Enumerative Techniques for Bishop Polynomials Generated by a θ^o Board With an $m \times n$ Array

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DOI: 10.29322/IJSRP.9.01.2019.p8599
http://dx.doi.org/10.29322/IJSRP.9.01.2019.p8599

Abstract: *The bishop polynomial on a board rotated in an angle of 45^o is considered a special case of the rook polynomial. Rook polynomials are a powerful tool in the theory of restricted permutations. It is known that the rook polynomial of any board can be computed recursively, using a cell decomposition technique of Riordan. This independent study examines counting problems of non-attacking bishop placements in the game of chess and its movements in the direction of $\theta = 45^o$ to capture pieces in the same direction as the bishop with restricted positions. In this investigation, we developed the total number of ways to arrange n bishops among m positions $(m \geq n)$ and also constructed the general formula of a generating function for bishop polynomial that decomposes into n disjoint sub-boards $B_1, B_2, \ldots B_n$ by using an $m \times n$ array board. Furthermore, we applied it to combinatorial problems which involve permutation with forbidden positions to construct bishop polynomials in a combinatorial way.*

Key words*: r-arrangement, combinatorial structures, Chess movements; Permutation; Arrangements with restrictions; Rook polynomials.*

1.0 Introduction

The rook polynomial is a powerful tool in the theory of restricted permutations (Abigail, 2004). However, in comparison with the bishop polynomial, a special case of the rook polynomial, has not been established (Skoch, 2015). Furthermore, Bishop (Rook) polynomials provide a method of enumerating permutations with forbidden position. Kaplansky and Riordan in 1946 started this study, with applications to card matching problems. Riordan, in his book is considered the first systematic analysis, and remains a classic treatment of the subject (Riordan, 1958). The series of papers by (Joichi, Goldman, & White, 1978; Goldman, Joichi, & White, 1977; Goldman, Joichi, Reiner, & White, 1976; White, Goldman, & Joichi, 1975) have expanded the field by applying more advanced combinatorial methods. More recently, (Laisin, 2018; Michaels, 2013) and (Ono, Haglund, & Sze, 1998; Haglund, 1996)also made investigations into various connections of rook polynomials to other parts of mathematics: hypergeometric series, enumeration of matrices over finite fields, and group representation theory. Furthermore, rook polynomials are also closely related to matching theory, chromatic theory and various other graph-theoretic topics (Chung & Graham, 1995; Farrell & Whitehead, 1991; Goldman., Joichi, Reiner, & White, 1976). In combinatorics proper, rook polynomials have been related to various permutation statistics (Butler, 1985) and the inverse problem has been solved for Ferrers boards in Mitchell, Preprint. It has long been known that the rook polynomial of any board can be computed recursively.

Nevertheless, in combinatorial mathematics, a bishop polynomial is a generating polynomial of the number of ways to place non-attacking bishops on a board that looks like a checkerboard; that is, no two bishops may be in the same diagonal. The board is any subset of the squares of an $m \times n$ rectangular board with m rows and n columns; we think of it as the squares in which one is allowed to put a bishop. Even though, Combinatorics is a young field of mathematics, starting to be an independent branch only in the 20th century. However, combinatorial methods and problems have been around ever since. Many combinatorial problems look entertaining or aesthetically pleasing and indeed one can say that roots of combinatorics lie in mathematical recreations and games. Nonetheless, this field has grown to be of great importance in today's world, because of its use for other fields like physical sciences, social sciences, biological sciences, information theory and computer science (Michaels, 2013; Berge, 1971).

The game of chess is an amazingly complicated game with a seemingly infinite number of scenarios. The rules that govern the game of chess have proven to be an attractive area of inquiry to mathematicians the world over. The game is played on an 8×8 checkered board and two players take turns moving their pieces around the board. The objective is to "checkmate" the opponent's king, which means the king can be attacked on the next turn, and no matter how the opponent moves his pieces, there is no way to prevent the king from being attacked. The two players, each starting with 8 pawns, 2 rooks, 2 knights, 2 bishops, 1 queen, and 1 king. These pieces differ only in the way they are allowed to move around the board and "attack" other pieces. For example, rooks can move and attack as many squares that are unoccupied along its row or column. Bishops can move and attack only along diagonals for as many squares that are unoccupied (Skoch, 2015). However, based on the rook movement, the theory of rook polynomials was introduced by Kaplansky and Riordan, and developed further by Riordan. The term "rook polynomial" was coined by John Riordan (Riordan, 1980). Despite the name's derivation from chess, the impetus for studying rook polynomials is their connection with counting with forbidden positions. The rook polynomial is a generating polynomial of the number of ways to place non-attacking rooks on a board that looks like a checkerboard; that is, no two rooks may be in the same row or column. The rook polynomial $R(x, B)$ (Laisin, 2018; Barbeau, 2003) of a board B is the generating function for the numbers of arrangements of non-attacking rooks, i.e.:

$$R(x, B) = \sum_{k=0}^{\infty} r_k(B)x^k$$

given that, r_k is the number of ways to place k non-attacking rooks on the board B.

Unlike the rook that moves in vertical rows and horizontal columns, the bishops move diagonally along the black or white squares on a chess board. Therefore, the bishop tends to capture pieces up or down along the diagonal it is placed, meaning that any piece along that diagonal can be captured by the bishop. To have non-attacking bishops on a board means that no two bishops must be on the same diagonal. To solve the problem of bishop placement with non-attacking bishops on an $m \times n$ board, the need to use the generating function of the bishop polynomial come into play.

If B is a board of size $m \times n$. The bishop polynomial of board B is denoted as $\mathfrak{B}(x, B)$ for the number of ways to place n non-attacking bishops on the board. The other pieces follow different rules for movement, but our study will focus on the strength of a bishop movements.

2.0 Preliminaries

Definition 2.1

Rook: A rook is a chess piece that moves horizontally or vertically and can take (or capture) a piece if that piece rests on a square in the same row or column while a bishop is a chess piece that moves diagonally and capture a piece if that piece rests on a square in the same diagonal (Chung & Graham, 1995; Goldman., Joichi, Reiner, & White, 1976; White, Goldman, & Joichi, 1975).

a. Board: A board B is an $m \times n$ array of n rows and m columns. When a board has a darkened square, it is said to have a forbidden position.
b. Bishop polynomial: A bishop polynomial on a board B, with forbidden positions is denoted as $\mathfrak{B}(x, B)$, given by

$$\mathfrak{B}(x, B) = \sum_{i=1}^{k} b_i(B)x^i,$$

where $\mathfrak{B}(y, B)$ has coefficients $b_i(B)$ representing the number of ways to place n non-attacking bishops on the board B. However, on an $m \times n$ board B, we have $b_0(B) = 1$ and the bishop polynomial with explicit coefficients is now:

$$\mathfrak{B}(x, B) = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} k!\, x^k = \sum_{k=0}^{\min(m,n)} \frac{n!\, m!}{k!\,(n-k)!\,(m-k)!} x^k.$$

Nevertheless, with the limitation that, " bishops must not attack each other" is removed, in this case one must choose any k squares from $m \times n$ arrays. Then, we have;

$$\binom{mn}{k} = \frac{(mn)!}{k!(mn-k)!} \text{ ways.}$$

Suppose $m \neq n$, then, the k bishops will differ in some way from each other, however, the results obtained will be multiplied by $k!$, for the k bishops. Then, we have;

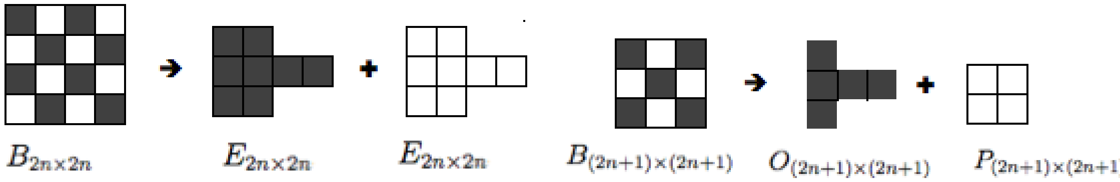
$$\binom{m}{k}\binom{n}{k} k! = \frac{n!m!}{k!(n-k)!(m-k)!} \text{ (Vilenkin, 1969).}$$

Definition 2.2 Square odd boards,

$B_{(2n+1)\times(2n+1)}$ can be broken down into two sub-boards: one with even number of cells – denoted as P board or $P_{(2n+1)\times(2n+1)}$- and the other with an odd number of cells – denoted as O board or $O_{(2n+1)\times(2n+1)}$.

Bishop polynomial of the P and O boards are expressed as $P_{(2n+1)\times(2n+1)}(x)$ and $O_{(2n+1)\times(2n+1)}(x)$

Thus, $B_{(2n+1)\times(2n+1)}(x) = P_{(2n+1)\times(2n+1)}(x) O_{(2n+1)\times(2n+1)}(x)$



Decomposition of $B_{(2n)\times(2n)}$ and $B_{(2n+1)\times(2n+1)}$ (Abigail, 2004; Shanaz , 1999).

Definition 2.3

P board coefficient is given by;

$$(b_k(P_{(2n-1)\times(2n-1)}(\mathrm{x})) = b_k(P_{(2n-3)\times(2n-3)}(\mathrm{x})) + 2(2\mathrm{n}-\mathrm{k}-1)\times b_{k-1}(P_{(2n-3)\times(2n-3)}(\mathrm{x})) + (2\mathrm{n}-\mathrm{k})(2\mathrm{n}-\mathrm{k}-1)\, b_{k-2}(P_{(2n-3)\times(2n-3)}(\mathrm{x}))$$

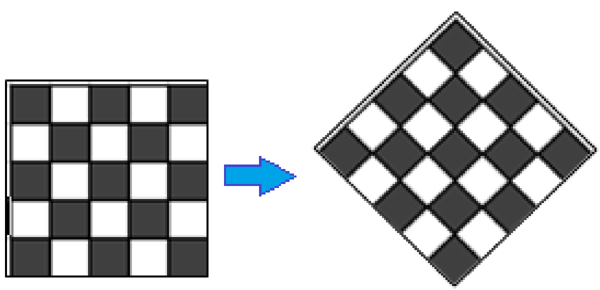
where $n \geq 3;\ 2 \leq k \leq 2n-3$ and $b_1(P_{1\times1}) = 1;\ b_1(P_{1\times1}) = 4; b_2(P_{3\times3}) = 2$. Note for all $\mathrm{k} > 2\mathrm{n}-2;\ b_k(P_{(2n-1)\times(2n-1)}(\mathrm{x})) = 0$ (Abigail, 2004; Shanaz , 1999)

2.2 The Bishop Polynomial

A bishop moves only diagonally without restriction in the distance of each move. Here, the bishop polynomial is the generating function of the number of arrangements of k non-attacking bishops (k-bishop placement) on a $m \times n$ board:

$$\mathfrak{B}(x,\ B) = \sum_{i=0}^{k} b_i(B)x^i$$

where b_k denotes the k^{th} coefficient of the bishop polynomial. While bishops move in diagonal rooks move in straight lines, they differ in the direction of movement. However, the movement of the two can be related through a 45^0 rotation of the board. Tracing out the path of a bishop after a 45^0 rotation, gives the path of a rook piece. In the Figure below, the bishop polynomial of board A is the rook polynomial of board B.



A B

Tilting board 45^0 converts bishop move (in A) to a rook one (in B). Given that a rook moves only vertically or horizontally, it can only occupy squares of the same colour (in board B above). Thus, the white board (consisting of white cells) and black board (consisting of black cells) are disjoint sub-boards of the overall board B. In particular, for even boards $B_{m\times n}$, note that $R(B_{white}) = R(B_{black})$. Therefore, $R(B) = R(B_{white}) \times R(B_{black})$ for all square board B (Skoch, 2015; Abigail, 2004).

Lemma 2.1 Forbidden Positions

The number of ways to arrange n-rooks among m-positions ($m \geq n$) such that order is maintained, is given by;

Case i. when $m \neq n$

$$r(x,B) = 1 - \frac{r_1(B)\binom{m-1}{n-i}}{\binom{m}{n}} + \frac{r_2(B)\binom{m-2}{n-i}}{\binom{m}{n}} - \cdots (-1)^i \frac{r_i(B)\binom{m-i}{n-i}}{\binom{m}{n}}$$

$$= \frac{1}{\binom{m}{n}} \sum_{i=0}^{n} (-1)^i r_i(B) \binom{m-i}{n-i}$$

Case ii. when $n = n$.

$$r(x,B) = 1 - \frac{r_1(B)\binom{n-1}{n-1}}{\binom{n}{n}} + \frac{r_2(B)\binom{n-2}{n-2}}{\binom{n}{n}} - \cdots (-1)^i \frac{r_i(B)\binom{n-i}{n-i}}{\binom{n}{n}}$$

$$= \frac{1}{\binom{n}{n}} \sum_{i=0}^{n} (-1)^i r_i(B) \binom{n-i}{n-i}$$ (Skoch, 2015).

Lemma 2.2

If B is a board of darkened squares that decomposes into two disjoint sub-boards $B_i : i = 1 \; and \; 2$, then $\mathfrak{B}(x,B) = \mathfrak{B}(x,B_1)\mathfrak{B}(x,B_2)$ (White, Goldman, & Joichi, 1975).

3.0 Main Results

Theorem 3.1

The number of ways to arrange n bishops among m positions $(m \geq n)$ through an angle of $\theta = 45^0$ for movement on the board with forbidden positions is;

$$\mathfrak{B}(y,B)P_{(m,\ n)} = \sum_{k=0}^{n} (-1)^k b_k^\theta P_{(m-k,\ n-k)}$$

Proof

The proof of theorem 3.1 follows immediately from Lemma 2.1 in arranging n bishops among m positions $(m \geq n)$ through a direction of movement in an angle of 45^0 with forbidden positions is as follows;

Case 1 $m > n$

$$\mathfrak{B}(y,B)P_{(m,\ n)} = P_{(m,\ n)} - b_1^\theta(B)P_{(m-1,n-1)} + b_2^\theta(B)P_{(m-2,n-2)} - b_3^\theta(B)P_{(m-3,n-3)} + \ldots (-1)^m b_m^\theta(B)P_{(m-n,0)}$$

$$= \sum_{k=0}^{m} (-1)^k b_k^\theta P_{(m-k,n-k)}$$

Case 2 $m = n$

$$\mathfrak{B}(y,B)P_{(n,\ n)} = P_{(n,\ n)} - b_1^\theta(B)P_{(n-1,n-1)} + b_2^\theta(B)P_{(n-2,n-2)} - b_3^\theta(B)P_{(n-3,n-3)} + \ldots (-1)^n b_n^\theta(B)P_{(0,0)}$$

$$= \sum_{k=0}^{n} (-1)^k b_k^\theta(B)P_{(n-k,n-k)} \qquad \blacksquare$$

Theorem 3.2 (n-disjoint sub-boards with movements through an angle of 45^0)

Suppose, B is an $n \times n$ board of darkened squares with bishops that move through a direction of an angle of $\theta = 45^0$ then, $\mathfrak{B}(x,B)$ for the disjoint sub-boards is;

$$\mathfrak{B}(x,B) = \sum_{i=0}^{n} \prod_{k=0}^{n} \mathcal{X}_{B_{j,K}}(x)^i b_i^\theta(B_j), \;\; j = 1,2,\ldots n$$

Proof,

The proof of theorem 3.2 follows immediately from Lemma 2.2 it follows that, the total number of bishops b_i^θ on the board is;

$$\mathfrak{B}(x,B) = \sum_{i=0}^{n} \prod_{k=0}^{n} \mathcal{X}_{B_{j,K}}(x)^i b_i^\theta(B_j), \;\; j = 1,2,\ldots n$$

$$= \mathfrak{B}(x,B_1) \times \mathfrak{B}(x,B_2) \times \ldots \times \mathfrak{B}(x,B_n)$$

The decomposition of the board into n disjoint sub-boards $B_1, B_2, \ldots B_n$, is;

$$\mathfrak{B}(x,B_1) = \sum_{i=0}^{n} (x)^i b_i^\theta(B_1) = 1 + xb_1^\theta(B_1) + x^2 b_2^\theta(B_1) + \cdots + x^n b_n^\theta(B_1)$$

$$\mathfrak{B}(x,B_2) = \sum_{i=0}^{n} (x)^i b_i^\theta(B_2) = 1 + xb_1^\theta(B_2) + x^2 b_2^\theta(B_2) + \cdots + x^n b_n^\theta(B_2)$$

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$$\mathfrak{B}(x,B_n)=\sum_{i=0}^{n}(x)^i b_i^\theta(B_n)=1+xb_1^\theta(B_n)+x^2b_2^\theta(B_n)+\cdots+x^nb_n^\theta(B_n)$$

The nth coefficient of
$\mathfrak{B}(x,B)=\mathfrak{B}(x,B_1)\mathfrak{B}(x,B_2)\ldots\mathfrak{B}(x,B_n)=\ b_0^\theta(B_1)b_n^\theta(B_2)b_{n-1}^\theta(B_3)\ldots b_{n-i+1}^\theta(B_n)+$
$b_1^\theta(B_1)b_{n-2}^\theta(B_2)b_{n-3}^\theta(B_3)\ldots b_{n-i}^\theta(B_n)+\cdots+b_n^\theta(B_1)b_{n-1}^\theta(B_2)b_{n-2}^\theta(B_3)\ldots b_0^\theta(B_n)$

This shows that when there is a non-attacking bishop on B_1 there are *n* bishops on B_2 and *n-1* bishops on B_n. This continues until we have ***n*** bishops on B_1, there are no bishops on B_2 and 1 bishop on B_n. Thus

$$\mathfrak{B}(x,B)=\mathfrak{B}(x,B_1)\mathfrak{B}(x,B_2)\ldots\mathfrak{B}(x,B_n)=\sum_{i=0}^{n}\prod_{k=0}^{n}\chi_{B_{j,K}}(x)^i b_i^\theta(B_j),\ \ j=1,2,\ldots n \quad \blacksquare$$

4.0 NUMERICAL APPLICATIONS

Example 4.1
A general matching service has eight positions; A, B, C, D, E, F, G, and H with eight army officers; *p*, *q*, *r*, *s*, *t*, *u*, *v* and *w*. After analyzing their personalities and strategies he decides that *p* should not be matched with A, B, D, F, and H, *q* should not be matched with B, D, F, G, and H, *r* should not be matched with F, G, and H, *v* should not be matched with E, F and H, *w* should not be matched with E, F, G, and H. Use a bishop polynomial to determine the number of ways in which the general can match his army.

Solution
We make figure 1 as the chessboard representing the permutation with forbidden positions as decided by the general.
Thus, we arrange figure 1 such that no army among the eight positions is non-attacking to each other through a direction of movement in an angle of $\theta=45^0$ with forbidden positions, thus, we have figure 2;
However, figure 2 is the arrangement of eight armies such that no two armies should attack each other. Now, applying the bishop polynomial on the board consists of the forbidden positions we get fig.2.

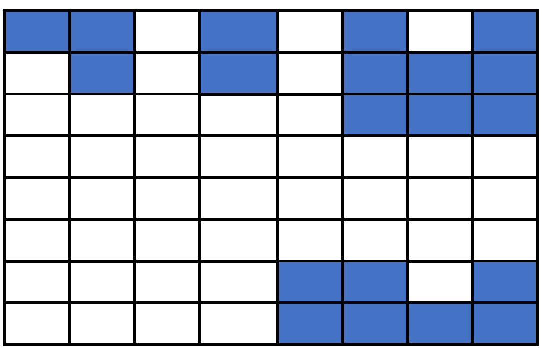


fig.1

⇒

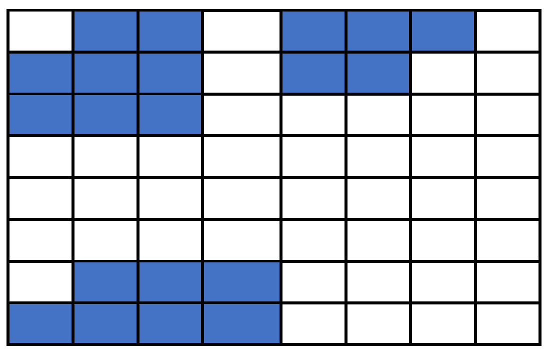
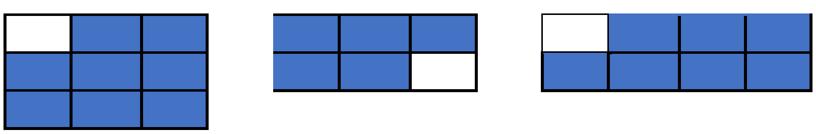


fig.2

Then, we have that, the 8 × 8 board can be decomposed into three disjoint sub-boards $B_i: i=1,2\ and\ 3$, such that, $\mathfrak{B}(x,B)=\mathfrak{B}(x,B_1)\mathfrak{B}(x,B_2)\mathfrak{B}(x,B_3)$.



B_1 B_2 B_3

Applying the bishop polynomial on the board with forbidden positions we have that $B_i: i=1,2\ and\ 3$, for each sub board is given by;
For the sub-board B_1, we have;

$$\mathfrak{B}(x,B_1)=\sum_{i=0}^{3}b_i^\theta(B_1)(x)^i=1+8x+3x^2+x^3$$

For the sub-board B_2, we have;

$$\mathfrak{B}(x,B_2)=\sum_{i=0}^{3} b_i^{\theta}(B_2)(x)^i = 1+5x+2x^2+x^3$$

For the sub-board B_3, we have;

$$\mathfrak{B}(x,B_3)=\sum_{i=0}^{3} b_i^{\theta}(B_3)(x)^i = 1+7x+2x^2+x^3$$

Thus, the number of bishops on the 8×8 board that decomposed into three disjoint sub-boards $B_i : i = 1, 2 \text{ and } 3$, is given by,

$$\mathfrak{B}(x,B)=\mathfrak{B}(x,B_1)\mathfrak{B}(x,B_2)\mathfrak{B}(x,B_3).$$

$$\mathfrak{B}(x,B)=(1+8x+3x^2+x^3)(1+5x+2x^2+x^3)(1+7x+2x^2+x^3)$$

Here, the bishop polynomial is a generating function of the number of arrangements of 9 non-attacking bishops (9-armies placement) on a 8×8 board, to give;

$$\mathfrak{B}(x,B)=1+20x+102x^2+123x^3+28x^4+213x^5+107x^6+36x^7+7x^8+x^9$$

Therefore, we have 9 armies to arrange in 9 positions $(m\geq n)$ through a direction of movement in an angle of 45^0 with forbidden positions. Thus, the total number of ways for the arrangement is given by;

$$\mathfrak{B}(x,B)P_{(9,9)}=\sum_{i=0}^{9}\prod_{k=0}^{3}\mathcal{X}_{B_{j,K}}(x)^i b_i^{\theta}(B_j)P_{(9-k,9-k),\ \ j=1,2,3}$$

$$=1+20+102+123+28+213+107+36+7+1=638\ ways$$

The general can match his army in 638 ways.

Conclusion

Bishop polynomials are not just interesting for their own sake. They have a variety of applications because they directly relate to permutations with restricted positions. This means that bishop polynomials can be used in everything from cryptography to combinatorial design theory.

In this paper we were able to develop the total number of ways to arrange n bishops among m positions $(m\geq n)$ and also to construct a bishop polynomial that decomposes into $n-$disjoint sub-boards $B_1, B_2, \ldots B_n$ by using an $m\times n$ array board. Furthermore, we applied these results to obtain the total number of bishops and the maximum number of arrangements for the k non-attacking bishops can be obtained on an $m\times n$ array board.

Recommendations

Further study can be carried out with the bishops on three-dimensional boards. In addition, further studies could also examine what would happen to rook and bishop polynomials by changing the shape of the boards.

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