ISSN 2250-3153

Integrals Involving H-function of Several Complex Variables

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Abstract- In this paper, the author presented certain integrals involving product of the multivariable H-function with exponential function, Gauss's hypergeometric function and Fox's function. The results derived here and basic in natural and many include a number of known and new results as particular cases.

Index Terms- Hypergeometric function, Exponential function,H-function, Multivariable H-function.

I. Introduction

The Gaussian hypergeometric function is of fundamental important in the theory of special functional. The important of this function lies in the first that most all of the commonly used function of applicable mathematics, mathematical physics, engineering and mathematical biology are expressible as its special cases.

The series

$$2^{F}{}_{1}(a,b;c:z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^u}{u!} \tag{1}$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \begin{cases} a(a+1)\dots\dots(a+n-1) & n \in N \\ 1 & n = 0 \end{cases} \tag{2}$$

is called the Gauss's hypergeometric series of the famous German mathematician Carl Friedrich Gauss (1777-1855) who is the year 1812 introduced the series. It is represented by the symbol $2^{F}{}_{1}(a,b;c:z)$ and is called the Gauss's hypergeometric function also.

In 1961 Charles Fox [2] introduced a function which is more general then the Meijer's G-function and this function is well known in the literature of special function as Fox's H-function or simply the H-function. This function is defined and represented by means of the following Mellin-Barnes type contour integral

$$H[z] = H_{p,q}^{m,n}[z] = H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_L \theta(s) z^s\, ds \tag{3}$$

where for convenience

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)} \tag{4}$$

and z is a suitable contour of the Mellin-Barnes type which runs from $-i\infty\ to + i\infty$, separating the poles of $\Gamma(b_j-\beta_j s), j = 1, \dots\dots, m$ from those of $\Gamma(1-a_j+\alpha_j s),\ j = 1, \dots\dots, n$. An empty product is interpreted as unity the integers m,n,p,q satisfy the inequalities $0 \le n \le p, 0 \le m \le q$; the coefficients $\alpha_j = (j = 1, \dots\dots, p), \beta_j = (j = 1, \dots\dots, q)$ are positive real numbers, and the complex parameter $a_j = (j = 1, \dots\dots, p), b_j = (j = 1, \dots\dots, q)$ are so constrained that no poles of the integrand coincide. Owing to the popularity of the special functions, those are defined in (1) and (3) (c.f [4],[3] and [5]), details regarding these are avoided.

The H-functionof several complexvariables, defined H. M. Srivastava and R. Panda [5], by means of the following Mellian-Barnes type contour integral

$$H[z_1,\dots,z_r] = H_{p,q:p_1,q_1;\dots;p_r,q_r}^{0,n:m_1,n_1;\dots;m_r,n_r}\left[\begin{matrix}z_1\\ \cdot\\ \cdot\\ \vdots\\ z_r\end{matrix}\left|\begin{matrix}(a_j,\alpha_j^1,\dots,\alpha_j^r)_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};\dots;(c_j^r,\gamma_j^r)_{1,p_r}\\(b_j,\beta_j^1,\dots,\beta_j^1)_{1,q}:(d_j^1,\delta_j^1)_{1,q_1};\dots;(d_j^r,\delta_j^r)_{1,q_r}\end{matrix}\right.\right]$$

$$= \frac{1}{(2\pi i)^r} \int_{L_1} \cdots \int_{L_r} \phi_1(\xi_1) \ldots \phi_r(\xi_r)\ \psi(\xi_1, \ldots, \xi_r) z_1^{\xi_1}, \ldots, z_r^{\xi_r}\ d\xi_1 \ldots d\xi_r \tag{5}$$

where

$$\psi(\xi_1, \ldots \ldots, \xi_r) = \frac{\prod_{j=1}^{n} \Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^i \xi_i)}{\prod_{j=m+1}^{p_i} \Gamma(a_j - \sum_{i=1}^{r} \alpha_j^i \xi_i) \prod_{j=1}^{q} \Gamma(1 - b_j + \sum_{i=1}^{r} \beta_j^i \xi_i)}$$

$$\phi_i \xi_i = \frac{\prod_{j=1}^{m_i} \Gamma(d_j^i - \delta_j^i \xi_i) \prod_{j=1}^{n_i} \Gamma(1 - c_j^i + \gamma_j^i \xi_i)}{\prod_{j=m+1}^{q_i} \Gamma(1 - d_j^i + \delta_j^i \xi_i) \prod_{j=n+1}^{p_i} \Gamma(c_j^i - \gamma_j^i \xi_i)} \tag{6}$$

where integers n, p, q, m_i, n_i, p_i and q_i are constrained by the inequalities $0 \le n \le p, q \ge 0$ $1 \le m_i \le q_i$ and $0 \le n_i \le p_i$; $\forall$ $i = 1, \ldots, r$; $z_1, \ldots, z_r$ are complex variable. L_i is a suitable contour of the Mellien-Barnes type running from $-i\infty$ to $+i\infty$, in complex ξ_1-plane. Details regarding existence conditions and various parametric restrictions ofH-function several complex variables.

II. Required Results

We shall require the following results in the sequel:
The Mellin transform of the H-function follows from the definition (3) in view of the well-known Mellin inversion theorem. We have

$$\int_0^\infty x^{s-1} H_{p,q}^{m,n} \left[ax \left| \begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right. \right] dx = a^{-s} \theta(-s)$$

$$= a^{-s} \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j - \alpha_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s) \prod_{j=n+1}^{p} \Gamma(a_j + \alpha_j s)} \tag{7}$$

where

$$A = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j > 0,$$

$$|\arg a| < \frac{1}{2} A\pi, \delta = \sum_{j=1}^{m} \beta_j - \sum_{j=1}^{m} \alpha_j > 0$$

and

$$-\max_{0 \le j \le m} [Re(b_j/\beta_j)] < \max_{0 \le j \le n} [Re\{(1 - a_j/\alpha_j)]$$.

Lemma 2.1 From Rainville [4], we have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k, n-k) \tag{8}$$

III. Main Results

In this section we have evaluated certain integrals involving product of the H-function several complex variables with exponential function, Gauss's hypergeometric function and Fox's H-function.

Theorem 1:

$$\int_0^t x^{\rho-1} (t-x)^{\sigma-1} e^{-zx} {}_2F_1\left(\alpha, \beta; \gamma; a x^{\zeta} (t-x)^{\eta}\right) \times H_{p,q:p_1,q_1;\ldots;p_r,q_r}^{0,n:m_1,n_1;\ldots;m_r,n_r} \left[\begin{matrix} y_1 x^{\mu_1} (t-x)^{\nu_1} \\ \vdots \\ y_r x^{\mu_r} (t-x)^{\nu_r} \end{matrix} \right]$$

$$\left.\begin{matrix}(a_j,\alpha_j^1,\ldots,\alpha_j^r)_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};\ldots;(c_j^r,\gamma_j^r)_{1,p_r}\\(b_j,\beta_j^1,\ldots,\beta_j^1)_{1,q}:(d_j^1,\delta_j^1)_{1,q_1};\ldots;(d_j^r,\delta_j^r)_{1,q_r}\end{matrix}\right]dx$$

$$= e^{-zt}t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{z^{u-k}}{(u-k)!}t^{(\zeta+\eta-1)k+u} \times H^{0,n+2\,:\,m_1,n_1;\ldots;m_r,n_r}_{p+2,q+1:\,p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix}y_1 t^{(\mu_1+\nu_1)}\\ \cdot\\ \cdot\\ \cdot\\ y_r t^{(\mu_r+\nu_r)}\end{matrix}\right|$$

$$\left(1-\rho-\zeta k;\prod_{i=1}^{r}\mu_i\right),(1-\sigma-(\eta-1)k-u;\prod_{i=1}^{r}\nu_i)(a_j,\alpha_j^1,\ldots,\alpha_j^r)_{1,p}:$$

$$(b_j,\beta_j^1,\ldots,\beta_j^1)_{1,q},(1-\rho-\sigma-(\zeta+\eta-1)k-u;\prod_{i=1}^{r}(\mu_i+\nu_i):$$

$$\left.\begin{matrix}(c_j^1,\gamma_j^1)_{1,p_1};\ldots;(c_j^r,\gamma_j^r)_{1,p_r}\\(d_j^1,\delta_j^1)_{1,q_1};\ldots;(d_j^r,\delta_j^r)_{1,q_r}\end{matrix}\right] \quad (9)$$

where

$$f(k)=\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k}{k!} \quad (10)$$

provided

(i) $\mu \geq 0, \nu \geq 0$ (not both zero simultaneously)

(ii) ζ and η are non-negative integers such that $\zeta+\eta \geq 1$

(iii) $A_i > 0, \beta < 0 : |\arg y| < \frac{1}{2}A_i\pi \quad \forall\ i \in 1,\ldots,r$ where

$$A_i=\sum_{j=1}^{n}\alpha_j-\sum_{j=n+1}^{p_i}\alpha_{ji}+\sum_{j=1}^{m}\beta_j-\sum_{j=m+1}^{q_i}\beta_{ji}$$

$$\beta_i=\frac{1}{2}(p_i-q_i)+\sum_{j=1}^{q_i}b_{ji}-\sum_{j=1}^{p_i}a_{ji}$$

(iv) $Re(\rho)+\mu\max_{0\leq j\leq m}[Re(b_j/\beta_j)]>0$

$Re(\sigma)+\nu\max_{0\leq j\leq m}[Re(b_j/\beta_j)]>0$

Proof:

$$e^{-zt}\int_0^t x^{\rho-1}(t-x)^{\sigma-1}e^{-zx}\,{}_2F_1(\alpha,\beta;\gamma;ax^{\zeta}(t-x)^{\eta})H^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}$$

$$\times\left[\begin{matrix}y_1x^{\mu_1}(t-x)^{\nu_1}\\ \cdot\\ \cdot\\ \cdot\\ y_rx^{\mu_r}(t-x)^{\nu_r}\end{matrix}\right|\left.\begin{matrix}(a_j,\alpha_j^1,\ldots,\alpha_j^r)_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};\ldots;(c_j^r,\gamma_j^r)_{1,p_r}\\(b_j,\beta_j^1,\ldots,\beta_j^1)_{1,q}:(d_j^1,\delta_j^1)_{1,q_1};\ldots;(d_j^r,\delta_j^r)_{1,q_r}\end{matrix}\right]dx$$

Now we replace $e^{(t-x)z}$ with $\sum_{u=0}^{\infty}\frac{(t-x)^u z^u}{u!}$ and express the hypergeometric function and H-function several complex variables with the help of (1) and (5) respectively, to get

$$e^{-zt}\int_0^t x^{\rho-1}(t-x)^{\sigma-1}\sum_{u=0}^{\infty}\frac{(t-x)^u z^u}{u!}\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k x^{\zeta k}\,(t-x)^{\eta k}}{k!}$$

$$\times\frac{1}{(2\pi i)^r}\int_{L_1}\cdots\int_{L_r}\phi_1(\xi_1)\ldots\phi_r(\xi_r)\,\psi(\xi_1,\ldots,\xi_r)y_1^{\xi_1}x^{\mu_1\xi_1}(t-x)^{\nu_1\xi_1},\ldots,y_r^{\xi_r}x^{\mu_r\xi_r}$$

$$(t-x)^{\nu_r\xi_r}\;d\xi_1\ldots d\xi_r\quad dx$$

$$=\;e^{-zt}\int_0^t x^{\rho-1}(t-x)^{\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k x^{\eta k}}{k!}\frac{(t-x)^{\eta k+u}z^u}{u!}$$

$$\times\frac{1}{(2\pi i)^r}\int_{L_1}\cdots\int_{L_r}\phi_1(\xi_1)\ldots\phi_r(\xi_r)\,\psi(\xi_1,\ldots,\xi_r)y_1^{\xi_1}x^{\mu_1\xi_1}(t-x)^{\nu_1\xi_1},\ldots,y_r^{\xi_r}x^{\mu_r\xi_r}$$

$$(t-x)^{\nu_r\xi_r}\;d\xi_1\ldots d\xi_r\quad dx$$

Now by using (8) the above result reduces to

$$=\;e^{-zt}\int_0^t x^{\rho-1}(t-x)^{\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k x^{\eta k}}{k!}\frac{(t-x)^{(\eta-1)k+u}z^{u-k}}{(u-k)!}$$

$$\times\frac{1}{(2\pi i)^r}\int_{L_1}\cdots\int_{L_r}\phi_1(\xi_1)\ldots\phi_r(\xi_r)\,\psi(\xi_1,\ldots,\xi_r)y_1^{\xi_1}x^{\mu_1\xi_1}(t-x)^{\nu_1\xi_1},\ldots,y_r^{\xi_r}x^{\mu_r\xi_r}$$

$$(t-x)^{\nu_r\xi_r}\;d\xi_1\ldots d\xi_r\quad dx$$

Interchanging the order of integration and summation, we obtain

$$=e^{-zt}\sum_{u=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{z^{u-k}}{(u-k)!}\;\times\frac{1}{(2\pi i)^r}\int_{L_1}\cdots\int_{L_r}\phi_1(\xi_1)\ldots\phi_r(\xi_r)\,\psi(\xi_1,\ldots,\xi_r)$$

$$\left\{\int_0^t x^{\rho+\zeta k+\sum_{i=1}^{r}\mu_i\xi_i-1}(t-x)^{\sigma+(\eta-1)k+u+\sum_{i=1}^{r}\nu_i\xi_i-1}dx\right\}y_1^{\xi_1},\ldots,y_r^{\xi_r}\;d\xi_1\ldots d\xi_r$$

where $f(x)$ is given by (10)

On substituting $x=ts$ in the inner x-integral, the above expression reduces to

$$=e^{-zt}t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{z^{u-k}}{(u-k)!}t^{(\zeta+\eta-1)k+u}\;\times\frac{1}{(2\pi i)^r}\int_{L_1}\cdots\int_{L_r}\phi_1(\xi_1)\ldots\phi_r(\xi_r)$$

$$\psi(\xi_1,\dots,\xi_r)t^{\sum_{i=1}^{r}(\mu_i+\nu_i)\xi_i}\left\{\int_0^1 s^{\rho+\zeta k+\sum_{i=1}^{r}\mu_i\xi_i-1}(1-s)^{\sigma+(\eta-1)k+u+\sum_{i=1}^{r}\nu_i\xi_i-1}\,ds\right\}$$

$$y_1^{\xi_1},\dots,y_r^{\xi_r}\,d\xi_1\dots d\xi_r$$

$$=e^{-zt}t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{z^{u-k}}{(u-k)!}t^{(\zeta+\eta-1)k+u}\frac{1}{(2\pi i)^r}\int_{L_1}\dots\int_{L_r}\phi_1(\xi_1)\dots\phi_r(\xi_r)$$

$$\psi(\xi_1,\dots,\xi_r)\frac{\Gamma\left(\rho+\zeta k+\sum_{i=1}^{r}\mu_{i\xi_i}\right)\quad\Gamma\left(\sigma+(\eta-1)k+u+\sum_{i=1}^{r}\nu_{i\xi_i}\right)}{\Gamma\left(\rho+\sigma+(\zeta+\eta-1)k+u+\sum_{i=1}^{r}(\mu_i+\nu_i)\xi_i\right)}$$

$$y_1^{\xi_1}t^{(\mu_1+\nu_1)\xi_1},\dots,y_r^{\xi_r}t^{(\mu_r+\nu_r)\xi_r}\quad d\xi_1\dots d\xi_r$$

Finally, interpreting the contour integral by virtue of (5), we obtain

$$=e^{-zt}t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{z^{u-k}}{(u-k)!}t^{(\zeta+\eta-1)k+u}\times H^{0,n+2:m_1,n_1;\dots;m_r,n_r}_{p+2,q+1:p_1,q_1;\dots;p_r,q_r}\left[\begin{matrix}y_1t^{(\mu_1+\nu_1)}\\ \cdot\\ \cdot\\ \cdot\\ y_rt^{(\mu_r+\nu_r)}\end{matrix}\right|$$

$$\left(1-\rho-\zeta k;\prod_{i=1}^{r}\mu_i\right),(1-\sigma-(\eta-1)k-u;\prod_{i=1}^{r}\nu_i)(a_j,\alpha_j^1,\dots,\alpha_j^r)_{1,p}:$$

$$(b_j,\beta_j^1,\dots,\beta_j^r)_{1,q},(1-\rho-\sigma-(\zeta+\eta-1)k-u;\prod_{i=1}^{r}(\mu_i+\nu_i):$$

$$\left.\begin{matrix}(c_j^1,\gamma_j^1)_{1,p_1};\dots;(c_j^r,\gamma_j^r)_{1,p_r}\\(d_j^1,\delta_j^1)_{1,q_1};\dots;(d_j^r,\delta_j^r)_{1,q_r}\end{matrix}\right]$$

Theorem 2:

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1}e^{-zx}{}_2F_1\left(\alpha,\beta;\gamma;ax^{\zeta}(t-x)^{\eta}\right)$$

$$\times H^{0,n:m_1,n_1;\dots;m_r,n_r}_{p,q:p_1,q_1;\dots;p_r,q_r}\left[\begin{matrix}y_1x^{-\mu_1}(t-x)^{-\nu_1}\\ \cdot\\ \cdot\\ \cdot\\ y_rx^{-\mu_r}(t-x)^{-\nu_r}\end{matrix}\left|\begin{matrix}(a_j,\alpha_j^1,\dots,\alpha_j^r)_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};\dots;(c_j^r,\gamma_j^r)_{1,p_r}\\(b_j,\beta_j^1,\dots,\beta_j^1)_{1,q}:(d_j^1,\delta_j^1)_{1,q_1};\dots;(d_j^r,\delta_j^r)_{1,q_r}\end{matrix}\right.\right]dx$$

$$= e^{-zt} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{n} f(k) \frac{z^{u-k}}{(u-k)!} t^{(\zeta+\eta-1)k+u} \times H^{0,n\,:\,m_1,n_1;\dots;m_r,n_r}_{p+1,q+2:\,p_1,q_1;\dots;p_r,q_r} \left[\begin{matrix} y_1 t^{-\mu_1-\nu_1} \\ \cdot \\ \cdot \\ \cdot \\ y_r t^{-\mu_r-\nu_r} \end{matrix} \middle| \begin{matrix} (a_j, \alpha_j^1, \dots, \alpha_j^r)_{1,p}, (\rho+\sigma+(\zeta+\eta-1)k-u; \prod_{i=1}^{r}(\mu_i+\nu_i)) : (c_j^1, \gamma_j^1)_{1,p_1}; \dots; (c_j^r, \gamma_j^r)_{1,p_r} \\ (\rho+\zeta k; \prod_{i=1}^{r}\mu_i), (\sigma+(\eta-1)k+u; \prod_{i=1}^{r}\nu_i)(b_j, \beta_j^1, \dots, \beta_j^r)_{1,q}, : (d_j^1, \delta_j^1)_{1,q_1}; \dots; (d_j^r, \delta_j^r)_{1,q_r} \end{matrix} \right] \tag{11}$$

provided

$Re(\rho) - \mu \max_{0\le j\le n} [Re\{(a_j - 1)/\alpha_j)\}] > 0,$

$Re(\sigma) - \nu \max_{0\le j\le n} [Re\{(a_j - 1)/\alpha_j)\}] > 0$,

along with the sets of conditions (i) to (ii) given with theorem-1 and $f(k)$ is given by (10).

Theorem 3:

$$\int_0^t x^{\rho-1} (t-x)^{\sigma-1} e^{-zx} {}_2F_1(\alpha, \beta; \gamma; a x^{\zeta} (t-x)^{\eta})$$

$$\times H^{0,n:m_1,n_1;\dots;m_r,n_r}_{p,q:p_1,q_1;\dots;p_r,q_r} \left[\begin{matrix} y_1 x^{\mu_1} (t-x)^{-\nu_1} \\ \cdot \\ \cdot \\ \cdot \\ y_r x^{\mu_r} (t-x)^{-\nu_r} \end{matrix} \middle| \begin{matrix} (a_j, \alpha_j^1, \dots, \alpha_j^r)_{1,p} : (c_j^1, \gamma_j^1)_{1,p_1}; \dots; (c_j^r, \gamma_j^r)_{1,p_r} \\ (b_j, \beta_j^1, \dots, \beta_j^1)_{1,q} : (d_j^1, \delta_j^1)_{1,q_1}; \dots; (d_j^r, \delta_j^r)_{1,q_r} \end{matrix} \right] dx$$

$$= e^{-zt} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{n} f(k) \frac{z^{u-k}}{(u-k)!} t^{(\zeta+\eta-1)k+u} \times H^{0,n+1\,:\,m_1,n_1;\dots;m_r,n_r}_{p+1,q+2:\,p_1,q_1;\dots;p_r,q_r} \left[\begin{matrix} y_1 t^{\mu_1-\nu_1} \\ \cdot \\ \cdot \\ \cdot \\ y_r t^{\mu_r-\nu_r} \end{matrix} \middle| \begin{matrix} \left(1-\rho-\zeta k; \prod_{i=1}^{r}\mu_i\right), (a_j, \alpha_j^1, \dots, \alpha_j^r)_{1,p} : \\ \left(\sigma+(\eta-1)k+u; \prod_{i=1}^{r}\nu_i\right), (b_j, \beta_j^1, \dots, \beta_j^r)_{1,q}, : (1-\rho-\sigma-(\zeta+\eta-1)k-u; \prod_{i=1}^{r}(\mu_i-\nu_i), \end{matrix} \right.$$

$$\left. \begin{matrix} (c_j^1, \gamma_j^1)_{1,p_1}; \dots; (c_j^r, \gamma_j^r)_{1,p_r} \\ (d_j^1, \delta_j^1)_{1,q_1}; \dots; (d_j^r, \delta_j^r)_{1,q_r} \end{matrix} \right] \tag{12}$$

provided $\mu > 0, \nu \ge 0$ such that $\mu - \nu \ge 0$,

$Re(\rho) + \mu \max_{0\le j\le m} [Re(b_j/\alpha_j)\}] > 0,$

$Re(\sigma) - \nu \max_{0\le j\le n} [Re\{(a_j - 1)/\alpha_j)\}] > 0$, along with the sets of conditions (i) to (iii) given with theorem-1 and $f(k)$ is given by (10).

Theorem 4:

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1}e^{-zx}{}_2F_1\left(\alpha,\beta;\gamma;ax^{\zeta}(t-x)^{\eta}\right)$$

$$\times H^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} y_1 x^{\mu_1}(t-x)^{-\nu_1} \\ \vdots \\ y_r x^{\mu_r}(t-x)^{-\nu_r}\end{matrix}\middle| \begin{matrix}(a_j,\alpha_j^1,\ldots,\alpha_j^r)_{1,p}:\left(c_j^1,\gamma_j^1\right)_{1,p_1};\ldots;\left(c_j^r,\gamma_j^r\right)_{1,p_r} \\ (b_j,\beta_j^1,\ldots,\beta_j^1)_{1,q}:\left(d_j^1,\delta_j^1\right)_{1,q_1};\ldots;\left(d_j^r,\delta_j^r\right)_{1,q_r}\end{matrix}\right]dx$$

$$= e^{-zt}t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{z^{u-k}}{(u-k)!}t^{(\zeta+\eta-1)k+u} \times H^{0,n+1\,:\,m_1,n_1;\ldots;m_r,n_r}_{p+2,q+1:\,p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} y_1 t^{\mu_1-\nu_1} \\ \vdots \\ y_r t^{\mu_r-\nu_r}\end{matrix}\right|$$

$$\left(1-\rho-\zeta k;\prod_{i=1}^{r}\mu_i\right),(a_j,\alpha_j^1,\ldots,\alpha_j^r)_{1,p}:\left(\rho+\sigma+(\zeta+\eta-1)k+u;\prod_{i=1}^{r}(\nu_i-\mu_i),\right.$$

$$\left(\sigma+(\eta-1)k+u;\prod_{i=1}^{r}\nu_i\right),(b_j,\beta_j^1,\ldots,\beta_j^r)_{1,q},:$$

$$\left.\begin{matrix}\left(c_j^1,\gamma_j^1\right)_{1,p_1};\ldots;\left(c_j^r,\gamma_j^r\right)_{1,p_r} \\ \left(d_j^1,\delta_j^1\right)_{1,q_1};\ldots;\left(d_j^r,\delta_j^r\right)_{1,q_r}\end{matrix}\right] \tag{13}$$

provided $\mu \geq 0, \nu > 0$ such that $\nu - \mu \geq 0$,

$Re(\sigma) - \mu \max_{0\leq j\leq n}[Re\{(a_j-1)/\alpha_j)\}] > 0$,

$Re(\rho) + \nu \max_{0\leq j\leq m}[Re(b_j/\alpha_j)\}] > 0,$

along with the sets of conditions (i) to (iii) given with theorem-1 and $f(k)$ is given by (10).

Theorem 5:

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1}e^{-zx}{}_2F_1\left(\alpha,\beta;\gamma;ax^{\zeta}(t-x)^{\eta}\right)$$

$$\times H^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} y_1 x^{-\mu_1}(t-x)^{\nu_1} \\ \vdots \\ y_r x^{-\mu_r}(t-x)^{\nu_r}\end{matrix}\middle| \begin{matrix}(a_j,\alpha_j^1,\ldots,\alpha_j^r)_{1,p}:\left(c_j^1,\gamma_j^1\right)_{1,p_1};\ldots;\left(c_j^r,\gamma_j^r\right)_{1,p_r} \\ (b_j,\beta_j^1,\ldots,\beta_j^1)_{1,q}:\left(d_j^1,\delta_j^1\right)_{1,q_1};\ldots;\left(d_j^r,\delta_j^r\right)_{1,q_r}\end{matrix}\right]dx$$

$$= e^{-zt} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{n} f(k) \frac{z^{u-k}}{(u-k)!} t^{(\zeta+\eta-1)k+u} \times H^{0,n+1\,:\, m_1,n_1;\dots;m_r,n_r}_{p+2,q+1:\, p_1,q_1;\dots;p_r,q_r} \left[\begin{matrix} y_1 t^{-\mu_1+\nu_1} \\ \cdot \\ \cdot \\ \cdot \\ y_r t^{-\mu_r+\nu_r} \end{matrix} \right|$$

$$\left(1-\sigma-(\eta-1)k-u; \prod_{i=1}^{r} \nu_i\right), (a_j, \alpha_j^1, \dots, \alpha_j^r)_{1,p} : \left(\rho+\sigma+(\zeta+\eta-1)k+u; \prod_{i=1}^{r}(\mu_i-\nu_i),\right.$$

$$\left(\rho+\zeta k; \prod_{i=1}^{r} \mu_i\right), (b_j, \beta_j^1, \dots, \beta_j^r)_{1,q},:$$

$$\left. \begin{matrix} (c_j^1,\gamma_j^1)_{1,p_1}; \dots; (c_j^r,\gamma_j^r)_{1,p_r} \\ (d_j^1,\delta_j^1)_{1,q_1}; \dots; (d_j^r,\delta_j^r)_{1,q_r} \end{matrix} \right] \qquad (14)$$

provided $\mu > 0, \nu \geq 0$ such that $\mu - \nu \geq 0$

$Re(\rho) + \mu \max_{0 \leq j \leq m} [Re(b_j/\alpha_j)\}] > 0,$

$Re(\sigma) - \nu \max_{0 \leq j \leq n} [Re\{(a_j-1)/\alpha_j)\}] > 0$,

along with the sets of conditions (i) to (iii) given with theorem-1 and $f(k)$ is given by (10).

Theorem 6:

$$\int_0^t x^{\rho-1} (t-x)^{\sigma-1} e^{-zx} {}_2F_1\left(\alpha,\beta;\gamma; a x^{\zeta}(t-x)^{\eta}\right)$$

$$\times H^{0,n:m_1,n_1;\dots;m_r,n_r}_{p,q:p_1,q_1;\dots;p_r,q_r} \left[\begin{matrix} y_1 x^{-\mu_1}(t-x)^{\nu_1} \\ \cdot \\ \cdot \\ \cdot \\ y_r x^{-\mu_r}(t-x)^{\nu_r} \end{matrix} \middle| \begin{matrix} (a_j,\alpha_j^1,\dots,\alpha_j^r)_{1,p} : (c_j^1,\gamma_j^1)_{1,p_1}; \dots; (c_j^r,\gamma_j^r)_{1,p_r} \\ (b_j,\beta_j^1,\dots,\beta_j^1)_{1,q} : (d_j^1,\delta_j^1)_{1,q_1}; \dots; (d_j^r,\delta_j^r)_{1,q_r} \end{matrix} \right] dx$$

$$= e^{-zt} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{n} f(k) \frac{z^{u-k}}{(u-k)!} t^{(\zeta+\eta-1)k+u} \times H^{0,n+1\,:\, m_1,n_1;\dots;m_r,n_r}_{p+1,q+2:\, p_1,q_1;\dots;p_r,q_r} \left[\begin{matrix} y_1 t^{-\mu_1+\nu_1} \\ \cdot \\ \cdot \\ \cdot \\ y_r t^{-\mu_r+\nu_r} \end{matrix} \right|$$

$$\left(1-\sigma-(\eta-1)k-u; \prod_{i=1}^{r} \nu_i\right), (a_j, \alpha_j^1, \dots, \alpha_j^r)_{1,p} :$$

$$\left(\rho+\zeta k; \prod_{i=1}^{r} \mu_i\right), (b_j, \beta_j^1, \dots, \beta_j^r)_{1,q},: \; \left(1-\rho-\sigma-(\zeta+\eta-1)k-u; \prod_{i=1}^{r}(\nu_i-\mu_i),\right.$$

$$\left.\begin{matrix} (c_j^1,\gamma_j^1)_{1,p_1};\dots;(c_j^r,\gamma_j^r)_{1,p_r} \\ (d_j^1,\delta_j^1)_{1,q_1};\dots;(d_j^r,\delta_j^r)_{1,q_r} \end{matrix}\right] \qquad (15)$$

provided $\mu \geq 0, \nu > 0$ such that $\nu - \mu \geq 0$,
$Re(\sigma) - \mu \max_{0\leq j\leq n}[Re\{(a_j-1)/\alpha_j)\}] > 0$,
$Re(\rho) + \nu \max_{0\leq j\leq m}[Re(b_j/\alpha_j)\}] >$
along with the sets of conditions (i) to (iii) given with theorem-1and $f(k)$ is given by (10).
The integrals (11) to (15) can be proved on lines similar to those of integral (9)

1. Particular Cases

Putting $r = 2, t = 1$ and $\eta = 0$ in (9)(11)(12)(13)(14) and (15) the following known as well as new results may be realised:

(i) Integral (9) reduces to the results:

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}e^{-zx}{}_2F_1\left(\alpha,\beta;\gamma;ax^{\zeta}(t-x)^{\eta}\right) \times H^{0,n:m_1,n_1;m_2,n_2}_{p,q:p_1,q_1;p_2,q_2}\left[\begin{matrix} y_1x^{\mu_1}(1-x)^{\nu_1} \\ y_1x^{\mu_2}(1-x)^{\nu_2} \end{matrix}\right|$$

$$\left.\begin{matrix} (a_j,\alpha_j^1,\alpha_j^2)_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};(c_j^2,\gamma_j^2)_{1,p_2} \\ (b_j,\beta_j^1,\beta_j^2)_{1,q}:(d_j^1,\delta_j^1)_{1,q_1};(d_j^2,\delta_j^2)_{1,q_2} \end{matrix}\right] dx$$

$$= e^{-z}\sum_{u=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{z^{u-k}}{(u-k)!} \times H^{0,n+2:m_1,n_1;m_2,n_2}_{p+2,q+1:p_1,q_1;p_2,q_2}\left[\begin{matrix} y_1 \\ y_2 \end{matrix}\right|$$

$$\left.\begin{matrix} (1-\rho-\zeta k;\mu_1,\mu_2),(1-\sigma+k-u;\nu_1,\nu_2)(a_j,\alpha_j^1,\alpha_j^2)_{1,p}: & (c_j^1,\gamma_j^1)_{1,p_1};(c_j^2,\gamma_j^2)_{1,p_2} \\ (b_j,\beta_j^1,\beta_j^2)_{1,q},(1-\rho-\sigma-(\zeta-1)k-u;\mu_1+\nu_1,\mu_2+\nu_2): & (d_j^1,\delta_j^1)_{1,q_1};(d_j^2,\delta_j^2)_{1,q_2} \end{matrix}\right] \qquad (16)$$

where,

$$f(x) = \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k}{k!}$$

(ii) Integral (11) reduces to the results:

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}e^{-zx}{}_2F_1\left(\alpha,\beta;\gamma;ax^{\zeta}(t-x)^{\eta}\right) \times H^{0,n:m_1,n_1;m_2,n_2}_{p,q:p_1,q_1;p_2,q_2}\left[\begin{matrix} y_1x^{-\mu_1}(1-x)^{-\nu_1} \\ y_1x^{-\mu_2}(1-x)^{-\nu_2} \end{matrix}\right|$$

$$\left.\begin{matrix} (a_j,\alpha_j^1,\alpha_j^2)_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};(c_j^2,\gamma_j^2)_{1,p_2} \\ (b_j,\beta_j^1,\beta_j^2)_{1,q}:(d_j^1,\delta_j^1)_{1,q_1};(d_j^2,\delta_j^2)_{1,q_2} \end{matrix}\right] dx$$

$$= e^{-z}\sum_{u=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{z^{u-k}}{(u-k)!} \times H^{0,n:m_1,n_1;m_2,n_2}_{p+1,q+2:p_1,q_1;p_2,q_2}\left[\begin{matrix} y_1 \\ y_2 \end{matrix}\right|$$

$$\left.\begin{matrix}(a_j,\alpha_j^1,\alpha_j^2)_{1,p},(\rho+\sigma+(\zeta-1)k-u;\mu_1+\nu_1,\mu_2+\nu_2):\left(c_j^1,\gamma_j^1\right)_{1,p_1};\left(c_j^2,\gamma_j^2\right)_{1,p_2}\\(\rho+\zeta k;\mu_1,\mu_2),(\sigma-k+u;\nu_1,\nu_2),(b_j,\beta_j^1,\beta_j^2)_{1,q},:\quad\left(d_j^1,\delta_j^1\right)_{1,q_1};\left(d_j^2,\delta_j^2\right)_{1,q_2}\end{matrix}\right] \quad (17)$$

where,

$$f(x)=\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k}{k!}$$

(iii) Integral (12) reduces to the results:

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}e^{-zx}{}_2F_1\left(\alpha,\beta;\gamma;ax^{\zeta}(t-x)^{\eta}\right)\times H^{0,n:m_1,n_1;m_2,n_2}_{p,q:p_1,q_1;p_2,q_2}\left[\begin{matrix}y_1x^{\mu_1}(1-x)^{-\nu_1}\\y_1x^{\mu_2}(1-x)^{-\nu_2}\end{matrix}\right|$$

$$\left.\begin{matrix}(a_j,\alpha_j^1,\alpha_j^2)_{1,p}:\left(c_j^1,\gamma_j^1\right)_{1,p_1};\left(c_j^2,\gamma_j^2\right)_{1,p_2}\\(b_j,\beta_j^1,\beta_j^2)_{1,q}:\left(d_j^1,\delta_j^1\right)_{1,q_1};\left(d_j^2,\delta_j^2\right)_{1,q_2}\end{matrix}\right]$$

$$=e^{-z}\sum_{u=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{z^{u-k}}{(u-k)!}\times H^{0,n+1:m_1,n_1;m_2,n_2}_{p+1,q+2:p_1,q_1;p_2,q_2}\left[\begin{matrix}y_1\\y_2\end{matrix}\right|$$

$$\begin{matrix}(1-\rho-\zeta k;\mu_1,\mu_2),(a_j,\alpha_j^1,\alpha_j^2)_{1,p}:\\(\sigma-k+u;\nu_1,\nu_2),(b_j,\beta_j^1,\beta_j^2)_{1,q},:(1-\rho-\sigma-(\zeta-1)k-u;\mu_1-\nu_1,\mu_2-\nu_2),\end{matrix}$$

$$\left.\begin{matrix}\left(c_j^1,\gamma_j^1\right)_{1,p_1};\left(c_j^2,\gamma_j^2\right)_{1,p_2}\\\left(d_j^1,\delta_j^1\right)_{1,q_1};\left(d_j^2,\delta_j^2\right)_{1,q_2}\end{matrix}\right] \quad (18)$$

where,

$$f(x)=\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k}{k!}$$

(iv) Integral (13) reduces to the results:

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}e^{-zx}{}_2F_1\left(\alpha,\beta;\gamma;ax^{\zeta}\right)\times H^{0,n:m_1,n_1;m_2,n_2}_{p,q:p_1,q_1;p_2,q_2}\left[\begin{matrix}y_1x^{\mu_1}(1-x)^{-\nu_1}\\y_1x^{\mu_2}(1-x)^{-\nu_2}\end{matrix}\right|$$

$$\left.\begin{matrix}(a_j,\alpha_j^1,\alpha_j^2)_{1,p}:\left(c_j^1,\gamma_j^1\right)_{1,p_1};\left(c_j^2,\gamma_j^2\right)_{1,p_2}\\(b_j,\beta_j^1,\beta_j^2)_{1,q}:\left(d_j^1,\delta_j^1\right)_{1,q_1};\left(d_j^2,\delta_j^2\right)_{1,q_2}\end{matrix}\right]dx$$

$$=e^{-z}\sum_{u=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{z^{u-k}}{(u-k)!}\times H^{0,n+1:m_1,n_1;m_2,n_2}_{p+2,q+1:p_1,q_1;p_2,q_2}\left[\begin{matrix}y_1\\y_2\end{matrix}\right|$$

$$\begin{matrix}(1-\rho-\zeta k;\mu_1,\mu_2),(a_j,\alpha_j^1,\alpha_j^2)_{1,p}:(\rho+\sigma+(\zeta-1)k+u;\nu_1-\mu_1,\nu_2-\mu_2),\\(\sigma-k+u;\nu_1,\nu_2),(b_j,\beta_j^1,\beta_j^2)_{1,q},:\end{matrix}$$

$$\left.\begin{matrix} (c_j^1,\gamma_j^1)_{1,p_1};(c_j^2,\gamma_j^2)_{1,p_2} \\ (d_j^1,\delta_j^1)_{1,q_1};(d_j^2,\delta_j^2)_{1,q_2} \end{matrix}\right] \quad (19)$$

where,

$$f(x)=\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k}{k!}$$

(v) Integral (14) reduces to the results:

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}e^{-zx}\,{}_2F_1(\alpha,\beta;\gamma;ax^{\zeta})$$

$$\times\ H^{0,n:m_1,n_1;m_2,n_2}_{p,q:p_1,q_1;p_2,q_2}\left[\begin{matrix} y_1x^{-\mu_1}(1-x)^{\nu_1} \\ y_1x^{-\mu_2}(1-x)^{\nu_2}\end{matrix}\middle|\begin{matrix}(a_j,\alpha_j^1,\alpha_j^2)_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};(c_j^2,\gamma_j^2)_{1,p_2} \\ (b_j,\beta_j^1,\beta_j^2)_{1,q}:(d_j^1,\delta_j^1)_{1,q_1};(d_j^2,\delta_j^2)_{1,q_2}\end{matrix}\right]dx$$

$$=e^{-z}\sum_{u=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{z^{u-k}}{(u-k)!}\times H^{0,n+1:m_1,n_1;m_2,n_2}_{p+2,q+1:p_1,q_1;p_2,q_2}\left[\begin{matrix}y_1\\y_2\end{matrix}\right|$$

$$(1-\sigma+k-u;\nu_1,\nu_2),(a_j,\alpha_j^1,\alpha_j^2)_{1,p}:(\rho+\sigma+(\zeta-1)k+u;\mu_1-\nu_1,\mu_2-\nu_2),$$
$$(\rho+\zeta k;\mu_1,\mu_2),(b_j,\beta_j^1,\beta_j^1)_{1,q},:$$

$$\left.\begin{matrix} (c_j^1,\gamma_j^1)_{1,p_1};(c_j^2,\gamma_j^2)_{1,p_2} \\ (d_j^1,\delta_j^1)_{1,q_1};(d_j^2,\delta_j^2)_{1,q_2} \end{matrix}\right] \quad (20)$$

where,

$$f(x)=\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k}{k!}$$

(vi) Integral (l5) leads to the result:

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}e^{-zx}\,{}_2F_1(\alpha,\beta;\gamma;ax^{\zeta})$$

$$\times\ H^{0,n:m_1,n_1;m_2,n_2}_{p,q:p_1,q_1;p_2,q_2}\left[\begin{matrix} y_1x^{-\mu_1}(1-x)^{\nu_1} \\ y_1x^{-\mu_2}(1-x)^{\nu_2}\end{matrix}\middle|\begin{matrix}(a_j,\alpha_j^1,\alpha_j^2)_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};(c_j^2,\gamma_j^2)_{1,p_2} \\ (b_j,\beta_j^1,\beta_j^2)_{1,q}:(d_j^1,\delta_j^1)_{1,q_1};(d_j^2,\delta_j^2)_{1,q_2}\end{matrix}\right]dx$$

$$=e^{-z}\sum_{u=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{z^{u-k}}{(u-k)!}\times H^{0,n+1:m_1,n_1;m_2,n_2}_{p+1,q+2:p_1,q_1;p_2,q_2}\left[\begin{matrix}y_1\\y_2\end{matrix}\right|$$

$$(1-\sigma+k-u;\nu_1,\nu_2),(a_j,\alpha_j^1,\alpha_j^2)_{1,p}:$$
$$(\rho+\zeta k;\mu_1,\mu_2),(b_j,\beta_j^1,\beta_j^2)_{1,q},:\ (1-\rho-\sigma-(\zeta-1)k-u;\nu_1-\mu_1,\nu_2-\mu_2),$$

$$\left.\begin{matrix}\left(c_j^1,\gamma_j^1\right)_{1,p_1};\left(c_j^2,\gamma_j^2\right)_{1,p_2}\\ \left(d_j^1,\delta_j^1\right)_{1,q_1};\left(d_j^2,\delta_j^2\right)_{1,q_2}\end{matrix}\right] \quad (22)$$

where,

$$f(x)=\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{a^k}{k!}$$

IV. Conclusion

The H-function of several complex variables, presented in this paper, is quite basic in nature. Therefore, on specializing the parameters of this function, we may obtain various other special functions such as Fox's H-function, Meijer's G-function, Wright's generalized Bessel function, Wright's generalized hypergeometric function, Mac-Robert's E-function, generalized hypergeometric function, Bessel function of first kind, modified Bessel function, Whittaker function, exponential function, binomial function etc. as its special cases, and therefore, various unified integral presentations can be obtained as special cases of our results.

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