Extension Of Fixed-Point Theorems In Extended B-Metric Spaces

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Abstract- In our paper, motivated by the extended b-metric space's concept, we prove two main results regarding fixed point theorems in extended b-metric spaces which holds true for both linear and non-linear contraction mappings. H. Aydi and S. Czerwik worked and showed that the results in generalized b-metric spaces hold true for Linear-Quasi and Non-Linear Contractions. Here in this paper we prove that the work of H. Aydi and S. Czerwik can further be applied to the extended b-metrics spaces Our results may further generalize some pre-existing results like quasi (s, r)-contractive multi-valued operators.

Index Terms- contraction mappings, fixed point theorems, extended b-metric space.

DOI: 10.29322/IJSRP.11.01.2021.p10914
http://dx.doi.org/10.29322/IJSRP.11.01.2021.p10914

1. INTRODUCTION

Bourbaki [1] and Bakhtin [2] initiated working on the concept of b-metric spaces. Czerwik [3], was the first to define the concept of b metric space formally while studying Banach theorems. Later on, Kamran T, and Samreen, M [4] worked on the triangular inequality used in the b-metric space and provide some sort of relaxation in the inequality and defined a new distance function, extended b-metric space [4] while working on the study of Fagin et al [5]. H. Aydi [6] and S. Czerwik [3] used similar type of triangular inequality for linear and non-linear contractions [7]. Here in this paper, we prove that the work of H. Aydi and S. Czerwik can further be enhanced to the extended b-metrics spaces. All these efforts enthralled and enchanted us to extend the idea of extended b-metric space into these results, linear and non-linear contractions.

2. PRELIMINARIES

Definition 2.1 Let $G \neq \emptyset$ set and $\delta: G \times G \longrightarrow \mathbb{R}^+$ be a distance function. Then we say that the pair (G, η) is a metric space if it satisfies:

$(m.1)$ $\delta(\eta_{1,}\eta_{2,}) = 0 \; iff \; \eta_1 = \eta_{2,}$

$(m.2)$ $\delta(\eta_{1,}\eta_{2,}) = \delta(\eta_{2\,,}\eta_1),$

$(m.3)$ $\delta(\eta_{1,}\eta_3) \leq \delta(\eta_{1,}\eta_{2,}) + \delta(\eta_{2\,,}\eta_3).$

for all $\eta_1,\eta_2,\eta_3 \ldots \in G$.

Definition 2.2 Consider a mapping $\Psi: G \longrightarrow G$ and for some arbitrary $\xi \in G$. Then if the image Ψ_ξ coincide with ξ i.e. $\Psi_\xi = \xi$, we say that $\xi \in G$ is a fixed point ([16], [17]) of Ψ.

Definition 2.3 Suppose (G, η) be a metric space. Then the mapping $\Psi: G \longrightarrow G$ is called a contraction [16] on G if there exist $\lambda \in [0, 1)$ with $\lambda < 1$ and for all $\eta_{1,}\eta_{2,} \in G$ we have:

$$\delta(\Psi_{\eta_{1,}}\Psi_{\eta_2}) \leq \lambda\delta(\eta_1, \eta_2).$$

Definition 2.4 Let (G, η) be a non-empty complete metric space with a contraction mapping $\Psi: G \longrightarrow G$. Then Ψ preserve a unique fixed point [17] $\xi \in G$ in G i.e. $\Psi_\xi = \xi$ and can be found by taking an arbitrary $\xi \in G$ followed by a sequence $\{\xi_j\}$ by $\xi_j = \{\Psi_{\xi_{j-1}}\}$ for $j \geq 1$.

If $\eta(\xi_j, \xi) \longrightarrow 0$ as $j \longrightarrow \infty$ then $\xi_j \longrightarrow \xi$.

Remark 2.1 The speed of convergence [17] can be described from the following inequalities:

$$\eta(\xi_j, \xi) \leq \frac{\mu^j}{1-\mu}\eta(\xi_0, \xi_1)$$

$$\eta(\xi_{j+1}, \xi) \leq \frac{\mu}{1-\mu}\eta(\xi_j, \xi_{j+1})$$

$$\eta(\xi_{j+1}, \xi) \leq \mu[\eta(\xi_j, \xi)]$$

Any value of μ satisfying the above inequality is called a Lipschitz constant [16] for Ψ and among these smallest values the smallest one is called "the best Lipschitz constant" for Ψ.

Definition 2.5 Let $\mu > 1$ and define ν by

$$\frac{1}{\mu} - \frac{1}{\nu} = 1$$

μ and ν are then called conjugate exponents.

Bakhtin [2] and Czerwik [3] introduced the concept of "b-metric space" as:

Definition 2.6 Let $G \neq \emptyset$ and $\lambda \in \mathbb{R}$ be fixed such that $\lambda \geq 1$. The elements of G are denoted by $\eta_1,\eta_2,\eta_3 \ldots$ and suppose that a distance function $\delta(\eta_{1,}\eta_{2,})$ where $0 \leq \delta(\eta_{1,}\eta_{2,}) < \infty$ is defined on G×G, the cartesian product of G. Then we say that the pair (G, δ) is a generalized b-metric space having generalized b-metric δ satisfying:

$(b.1)$ $\delta(\eta_{1,}\eta_{2,}) = 0 \; iff \; \eta_1 = \eta_{2,}$

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$(b.2)$ $\delta(\eta_1,\eta_2) = \delta(\eta_2,\eta_1)$,
$(b.3)$ $\delta(\eta_1,\eta_3) \le \lambda[\delta(\eta_1,\eta_2) + \delta(\eta_2,\eta_3)]$.
for all $\eta_1,\eta_2,\eta_3 \ldots \in G$.

Example 2.1 Consider the space $G = L_p[0,1]$ of all real-valued functions $\alpha(s), 0 \le s \le 1$ s.t

$$\int_0^1 |\alpha(s) - \beta(s)|^p < \infty,$$

where $p \in (0,1)$. Define a distance function $B_m: G \times G \longrightarrow \mathbb{R}^+$ on G as:

$$B_m(g_1, g_2) = (\int_0^1 |\alpha(s) - \beta(s)|^p ds)^{1/p} < \infty$$

The distance function B_m together with G, (G, B_m), is a b-metric space [3], [8], [9] having co-efficient $\lambda = 2^{1/p}$.

Example 2.2 On $= L_p(\mathbb{R})$, $p \in (0,1)$ and $L_p(\mathbb{R}) = \{(\xi_j) \subset \mathbb{R} : \sum_{j=1}^{\infty} |\xi_j|^p < \infty\}$, define a distance function $B_m: G \times G \longrightarrow \mathbb{R}^+$ as:

$$B_m(g_1, g_2) = (\sum_{j=1}^{\infty} |\xi_j - \eta_j|)^p)^{1/p}$$

Where $g_1 = (\xi_j)$ and $g_2 = (\eta_j)$, then(G, B_m) is a b-metric space [3], [8] and [9] having coefficient $\lambda = 2^{1/p}$.
The given examples reveal that class of b-metric spaces are much greater than that of metric spaces. For λ=1, idea of b-metric spaces overlaps with that of metric spaces. For more results see [8], [9] and [10].

Definition 2.2 Suppose $G \neq \emptyset$ set and $\varphi: G \times G \longrightarrow [1, \infty)$ and a distance mapping $B_\varphi: G \times G \longrightarrow [0, \infty)$ satisfies:
($B_\varphi.1$**)** $B_\varphi(\eta_1,\eta_2) = 0 \; iff \; \eta_1 = \eta_2$
($B_\varphi.2$**)** $B_\varphi(\eta_1,\eta_2) = B_\varphi(\eta_1,\eta_2)$
($B_\varphi.3$**)** $B_\varphi(\eta_1,\eta_3) \le \varphi(\eta_1,\eta_2)[B_\varphi(\eta_1,\eta_2) + B_\varphi(\eta_2,\eta_3)]$.
$\forall \eta_1,\eta_2,\eta_3 \in G$.
The distance function B_m together with the set G, (G, B_m), is known as extended b-metric space [4] abbreviated as EBM space later on.

Remark 2.1 If we put $\varphi(g_1, g_2) = \lambda$ for $\lambda \ge 1$, then definition 2.2 coincide with definition 2.1.

Example 2.3 Consider the space $S = C([a,b], \mathbb{R})$ of real-valued functions which are continuous over $I = [a,b]$. By defining the distance function $B_\varphi: S \times S \longrightarrow \mathbb{R}^+$ as:

$$B_\varphi(\eta_1, \eta_2) = sup_{\xi \in [a,b]} |\eta_1(\xi) - \eta_2(\xi)|^2$$

Then S is a complete extended b-met space [4] having following metric:

$$\varphi(\eta_1, \eta_2) = |\eta_1(\xi) - \eta_2(\xi)| + 2 \text{ and } \varphi: S \times S \longrightarrow [1, \infty).$$

Definition 2.3 Consider the EBM space (G, B_m). The sequence $\{\xi_j\}$ in G is:
$(\boldsymbol{i})$ B_m −convergent [11] if and only if for some $\xi \in G$ we must have
$B_m(\xi_j, \xi) \longrightarrow 0$ while $j \longrightarrow \infty$.
In this respect, we then write $\lim_{n \to \infty} \xi_j = \xi$;
$(\boldsymbol{ii})$ B_m −Cauchy [11] if and only if $B_m(\xi_j, \xi_k) \longrightarrow 0$ as $j, k \longrightarrow \infty$;
$(\boldsymbol{iii})$ The pair (G, B_m) is said to be complete [11] if and only if every Cauchy sequence is B_m −convergent in G.
As customary, the symbols $\mathbb{N}$, $\mathbb{Z}^+$and $\mathbb{R}^+$ or by $[0, \infty)$ define their usual meanings of mathematics.

The nth-iteration of a function $\Psi; G \longrightarrow G$ is denoted by Ψ^n with $\Psi(\xi) = \Psi_\xi$ i.e. $\Psi^0(\xi) = \xi, \ldots, \Psi^n = \Psi o \Psi^n$ and $\Psi(\varphi) = \Psi o \varphi$ with usual meanings.

3. MAIN RESULTS

In this part of our main result, we establish fixed-point theorems for Linear Quasi and non-linear contractions defined on extended b-metric spaces.

3.1 LINEAR QUASI-CONTRACTIONS

We start the result with the following theorem:

Theorem 3.1.1 Consider a non-empty complete extended b-metric space (G, B_m). Suppose $\Psi: G \longrightarrow G$ be continuous and satisfies the condition:

$$B_m(\Psi_g, \Psi_g{}^2) \le \alpha B_m(g, \Psi_g)$$

(1)
for $g \in G$, such that $B_m(g, \Psi_g) < \infty$ and

$$\alpha\lambda = r < 1$$

Further suppose that $g \in G$ be an arbitrary and fixed, then the following alternatives hold i.e.
(A) $\forall i \in \mathbb{Z}^+$, $B_m(\Psi_g{}^i, \Psi_g{}^{i+1}) = \infty$ or
(B) $\exists j \in \mathbb{Z}^+$ such that $B_m(\Psi_g{}^j, \Psi_g{}^{j+1}) < \infty$. Further in (B), we have
$(\boldsymbol{i})$ the sequence $\{\Psi_g{}^k\}$ is Cauchy in G.
$(\boldsymbol{ii})$ there exists ω ∈ G such that $\sum_{k \to \infty} B_m(\Psi_g{}^k, \omega) = 0$ and $\Psi_g = \omega$.

Proof. Consider the case (B); then from (1), we have,

$$B_m(\Psi_g{}^{j+1}, \Psi_g{}^{j+2}) \le \alpha B_m(\Psi_g{}^j, \Psi_g{}^{j+1}) < \infty$$

And by induction,

$$B_m(\Psi_g{}^{j+i}, \Psi_g{}^{j+i+1}) \le \alpha^i B_m(\Psi_g{}^j, \Psi_g{}^{j+1}), \tag{2}$$

for $i = 0,1,2, \ldots$
Thus, for non-negative integers $i, n \in \mathbb{N}$, (2) gives,

$$\begin{aligned} B_m(\Psi_g{}^{j+i}, \Psi_g{}^{j+i+n}) &\le \alpha B_m(\Psi_g{}^{j+i}, \Psi_g{}^{j+i+1}) + \cdots + \\ &+\alpha^{n-1} B_m(\Psi_g{}^{j+i+n-2}, \Psi_g{}^{j+i+n-1}) + \\ &+\alpha^n B_m(\Psi_g{}^{j+i+n-1}, \Psi_g{}^{j+i+n}) \\ &\le \lambda\alpha^i B_m(\Psi_g{}^j, \Psi_g{}^{j+1}) + \cdots + \\ &+\lambda^{n-1}\alpha^{i+n-2} B_m(\Psi_g{}^j, \Psi_g{}^{j+1}) + \\ &+\lambda^n \alpha^{i+n-1} B_m(\Psi_g{}^j, \Psi_g{}^{j+1}) \\ &\le \lambda\alpha^i [1 + (\lambda\alpha) + (\lambda\alpha)^2 + \cdots + (\lambda\alpha)^{n-1}] B_m(\Psi_g{}^j, \Psi_g{}^{j+1}) \\ &\le \lambda\alpha^i \textstyle\sum_{k=0}^{\infty} (\lambda\alpha)^m \, B_m(\Psi_g{}^j, \Psi_g{}^{j+1}) \\ &\le \frac{\lambda\alpha^i}{1 - \lambda\alpha} B_m(\Psi_g{}^j, \Psi_g{}^{j+1}) \end{aligned}$$

Thus,

$$B_m(\Psi_g{}^{j+i}, \Psi_g{}^{j+i+n}) \le \frac{\lambda\alpha^i}{1-\lambda\alpha} B_m(\Psi_g{}^j, \Psi_g{}^{j+1}) \tag{3}$$

for non-negative integers $i, n \in \mathbb{N}$.
Inequality (3) shows $\{\Psi_g{}^j\}$ be a Cauchy sequence in G.
As G be complete there must exists ω ∈ G such that $\lim_{i \to \infty} B_m(\Psi_g{}^i, \omega) = 0$. Since Ψ is assumed to be continuous w.r.t B_m, therefore:

$$\omega = \Psi_g{}^{i+1} = \Psi\left(\lim_{i\to\infty}\Psi_g{}^{i}\right) = \Psi_g$$

Showing that Ψ has ω as fixed point, that concludes our first result.

Remark 3.1.1 Ψ may have more than one fixed point in G. Else take $\Psi_g = g$.

Remark 3.1.2 As a function of one variable, the mapping B_m need not to be continuous [12] and [13] in case if it is, then the inequality (3) gives the following estimation

$$B_m(\Psi_g{}^k, \omega) \leq \frac{\lambda\alpha^i}{1-\lambda\alpha} B_m(\Psi_g{}^j, \Psi_g{}^{j+1})$$

Remark 3.1.3 An operator Ψ satisfying inequality (1) not necessarily be continuous [10]. In fact, if Ψ satisfies

$$B_m\left(\Psi_{g_1}, \Psi_{g_2}\right) \leq \frac{\alpha}{2}\left[B_m\left(g_1, \Psi_{g_1}\right) + B_m\left(g_2, \Psi_{g_2}\right)\right]$$

$g_1, g_2 \in G$.

then Ψ satisfies (1) as well.

Remark 3.1.4 The above theorem 2.1 generalize many results including Luxemburg [14], Diaz [13], Margolis [11], Czerwik, S., & Król, K [3].

3.2 NON-LINEAR CONTRACTIONS

In this part of the paper, we conclude the second theorem of our main results.

Theorem 3.2.1. Consider the complete extended b-metric space (G, B_m). Assume $\Psi: G \longrightarrow G$ be continuous and satisfies

$$B_m\left(\Psi_{g_1}, \Psi_{g_2}\right) \leq \varphi[B_m(g_1, g_2)]$$

(4)

for $g_1, g_2 \in G$ and

$$B_m(g_1, g_2) < \infty$$

with a non-decreasing function $\Omega: [0,\infty) \longrightarrow [0,\infty)$ such that

$$\lim_{i\to\infty}\Omega^i(\rho) = 0, \text{ for } \rho > 0$$

Let for an arbitrary but fixed $g \in G$, the following alternatives hold i.e. either

(C) for each non-negative integer $i \in \mathbb{N}_0$

$$B_m(\Psi_g{}^i, \Psi_g{}^{i+1}) = \infty:$$

(D) $\exists\, j \in \mathbb{Z}^+$ such that $B_m(\Psi_g{}^j, \Psi_g{}^{j+1}) < \infty$. Further in (D), we have

$(\boldsymbol{iii})$ the sequence $\{\Psi_g{}^k\}$ is Cauchy in G.

$(\boldsymbol{iv})$ there exists ω ∈ G such that $\lim_{i\to\infty} B_m(\Psi_\xi{}^i, \omega) = 0$ and $\Psi_\omega = \omega$.

$(\boldsymbol{v})$ ω is a unique fixed point of Ψ in the bounded set

$$S = \{\xi \in G: B_m(\Psi_g{}^i, \xi) < \infty,$$

$(\boldsymbol{vi})$ for each $\xi \in S$

$$\lim_{i\to\infty} B_m(\Psi_\xi{}^i, \omega) = 0.$$

Further if B_m is continuous w.r.t one variable and the infinite series sum $\sum_{j=1}^{\infty}\lambda^n\Omega_\xi{}^j < \infty$ for $\xi > 0$. Then for arbitrary $\xi \in S$ implies

$$B_m(\Psi_\xi{}^k, \omega) \leq \sum_{j=0}^{\infty}\lambda^{j+1}\Omega_\xi{}^{j+k}[B_m(\Omega_\xi, \xi)], \text{ for}$$

some $k \in \mathbb{N}$.

Proof: For arbitrary $g \in G$ and $\epsilon > 0$ we choose $m \in \mathbb{N}$ is such that $\Omega^m{}_\epsilon < \frac{\epsilon}{2\lambda}$. Put $T = \Psi^m, \alpha = \Omega^m$ and $g_n = T_g{}^n$ for $m \in \mathbb{N}$. Then for each $g_1, g_2 \in G$ with $B_m(g_1, g_2) < \infty$, we obtain

$$B_m(T_{g_1}, T_{g_2}) \leq \Omega^n[B_m(g_1, g_2)] = \alpha[B_m(g_1, g_2)]$$

(5)

As $\Psi_g{}^j, \Psi_g{}^{j+1} \in S$, so (X, B_m) be a complete EBM space. Considering $\Psi: S \longrightarrow S$ and $\xi \in S$ then from (v) $B_m(\Psi_g{}^i, \xi) < \infty$,so

$$B_m(\Psi_g{}^i, \Psi_\xi) \leq \lambda[B_m(\Psi_g{}^i, \Psi_g{}^{i+1}) + B_m(\Psi_g{}^{i+1}, \Psi_\xi)]$$
$$\leq \lambda[\epsilon_1 + \Omega\{B_m(\Psi_g{}^i, \xi)\}]$$
$$B_m(\Psi_g{}^i, \Psi_\xi) \leq \lambda[\epsilon_1 + \epsilon_2] < \infty,$$

for $\epsilon_1, \epsilon_2 > 0$.

Now consider $T: S \longrightarrow S$. For arbitrary $\xi \in S, \{T_\xi{}^i\} \subset S$ for $i \in \mathbb{Z}^+$. We want to prove that $\{T_\xi{}^i\}$ is Cauchy in S. Put $g_i = T_\xi{}^i$, we obtain:

$$B_m(T_\xi, T_\xi{}^2) \leq \alpha[B_m(T_\xi, \xi)]$$

And successively induction yields

$$B_m(T_\xi{}^i, T_\xi{}^{i+1}) \leq \alpha^i[B_m(T_\xi, \xi)]$$

Or $\quad B_m(g_i, g_{i+1}) \leq \alpha^i[(T_\xi, \xi)]$

Which implies $B_m(g_i, g_{i+1}) \longrightarrow 0$ as $i \longrightarrow \infty$.

Set i as such that $B_m(g_i, g_{i+1}) < \frac{\epsilon}{2\lambda}$.

Then for arbitrary $\theta \in U(g_i, \epsilon) = \{g \in G: B_m(g_i, g) \leq \varepsilon\}$, we get

$$B_m(T_\theta, T_{g_i}) \leq \alpha\,[B_m(\theta, g_i)] \leq \alpha_\varepsilon < \Omega^m < \frac{\epsilon}{2\lambda}$$
$$B_m(T_{g_i}, g_i) = B_m(g_{i+1}, g_i) < \frac{\epsilon}{2\lambda}$$

Hence

$$B_m(T_\theta, T_{g_i}) < \lambda\left(\frac{\epsilon}{2\lambda} = \frac{\epsilon}{2\lambda}\right) = \varepsilon$$
$$B_m(T_\theta, T_{g_i}) < \varepsilon$$

Which shows that T maps $U(g_i, \epsilon)$ into itself. Thus

$$B_m(g_j, g_k) < 2\lambda\varepsilon,$$

for $j, k \geq i$

Hence, the sequence $\{T_\xi{}^i\} \in S$ and it is Cauchy. S being complete, it must has an element ω such that $\omega \in S \subset G$ $\ni g_j \longrightarrow \omega$ as $j \longrightarrow \infty$. Also (4) implies the continuity of T, so we have

$$T_\omega = \lim_{j\to\infty} T_{g_j} = \lim_{j\to\infty} g_{j+1} = \omega$$

Showing that T has ω as a fixed point. The uniqueness of ω in S can be seen if we consider $\alpha_\xi = \Omega_\xi{}^i$ for arbitrary $\xi > 0$.

Furthermore, the inequality (5) shows that Ψ is continuous on S, so

$$\Psi_\omega = \lim_{j\to\infty} T_\xi{}^j = \lim_{j\to\infty} T^j(\Psi_\omega) = \omega \qquad \text{(A)}$$

.Which also shows ω a fixed point of Ψ. As (7) implies that this fixed point is unique in S, so combining (4) and the above equality (A) we finally have for each $\xi \in S$ and each $j = 0, 1, 2, \ldots i-1$

$$\Psi_\xi{}^k = \Psi_\xi{}^{sj+t} = T^s(\Psi_\xi{}^t) \longrightarrow \omega \text{ as } s \longrightarrow \infty$$

Thus

$$B_m(\Psi_\xi{}^k, \omega) \longrightarrow 0 \text{ as } k \longrightarrow \infty$$

Which shows that for each $\in S$, we get (vi).

And thus, at last for arbitrary $\xi \in S$ and $i, j \in \mathbb{Z}^+$ we get

$$B_m\left({\Psi_\xi}^k, {\Psi_\xi}^{k+i}\right) \leq \lambda[B_m\left({\Psi_\xi}^k, {\Psi_\xi}^{k+1}\right) + B_m\left({\Psi_\xi}^{k+1}, {\Psi_\xi}^{k+i}\right)]$$
$$\leq \lambda[B_m\left({\Psi_\xi}^k, {\Psi_\xi}^{k+1}\right)] + \cdots + \lambda^i[B_m\left({\Psi_\xi}^{k+i-1}, {\Psi_\xi}^{k+i}\right)]$$
$$\leq \lambda\Omega^i[B_m\left(\Psi_\xi, \xi\right)] + \cdots$$
$$+ \lambda^i\Omega^{k+i-1}[B_m\left(\Psi_\xi, \xi\right)]B_m\left({\Psi_\xi}^k, {\Psi_\xi}^{k+i}\right)$$
$$\leq \sum_{t=0}^{\infty} \lambda^{t+1}\Omega^{k+t}\,[B_m\left(\Psi_\xi, \xi\right)]$$

Thus, if B_m is continuous and for arbitrary $\xi \in S$, $k, t \in \mathbb{Z}^+$ and $i \longrightarrow \infty$, then we have

$$B_m\left({\Psi_\xi}^k, \omega\right) \leq \sum_{t=0}^{\infty} \lambda^{t+1}\Omega^{k+t}\,[B_m\left(\Psi_\xi, \xi\right)]$$

That winds up our proof.

If G is a b-metric space and $S = G$ then the Theorem 2.1 implies:

Corollary 3.2.1 If $(G,\, B_m)$ be a complete extended b-metric space and $\Psi: G \longrightarrow G$ satisfies

$$B_m\left(\Psi_{g_1}, \Psi_{g_2}\right) \leq \varphi[B_m(g_1, g_2)]$$

$g_1, g_2 \in G$.

Where $\Omega: [0,\infty) \longrightarrow [0,\infty)$ is such that:

$(\boldsymbol{i})$ Ω is a non-decreasing

$(\boldsymbol{ii}$ $\lim_{i\to\infty} \Omega^i(\xi) = 0$, for $\xi > 0$

Then Ψ has unique fixed point $\omega \in G$ with $\lim_{i\to\infty} B_m\left({\Psi_\xi}^i, \omega\right) = 0 \;\forall\, g \in G$.

Remark 3.2.1 For more details about corollary see [12] and [15].

3. ACKNOWLEDGMENT

We wish especial thanks to Dr. Tayyab Kamran and Hassen Ayedi, University of Dammam Saudi Arabia for their guidance and support and at last but not least the authors are also grateful to anonymous referee/referees for giving him/their valuable time to read our paper.

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