v}} \times \vec{\boldsymbol{r}}] \quad (\mathbf{4.2}) \end{aligned}$$

As we see from (4.2) that the result is a general quaternion with a scalar and a vector component.

As special case: If $\hat{\mathbf{v}}$ perpendicular to $\vec{\boldsymbol{r}}$, the dot product term $\boldsymbol{\lambda}\hat{\mathbf{v}}.\vec{\boldsymbol{r}}$ in (4.2) vanish, and we have left with the pure quaternion

$$\boldsymbol{P}' = [\mathbf{0}, \boldsymbol{w}\vec{\boldsymbol{r}} + \boldsymbol{\lambda}\hat{\mathbf{v}} \times \vec{\boldsymbol{r}}] \quad (4.3)$$

$\vec{\boldsymbol{r}}$ is perpendicular to $\hat{\mathbf{v}}$, and $\hat{\mathbf{v}} \times \vec{\boldsymbol{r}}$ is perpendicular to the plane containing $\vec{\boldsymbol{r}}$ and $\hat{\mathbf{v}}$. Now because $\hat{\mathbf{v}}$ is a unit vector.

In general Case: We begin by defining a unit-norm quaternion $\boldsymbol{Q}$: $\boldsymbol{Q} = [\boldsymbol{w}, \boldsymbol{\lambda}\hat{\mathbf{v}}]$ where $\boldsymbol{w^2} + \boldsymbol{\lambda^2} = \mathbf{1}$, The vector $\vec{\boldsymbol{r}}$ to be rotated is encoded as a pure quaternion $\boldsymbol{P} = [\mathbf{0}, \vec{\boldsymbol{r}}]$, and the inverse quaternion $\boldsymbol{Q^{-1}} = [\boldsymbol{w}, -\boldsymbol{\lambda}\hat{\mathbf{v}}]$.

Therefore, the product $\boldsymbol{QPQ^{-1}}$ is

$$\begin{aligned} \boldsymbol{QPQ^{-1}} &= [\boldsymbol{w}, \boldsymbol{\lambda}\hat{\mathbf{v}}][\mathbf{0}, \vec{\boldsymbol{r}}][\boldsymbol{t}, -\boldsymbol{\lambda}\hat{\mathbf{v}}] \\ &= [-\boldsymbol{\lambda}\hat{\mathbf{v}}.\vec{\boldsymbol{r}}, \boldsymbol{w}\vec{\boldsymbol{r}} + \boldsymbol{\lambda}\hat{\mathbf{v}} \times \vec{\boldsymbol{r}}][\boldsymbol{w}, -\boldsymbol{\lambda}\hat{\mathbf{v}}] \\ &= [\mathbf{0}, \mathbf{2}\boldsymbol{\lambda^2}(\hat{\mathbf{v}}.\vec{\boldsymbol{r}})\hat{\mathbf{v}} + (\boldsymbol{t^2} - \boldsymbol{\lambda^2})\vec{\boldsymbol{r}} + \mathbf{2}\boldsymbol{\lambda t}\hat{\mathbf{v}} \times \vec{\boldsymbol{r}}] (4.4) \end{aligned}$$

Now, if we want this product to actually rotate the vector by θ, then we must build this in from the outset by halving θ in $\boldsymbol{Q}$.

Hamilton calls the angle ($0 \le \theta \le \pi$) the angle of , and calls the unit vector $\hat{\mathrm{v}}$ the axis of $\boldsymbol{Q}$.

Theorem: For any unit quaternion

$$\boldsymbol{Q} = [w, \vec{\mathrm{r}}] = \cos \tfrac{1}{2}\theta + \hat{\mathrm{v}} \sin \tfrac{1}{2}\theta$$

and for any vector $\vec{\mathrm{v}} \in \mathbb{R}^3$ the action of operator L_q on $\vec{\mathrm{v}}$ is equivalent to rotation of the vector through angle θ about $\hat{\mathrm{v}}$ as the axis of rotation

$$L_{\boldsymbol{Q}}(\vec{\mathrm{v}}) = \boldsymbol{Q}\vec{\mathrm{v}}\boldsymbol{Q}^*$$

We substitute the unit quaternion in this form to obtain the resulting vector form rotating a vector $\vec{\mathrm{v}}$ about the axis $\hat{\mathrm{v}}$ through θ:

$L_{\boldsymbol{Q}}(\vec{\mathrm{v}}) = \boldsymbol{QPQ^{-1}} =$
$[\mathbf{0}, (\mathbf{1} - \cos\boldsymbol{\theta})(\hat{\mathbf{v}}.\vec{\boldsymbol{r}})\hat{\mathbf{v}} + \cos\boldsymbol{\theta}\,\vec{\boldsymbol{r}} + \sin\boldsymbol{\theta}\,\hat{\mathbf{v}} \times \vec{\boldsymbol{r}}]$(4.4)

The product $\boldsymbol{QPQ^{-1}}$ was discovered by Hamilton who failed to publish the result. Cayley, also discovered the product and published the result in 1845 [8]. However, Altmann notes that “in Cayley’s collected papers he concedes priority to Hamilton” [2], which was a nice gesture. However, the person who had recognised the importance of the half-angle parameters in (3.8) before Hamilton and Cayley was Rodrigues—who published a solution that was not seen by Hamilton, but apparently, was seen by Cayley.

Quaternions in Matrix Form of $\boldsymbol{QPQ^{-1}}$.

The are two methods: the first is a simple vectorial method which translates the vector equation representing $\boldsymbol{QPQ^{-1}}$ directly into matrix form. The second method uses matrix algebra to develop a rather cunning solution.

Vector Method

For the vector method it is convenient to describe the unit-norm quaternion as

$\boldsymbol{Q} = [\boldsymbol{w}, \vec{\mathbf{v}}] = [\boldsymbol{w}, x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}]$, where $\boldsymbol{t^2} + |\vec{\mathbf{v}}|^2 = \mathbf{1}$

and the pure quaternion as

$$\boldsymbol{P} = [\mathbf{0}, \vec{\boldsymbol{u}}] = [\mathbf{0}, \mathbf{u}_x\boldsymbol{i} + \mathbf{u}_y\boldsymbol{j} + \mathbf{u}_z\boldsymbol{k}]$$

A simple way to compute $\boldsymbol{QPQ^{-1}}$ is to use (4.3) and substitute $|\vec{\mathbf{v}}|$ for λ

$$\begin{aligned} \boldsymbol{QPQ^{-1}} &= [\mathbf{0}, \mathbf{2}\boldsymbol{\lambda^2}(\hat{\mathbf{v}}.\vec{\boldsymbol{u}})\hat{\mathbf{v}} + (\boldsymbol{w^2} - \boldsymbol{\lambda^2})\vec{\boldsymbol{u}} + \mathbf{2}\boldsymbol{\lambda w}\hat{\mathbf{v}} \times \vec{\boldsymbol{u}}] \\ &= [\mathbf{0}, \mathbf{2}|\vec{\mathrm{v}}|^2(\hat{\mathbf{v}}.\vec{\boldsymbol{u}})\hat{\mathrm{v}} + (\boldsymbol{w^2} - |\vec{\mathrm{v}}|^2)\vec{\boldsymbol{u}} + \mathbf{2}|\vec{\mathrm{v}}|\boldsymbol{w}\hat{\mathbf{v}} \times \vec{\boldsymbol{u}}] \end{aligned}$$

Next, we substitute $\vec{\mathbf{v}}$ for $|\vec{\mathbf{v}}|\hat{\mathbf{v}}$:

$$\boldsymbol{QPQ^{-1}} = [\mathbf{0}, \mathbf{2}(\vec{\mathbf{v}}.\vec{\boldsymbol{u}})\vec{\mathbf{v}} + (\boldsymbol{w^2} - |\vec{\mathrm{v}}|^2)\vec{\boldsymbol{u}} + \mathbf{2}\boldsymbol{w}\vec{\mathbf{v}} \times \vec{\boldsymbol{u}}]$$

Since we are working with unit-norm quaternions to prevent scaling

$$w^2 + |\vec{v}|^2 = 1 \Rightarrow w^2 - |\vec{v}|^2 = 2w^2 - 1$$

Therefore,

$$QPQ^{-1} = [0, 2(\vec{v}.\vec{u})\vec{v} + (2w^2 - 1)\vec{u} + 2w\vec{v} \times \vec{u}]$$

If we let $P' = QPQ^{-1}$, which is a pure quaternion, we have

$$[0, \vec{u}] = [0, 2(\vec{v}.\vec{u})\vec{v} + (2w^2 - 1)\vec{u} + 2w\vec{v} \times \vec{u}]$$

$$\vec{u} = 2(\vec{v}.\vec{u})\vec{v} + (2w^2 - 1)\vec{u} + 2w\vec{v} \times \vec{u} \quad (4.5)$$

We are only interested in the rotated vector $\vec{v}$ comprising the three terms $2(\vec{v}.\vec{u})\vec{v}$,
$(2w^2 - 1)\vec{u}, 2w\vec{v} \times \vec{u}$, which can be represented by three individual matrices and
summed together.

$$2(\vec{v}.\vec{u})\vec{v} = 2(x\mathrm{u}_x + y\mathrm{u}_y + z\mathrm{u}_z)(xi + yj + zk)$$

$$\begin{bmatrix} 2x^2 & xy & xz \\ xy & 2y^2 & yz \\ xz & yz & 2z^2 \end{bmatrix}\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

$$(2w^2 - 1)\vec{u} = (2w^2 - 1)u_x i + (2w^2 - 1)u_y j + (2w^2 - 1)u_z k$$

$$(2w^2 - 1)\vec{u} = \begin{bmatrix} 2w^2 - 1 & 0 & 0 \\ 0 & 2w^2 - 1 & 0 \\ 0 & 0 & 2w^2 - 1 \end{bmatrix}\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

$$2w\vec{v} \times \vec{u} = 2w\left((\mathrm{u}_z y - z\mathrm{u}_y)i + (\mathrm{u}_x z - z\mathrm{u}_x)j + (\mathrm{u}_y x - x\mathrm{u}_y)k\right)$$

$$(2w^2 - 1)\vec{u} = \begin{bmatrix} 0 & -2wz & 2wy \\ 2wz & 0 & -2wx \\ -2wy & 2wx & 0 \end{bmatrix}\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

Adding these matrices together: $\vec{u} = Q\vec{v}Q^*$

$$\begin{bmatrix} 2(w^2 + x^2) - 1 & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & 2(w^2 + y^2) - 1 & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & 2(w^2 + z^2) - 1 \end{bmatrix}\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \quad (4.6)$$

Or $\vec{u} = Q\vec{v}Q^*$

$$\begin{bmatrix} 1 - 2(y^2 + z^2) & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & 1 - 2(x^2 + z^2) & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & 1 - 2(x^2 + y^2) \end{bmatrix}\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \quad (4.7)$$

Or $\vec{u} = Q\vec{v}Q^*$

$$\begin{bmatrix} w^2 + x^2 - y^2 - z^2 & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & w^2 - x^2 + y^2 - z^2 & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & w^2 - x^2 - y^2 + z^2 \end{bmatrix}\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

(4.8)

If we reverse the product. To compute the vector part of $Q^{-1}PQ$ all that we have to do is reverse the sign of $\vec{v} \times \vec{u}$, $\vec{u} = Q^*\vec{v}Q$

$$\begin{bmatrix} 2(w^2 + x^2) - 1 & 2(xy + wz) & 2(xz - wy) \\ 2(xy - wz) & 2(w^2 + y^2) - 1 & 2(yz + wx) \\ 2(xz + wy) & 2(yz - wx) & 2(w^2 + z^2) - 1 \end{bmatrix}\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \quad (4.9)$$

Or $\vec{u} = Q^*\vec{v}Q$

$$\begin{bmatrix} 1 - 2(y^2 + z^2) & 2(xy + wz) & 2(xz - wy) \\ 2(xy - wz) & 1 - 2(x^2 + z^2) & 2(yz + wx) \\ 2(xz + wy) & 2(yz - wx) & 1 - 2(x^2 + y^2) \end{bmatrix}\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \quad (4.10)$$

we Observe that (4.8) is the transpose of (4.6), and (4. 9) is the transpose of (4.7)

Where

$$[0, \vec{u}] = Q^{-1}pQ$$

Problem.1

Consider a rotation about an axis defined by (1,1,1) through angle $2\pi/3$ about the axis .

We define a unit vector $\hat{u} = \frac{1}{\sqrt{3}}(1,1,1)$

Let rotation angle $\theta = 2\pi/3$, then Q of rotation define as: $Q = \cos ½\theta + \sin ½\theta\, \hat{v}$

$Q = \left[½, \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{3}}(1,1,1)\right] = [½, ½(1,1,1)]$, then

$L_Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, let us compute the effect of rotation on the basic vector $i = (1,0,0)$, we obtain the resulting vector

$$L_Q\vec{v} = L_Q i = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = j \;.\; [13]$$

Problem.2

To rotate point P(0,1,1) about y- axis through angle $\theta = 90^\circ$.

The Q of rotation define as:

$Q = \cos ½\theta + \sin ½\theta\, \hat{v} = \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}j\right]$

Substituting $w = \frac{\sqrt{2}}{2}, x = 0 , y = \frac{\sqrt{2}}{2}, z = 0$

There fore

$$L_Q\vec{u} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = i + j. \; [5]$$

Matrix Method

It is obvious that $L_{Q^*} = L_Q^T$,the transpose of L_Q, and similarly, $R_{P^*} = R_P^T$.

Since $QQ^* = |Q|$, the matrix $L_Q L_Q^T$is the diagonal matrix $|Q|I$(where $I$is the identity 4×4 matrix), and similarly the matrix $R_P R_P^T$ is the diagonal matrix $|P|I$. Since $L_Q$and L_Q^T have the same determinant, we deduce that

$\det(L_Q)^2 = |Q|^4$,and thus

$\det(L_Q) = \pm|Q|^2$ .Then

$$det(L_Q) = w^2 + x^2 + y^2 + z^2 \quad (5.11)$$

This shows that when Q is a unit quaternion, $L_Q$is a rotation matrix, and similarly when P is a unit quaternion, R_P is a rotation matrix .

Define the map $\varphi: \mathbb{H} \times \mathbb{H} \to \mathbb{R}$as follows:

$$\varphi(Q,P) = ½Tr(Q\bar{P}) = ww' + xx' + yy' + zz' \quad (5.12)$$

It is easily verified that φ is bilinear, symmetric, and definite positive. Thus, the quaternions form a Euclidean space under the inner product defined by φ (see Berger [12], Dieudonn´ e [46], Bertin [15]).

It is immediate that under this inner product, the norm of a quaternion Q is just $\sqrt{|Q|}$. As a Euclidean space,$\mathbb{H}$is isomorphic to E^4. It is also immediate that the subspace $\mathbb{H}_p$ of pure quaternions is orthogonal to the space of “real quaternions”ℝ1. The subspace $\mathbb{H}_p$ of pure quaternions inherits a Euclidean structure, and this subspace is isomorphic to the Euclidean space E^3. Since Hand $E^4$are isomorphic Euclidean spaces, their groups of rotations $SO(\mathbb{H})$and $SO(4)$ are isomorphic, and we will identify them.Similarly, we will identify $SO(\mathbb{H}_p)$and $SO(3)$.

Quaternions and Rotations in SO(3)

The group of $SO(3)$is isomorphic to the group U(2) of quaternions $exp(\hat{v}\theta) = \cos\theta + \hat{v}\sin\theta$ of unit length.

Consider the sphere S^3in $\mathbb{R}^4$, and special unitary 2×2martix $SU(2)$
Recall that the 3-sphere $S^3$is the set of points $(x,y,z,t) \in \mathbb{R}^4$such that

$$x^2+y^2+z^2+t^2=1$$

and that the real projective space $\mathbb{R}P^3$ is the quotient of S^3modulo the equivalence relation that identifies antipodal points (where $(x,y,z,t)$and $(x,-y,-z,-t)$)are antipodal points).
The group $SO(3)$ of rotations of $\mathbb{R}^3$ is intimately related to the 3-sphere $S^3$and to the real projective space $\mathbb{R}P^3$.The key to this relationship is the fact that rotations can be represented by quaternions, discovered by Hamilton in 1843. Historically, the quaternions were the first instance of a skew field. As we shall see, quaternions represent rotations in $\mathbb{R}^3$. Then every plane rotation $\rho\theta$ by an angle $\theta$is represented by multiplication by the quaternion $exp(\hat{v}\theta) \in SU(2)$, in the sense that for all $Q,Q' \in \mathbb{H}$: $Q' = \rho\theta(Q)$ iff $Q' = exp(\hat{v}\theta)Q$

If$Q = xi + yj + zk$ and $P = x'i + y'j + z'k$ are pure quaternions, identifying Q and P with the corresponding vectors in $\mathbb{R}^3$,the inner product $Q\cdot P$and the cross product $Q\times P$ make sense, and letting $[0, Q\times P]$ denote the quaternion whose first component is 0 and whose last three components are those of $Q\times P$, we have the remarkable identity

$$\mathrm{QP} = -(Q\cdot P)1 + [[0, Q\times P] \quad (4.13)$$

More generally, given quaternion
$Q = t1 + xi + yj + zk$, we can write it
as $= [w,(x,y,z)]$, where w is called the scalar part of Q and (x,y,z)thepure part of Q. Then if $Q = [w,\vec{v}]$and $P = [w',\vec{v}']$it is easily seen that the quaternion product QPcan be expressed as

$$QP = \left[ww' - \vec{v}.\vec{v}', w\vec{v}'' + w'\vec{v} + \vec{v}\times\vec{v}'\right] \, (4.14)$$

The above formula for quaternion multiplication allows us to show the
following fact. Let $Z \in \mathbb{H}$, and assume that $ZQ = QZ$ for all $Q \in \mathbb{H}$.We claim that the pure part of Z is null, i.e., $Z = w1$ for some $w \in \mathbb{R}$.
Indeed ,writing $Z = [w,\vec{u}]$, if $\vec{u} \neq 0$
there is at least one non null pure quaternion$Q = [0,\vec{v}]$ such that $\vec{u}\times\vec{v} \neq 0$.
For arbitrary quaternions $Q = [w,\vec{v}]$and
$P = [w,\vec{v}']$, satisfies Lie algebra

$$[Q,P] = QP - PQ = [0,2(\vec{v}\times\vec{v}')] \quad (4.15)$$

and that for pure quaternions $Q,P \in \mathbb{H}_p$

$$2(Q\cdot P))1 = -(QP+PQ). \qquad (4.16)$$

Since quaternion multiplication for a given Q is bilinear, the map $P \mapsto QP$ is linear, and similarly for a given P, the map $Q \mapsto QP$ is linear. It is immediate that if the matrix of the first map is L_Q and the matrix of the second map is R_P, then

$$QP = L_q P = \begin{bmatrix} w & -x & -y & -z \\ x & w & z & -y \\ y & -z & w & x \\ z & y & -x & w \end{bmatrix} \begin{bmatrix} w' \\ x' \\ y' \\ z' \end{bmatrix} \, (4.17)$$

and $QP =$

$$R_p Q = \begin{bmatrix} w' & -x' & -y' & -z' \\ x' & w' & z' & -y' \\ y' & -z' & w' & x' \\ z' & y' & -x' & w' \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \, (4.18)$$

We observe that the columns (and the rows) of the above matrices are orthogonal. Thus, when Q and P are unit quaternions, both L_Q and R_P are orthogonal matrices .
Multiple Rotations
Say a vector or frame of reference is subjected to two rotations specified by Q_1
followed by Q_2. There is a temptation to convert both quaternions to their respective
matrix and multiply the matrices together. However, this not the most efficient way
of combining the rotations. It is best to accumulate the rotations as quaternions and
then convert to matrix notation, if required.
To illustrate this, consider the pure quaternion P subjected to the first quaternion Q_1

$$Q_1 P Q_1^*$$

followed by a second quaternion Q_2

$$Q_2(Q_1 P Q_1^*)Q_2^*$$

which can be expressed as

$$Q_2(Q_1 P Q_1^*)Q_2^* = (Q_2 Q_1)P(Q_2 Q_1)^* \quad (4.17)$$

Extra quaternions can be added accordingly.

$$Q_3[Q_2(Q_1 P Q_1^*)Q_2^*]Q_3^* = Q_3[(Q_2 Q_1)P(Q_2 Q_1)^*]Q_3^*$$
$$Q_3(Q_2 Q_1)P(Q_2 Q_1)^* Q_3^* = (Q_3 Q_2 Q_1)P(Q_3 Q_2 Q_1)^*$$
$$Q_3[Q_2(Q_1 P Q_1^*)Q_2^*]Q_3^* = (Q_3 Q_2 Q_1)P(Q_3 Q_2 Q_1)^*$$

Eigenvalue and Eigenvector
Although there is no doubt that (4.6) is a rotation matrix, we can secure further
evidence by calculating its eigenvalue and eigenvector. The eigenvalue should be θ
where

$$\begin{aligned} Tr(Q\vec{v}Q^{-1}) &= \mathbf{1} - \mathbf{2}(y^2+z^2) + \mathbf{1} - \mathbf{2}(x^2+z^2) + \mathbf{1} - \mathbf{2}(x^2+y^2) \\ &= 4w^2 + 2(w^2+x^2+y^2+z^2) - 3 = 4w^2 - 1 \\ &= 4\cos^2 ½\theta - 1 = 4\cos\theta + 4\sin^2 ½\theta - 1 \\ &= 4\cos\theta + 2 - 2\cos\theta - 1 = 1 + 2\cos\theta \end{aligned}$$

and

$$\cos\theta = ½(Tr(Q\vec{v}Q^{-1}) - 1) \qquad (4.18)$$

To compute the eigenvector of (4.6) we use the three equations

$$\begin{aligned} x_v &= (a_{22}-1)(a_{33}-1) - a_{23}a_{32} \\ y_v &= (a_{11}-1)(a_{33}-1) - a_{13}a_{31} \quad (4.19) \\ z_v &= (a_{11}-1)(a_{22}-1) - a_{12}a_{21} \end{aligned}$$

Therefore,
$x_v = 4x^2, y_v = 4y^2$ and $z_v = 4z^2$, which confirm that the eigenvector has components associated with the quaternion's vector. The square terms should be no surprise, as the triple QPQ^{-1}
includes the product of three quaternions.

Frames of Reference
The product QPQ^{-1} is used for rotating points about the vector associated with the
Quaternion Q, whereas the triple $Q^{-1}PQ$ can be used for rotating points about the
same vector in the opposite direction.
But this reverse rotation is also equivalent
to a change of frame of reference.

Interpolating Quaternions

Like vectors, quaternions can be interpolated to compute an in-between quaternion.

However, whereas two interpolated vectors results in a third vector that is readily visualised, two interpolated quaternions results in a third quaternion that acts as a rotor, and is not immediately visualised.

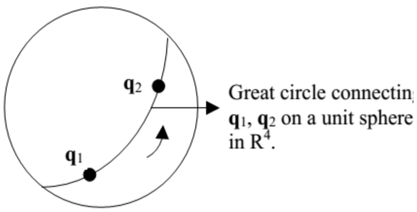

Fig. 4 Spherical linear interpolation between two unit

The spherical interpolant for vectors is

$$\vec{\mathrm{v}} = \frac{\sin(1-t)\theta}{\sin\theta}\mathrm{v}_1 + \frac{\sin s\theta}{\sin\theta}\mathrm{v}_2 \quad (4.19)$$

where θ is the angle between the vectors, The spherical linear interpolation between Q_1 and Q_2 is given by: $\mathrm{Slerp}(\mathrm{Q}_1, \mathrm{Q}_2; t)$

$$Q = \frac{\sin(1-t)\theta}{\sin\theta}\mathrm{Q}_1 + \frac{\sin s\theta}{\sin\theta}\mathrm{Q}_2, 0 \le t < 1 \quad (4.20)$$

So, given

$$\mathrm{Q}_1 = [w_1, x_1 i + y_1 j + z_1 k]$$
$$\mathrm{Q}_2 = [w_2, x_2 i + y_2 j + z_2 k]$$

θ is obtained by taking the 4D dot product of Q_1 and Q_2:

$$\cos\theta = \frac{\mathrm{Q}_1\mathrm{Q}_2}{|\mathrm{Q}_1||\mathrm{Q}_2|} \qquad (4.21)$$

$$\cos\theta = \frac{w_1w_2 + x_1x_2 + y_1y_2 + z_1z_2}{|\mathrm{Q}_1||\mathrm{Q}_2|}$$

and if we are working with unit-norm quaternions, then

$$\cos\theta = w_1w_2 + x_1x_2 + y_1y_2 + z_1z_2$$

Converting a Rotation Matrix to a Quaternion

The matrix transform equivalent to qpq^{-1} is:

$$QPQ^{-1} = \begin{bmatrix} \mathbf{1-2}(y^2+z^2) & \mathbf{2}(xy-\boldsymbol{w}z) & \mathbf{2}(xz+\boldsymbol{w}y) \\ \mathbf{2}(xy+\boldsymbol{w}z) & \mathbf{1-2}(x^2+z^2) & \mathbf{2}(yz-\boldsymbol{w}x) \\ \mathbf{2}(xz-\boldsymbol{w}y) & \mathbf{2}(yz+\boldsymbol{w}x) & \mathbf{1-2}(x^2+y^2) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad (4.22.1)$$

Inspection of the matrix shows that by combining various elements we can isolate the terms of a quaternion t, x, y, z. For example, by adding the terms $a_{11} + a_{22} + a_{33}$ we obtain

$$a_{11} + a_{22} + a_{33} = 4t^2 - 1$$

therefore,

$$t = \pm½\sqrt{1 + a_{11} + a_{22} + a_{33}} \qquad (4.22.2)$$

To find x, y and z we use

$$x = \frac{1}{4t}(a_{32} - a_{23})$$
$$y = \frac{1}{4t}(a_{13} - a_{31}) \qquad (4.22.3)$$
$$z = \frac{1}{4t}(a_{21} - a_{12})$$

Euler Angles to Quaternion

Perhaps the most important property of quaternions isthat they can characterize rotations in a three dimensional space. The conventional way of representing three-dimensional rotations is by using a set of Euler angles $\{\psi, \varphi, \theta\}$ which denote rotations about independent coordinate axes. The function that maps this transformation to its corresponding rotation matrix $\boldsymbol{R_{zyx}}: \mathbb{R}^3 \to SO(3)$ is:[8]

$$\vec{u} = Q_z Q_y Q_x \vec{\mathrm{v}} = R_{\psi,z} R_{\varphi,y} R_{\theta,x} \vec{\mathrm{v}}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(4.23)[3]

Expanding, we find w, x', y' and z', we get

$$\begin{aligned} w &= \cos½\psi\cos½\varphi\cos½\theta + \sin½\psi\sin½\varphi\sin½\theta \\ x' &= \cos½\psi\cos½\varphi\sin½\theta - \sin½\psi\sin½\varphi\cos½\theta \\ y' &= \cos½\psi\sin½\varphi\cos½\theta - \sin½\psi\cos½\varphi\sin½\theta \\ z' &= \sin½\psi\cos½\varphi\cos½\theta - \cos½\psi\sin½\varphi\sin½\theta \end{aligned} \quad (4.24)$$

where

$$q = [w, x'i + y'j + y'k] \qquad (4.25)$$

And the three rotation transforms

$$R_{\theta,x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_{\Phi,y} = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix} \qquad (4.26)$$

$$R_{\psi,z} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then

$$R_{\psi,z}, R_{\varphi,y}, R_{\theta,x} = \begin{bmatrix} \mathbf{C\psi C\varphi} & \mathbf{-S\psi C\theta + C\psi S\varphi S\theta} & \mathbf{S\psi \sin\theta + C\psi S\varphi C\theta} \\ \mathbf{S\psi \cos\varphi} & \mathbf{C\psi C\theta + S\psi S\varphi S\theta} & \mathbf{-C\psi S\theta + S\psi S\varphi C\theta} \\ \mathbf{-S\varphi} & \mathbf{C\varphi S\theta} & \mathbf{C\varphi C\theta} \end{bmatrix}$$

(4.27)

Then the most general rotation of a rigid body with a fixed point can be achieved by a single rotation about an axis through the fixed point. If $v = (l, m, n)$ denote a vector in 3-D space, then a rotational transformation by an angle θabout this vector is given by [3]

$$\begin{bmatrix} \boldsymbol{x'} \\ \boldsymbol{y'} \\ \boldsymbol{z'} \end{bmatrix} = \begin{bmatrix} \boldsymbol{l^2(1-C\theta)+C\theta} & \boldsymbol{lm(1-C\theta)-nS\theta} & \boldsymbol{nl(1-C\theta)+mS\theta} \\ \boldsymbol{lm(1-C\theta)+nS\theta} & \boldsymbol{m^2(1-C\theta)+C\theta} & \boldsymbol{mn(1-C\theta)-lS\theta} \\ \boldsymbol{nl(1-C\theta)-mS\theta} & \boldsymbol{mn(1-C\theta)+lS\theta} & \boldsymbol{n^2(1-C\theta)+C\theta} \end{bmatrix}$$

(5.28)

Where, $\vec{u} = (x', y', z')$, $\vec{\mathrm{v}} = (x, y, z)$ and

$$Q = (\cos½\theta + l\sin½\theta + +m\sin½\theta + n\sin½\theta)$$

(5.29)

II. Conclusion

In this paper, compact overview of mathematical constructs used most often quaternion rotation operator acting on any vector $\vec{v}$ in the three-dimensional space. Thisoperator plays an important role in classical mechanics as well as in computer graphics.

Vector transformations using unit quaternions are presented. These transformations are used as foundation for new, simple and compact direct kinematic algorithm in unit quaternion space.

Developed algorithm is demonstrated on human centrifuge which is modelled as 3 dimentional of manipulator with rotational joints.

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Authors

First Author – Emran Sasi Althabit*, PH.D of mathematics, SUDAN UNIVERISITY SEEINCE AND TECHNOLOGY (Khartoum), omransasi045@gmail.com +218926083695
Second Author – Ismail Mustafa Mohamed**, PH.D in mathematics, AL NEELAN UNIVERISITY (Khartoum), abuahd91@gmail.com , +249124699207