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Operator Theory on Riemannian Differentiable Manifolds

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Abstract- In this paper is In this paper some fundamental theorems , definitions in Riemannian geometry to pervious of differentiable manifolds which are used in an essential way in basic concepts of Riemannian geometry, we study the defections, examples of the problem of differentially projection mapping parameterization system by strutting rank n on surfaces $n-k$ dimensional is sub manifolds space . A manifolds is a generalization of curves and surfaces to higher dimension, it is Euclidean in E^R in that every point has a neighbored, called a chart homeomorphic to an open subset of R^n ,

Index Terms- [1]A topological space M is an n-dimensional (topological) manifold with boundary $\partial M \subset M$ - N and M be manifolds as $f : M \to R$ is be a continuous mapping f is called a homeomorphism [2] a continuous inverse $f^{-1} : M \to N$ is called homeomorphic – Topological manifold M which the transition maps $\varphi_{i\,j} = \varphi_j \circ \varphi_i$ for pairs φ_j, φ_i in the atlas are diffeomorphisms is called (differentiable or smooth) [3] $rk(f) = m$ an immersion that is injective or 1-1 is a homeomorphism onto (surjetive mapping) its image $f(N) \subset M$ is respect to subspace topology is called (a smooth embedding) [4] a $p \in M$ define the tangent space $T_p M$ as the space of all equivalence classes $[\gamma]$ [5]At some $p \in M$ the cotangent space $T^*{}_p M$ is defined as the dual vector space $T_p M$.

I. INTRODUCTION

The Riemannian manifold with boundary, in the Euclidean domain the interior geometry is given ,flat and trivial, and the interesting phenomena come from the shape of the boundary ,Riemannian manifolds have no boundary, and the geometric phenomena are those of the interior . The present paper is an introduction, so we have to refrain from saying too must . For example, we will mainly consider compact Riemannian manifolds . The manifolds to investigated which are manifolds of systems of differential polynomials in a single unknown , possess a degree of analogy to bounded sets of numbers . They are manifolds which may be said (not to contain infinity as a solution) more definitely, zero is not a limit of reciprocals of solutions. For manifolds of this type, which will be called limited , operations of addition, multiplication and differentiation will be studied. Given two manifolds M_1 and M_2 their arithmetic sum is secured by completing into a manifold the totality of function each of which is in some area , the sum of a solution in M_1 and a solution in M_2 , multiplication is defined similarly, it turns out that if M_1 and M_2 are general solutions of equations of the first order , and are limited , their sum and product are limited .

On differentiable manifolds, these are higher dimensional analogues of surfaces and image to have but we shouldn't think of a manifold as always sitting inside a fixed Euclidean space like this one, but rather as an abstract object . One of the historical driving forces of the theory was general relativity, where the manifold is four-dimensional space-time, wormholes and all a field of co frames on M or an open set U of M , an oriented vector space is a vector space plus an equivalence class of allowable bases choose a basis to determine the orientations those equivalents to will be called oriented or positively oriented bases or frames this concept is related to the choice of a basis Ω of $\Lambda^n(V)$, say that M is oriented if is possible to define a C^∞ n-form Ω on Ω which is not zero at any point in which case M is said to be oriented by the choice . A differentiable structures is topological is a manifold it an open covering U_α where each set U_α is homeoomorphic, via some homeomorphism h_α to an open subset of Euclidean space R^n , let M be a topological space , a chart in M consists of an open subset $U \subset M$ and a homeomorphism h of U onto an open subset of R^m , a C^r atlas on M is a collection (U_α, h_α) of charts such that the U_α cover M and h_B, h_α^{-1} the differentiable vector fields on a differentiable manifold.

Tangent space as defined tangent space to level surface γ be a curve is in R^n, $\gamma : t \to \left(\gamma^1(t), \gamma^2(t), ..., \gamma^n(t)\right)$ a curve can be described as vector valued function converse a vector valued function given curve , the tangent line at the point

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II. BASIC ON RIEMANNIAN MANIFOLD

2.1 Basic on Riemannian Manifold

Definition 2.1.2 [Topological Manifold]

A topological manifold M of dimension n, is a topological space with the following properties:
(a) M is a Hausdorff space . For ever pair of points $p, g \in M$, there are disjoint open subsets $U, V \subset M$ such that $p \in U$ and $g \in V$.
(b) M is second countable . There exists accountable basis for the topology of M . (c) M is locally Euclidean of dimension n Every point of M has a neighborhood that is homeomorphic to an open subset of R^n .

Definition 2.2.3 [Coordinate Charts]

A coordinate chart or just a chart on a topological $n-$manifold M is a pair (U, φ), Where U is an open subset of M and $\varphi : U \to \widetilde{U}$ is a homeomorphism from U to an open subset $\widetilde{U} = \varphi\ (U) \subset R^n$.

Examples 2.1.4 : [Topological Manifolds] Spheres:

Let S^n denote the (unit) $n-$sphere, which is the set of unit vectors in R^{n+1}: $S^n = \{x \in R^{n+1} : \left| x \right| = 1\}$ with the subspace topology, S^n is a topological $n-$manifold.

Definition 2.1.5 [Projective spaces]

The $n-$dimensional real (complex) projective space, denoted by $P_n(R)$ *or* $P_n(C)$), is defined as the set of 1-dimensional linear subspace of R^{n+1} *or* C^{n+1}), $P_n(R)$ *or* $P_n(C)$ is a topological manifold.

Definition 2.1.6:

For any positive integer n, the $n-$torus is the product space $T^n = (S^1 \times \ldots \times S^1)$ .It is an $n-$dimensional topological manifold. (The 2-torus is usually called simply the torus).

Definition2.1.7 [Boundary of a manifold]

The boundary of a line segment is the two end points; the boundary of a disc is a circle. In general the boundary of an $n-$manifold is a manifold of dimension $(n-1)$, we denote the boundary of a manifold M as ∂M . The boundary of boundary is always empty, $\partial\partial M = \phi$

Lemma 2.1.8

(a) Every topological manifold has a countable basis of Compact coordinate balls. (B) Every topological manifold is locally compact.

Definitions 2.1.9 [Transition Map]

Let M be a topological space n-manifold. If $(U, \varphi), (V, \psi)$ are two charts such that $U \cap V \neq \phi$, the composite map

(2.1) $$\psi \circ \varphi^{-1} : \varphi\ (U \cap V) \to \psi\ \ (U \cap V)$$

is called the transition map from φ to ψ .

Definition 2.1.10 [A smooth Atlas]

An atlas A is called a smooth atlas if any two charts in A are smoothly compatible with each other. A smooth atlas A on a topological manifold M is maximal if it is not contained in any strictly larger smooth atlas. (This just means that any chart that is smoothly compatible with every chart in A is already in A.

Definition 2.1.11 [A smooth Structure]

A smooth structure on a topological manifold M is maximal smooth atlas. (Smooth structure are also called differentiable structure or C^∞ structure by some authors).

Definition 2.1.12 [A smooth Manifold]

A smooth manifold is a pair $(M$, A), where M is a topological manifold and A is smooth structure on M. When the smooth structure is understood, we omit mention of it and just say M is a smooth manifold.

Definition 2.1.13

Let M be a topological manifold. (a) Every smooth atlases for M is contained in a unique maximal smooth atlas.(b) Two smooth atlases for M determine the same maximal smooth atlas if and only if their union is smooth atlas.

Definition 2.1.14

Every smooth manifold has a countable basis of pre-compact smooth coordinate balls. For example the General Linear Group The general linear group $GL(n,R)$ is the set of invertible $n\times n$ -matrices with real entries. It is a smooth n^2 -dimensional manifold because it is an open subset of the n^2 - dimensional vector space $M(n,R)$, namely the set where the (continuous) determinant function is nonzero.

Definition 2.1.15 [Tangent Vectors On A manifold]

Let M be a smooth manifold and let p be a point of M. A linear map $X: C^{\infty}(M)\to R$ is called a derivation at p if it satisfies :

(2.2) $$X(fg)=f(p)Xg+g(p)Xf$$

for all $f,g\in C^{\infty}(M)$. The set of all derivation of $C^{\infty}(M)$ at p is vector space called the tangent space to M at p, and is denoted by [T_pM]. An element of T_pM is called a tangent vector at p.

Lemma 2.1.16 [Properties of Tangent Vectors.]

Let M be a smooth manifold, and suppose $p\in M$ and $X\in T_pM$. If f is a const and function, then $Xf=0$. If $f(p)=g(p)=0$, then $X(fp)=0$.

Definition2.1.17 [Tangent Vectors to Smooth Curves]

If γ is a smooth curve (a continuous map $\gamma: J\to M$,where $J\subset R$ is an interval) in a smooth manifold M, we define the tangent vector to γ at $t_o\in J$ to be the vector

(2.3) $$\gamma'(t_o)=\gamma_*\left(\frac{d}{dt}|_{t_o}\right)\in T_{\gamma(t_o)}M$$, where $d/dt|_{t_o}$ is the

standard coordinate basis for $T_{t_o}R$. Other common notations for the tangent vector to γ are $\left[\gamma^*(t_o),\frac{d\gamma}{dt}(t_o)\right]$ and $\left[\frac{d\gamma}{dt}|_{t=t_o}\right]$. This tangent vector acts on functions by :

(2.4) $$\gamma'(t_o)\ f=\left(\ \gamma_*\frac{d}{dt}|_{t_o}\right)\ f=\frac{d}{dt}|_{t_o}\left(f\circ\gamma\right)=\frac{d(f\circ\gamma)}{dt}(t_o).$$

Lemma 2.1.18 [Smooth manifold]

Let M be a smooth manifold and $p\in M$. Every $X\in(T_pM)$ is the tangent vector to some smooth curve in M.

Definition 2.1.19 [Lie Groups]

A Lie group is a smooth manifold G that is also a group in the algebraic sense, with the property that the multiplication map $m: G\times G\to G$ and inversion map $m: G\to G$, given by $m\ (g,h)=g\,h,\quad i\ (g)=g^{-1}$, are both smooth. If G is a smooth manifold with group structure such that the map $G\times G\to G$ given by $(g,h)\to gh^{-1}$ is smooth, then G is a Lie group. Each of the following manifolds is a lie group with indicated group operation. (a) The general linear group $GL(n,R)$ is the set of invertible $n\times n$ matrices with real entries. It is a group under matrix multiplication, and it is an open sub-manifold of the vector space $M(n,R)$, multiplication is smooth because the matrix entries of A and B. Inversion is smooth because Cramer's rule expresses the entries of A^{-1} as rational functions of the entries of A. The $n-$torus $T^n=(S^1\times\ldots\times S^1)$ is an $n-$dimensional a Belgian group.

Definition 2.1.20 [Lie Brackets]

Let V and W be smooth vector fields on a smooth manifold M . Given a smooth function $f : M \to R$, we can apply V to f and obtain another smooth function Vf , and we can apply W to this function, and obtain yet another smooth function $(WV)f = W(Vf)$. The operation $f \to WV\ f$, however, does not in general satisfy the product rule and thus cannot be a vector field, as the following for example shows let $V = \left(\frac{\partial}{\partial x}\right)$ and $W = \left(\frac{\partial}{\partial y}\right)$ on R^n , and let $f(x,y) = x,\ g(x,y) = y$. Then direct computation shows that $VW(f\ g) = 1$, while $(f\ V\ W\ g + g\ V\ W\ f) = 0$, so VW is not a derivation of $C^{\infty}(R^2)$. We can also apply the same two vector fields in the opposite order, obtaining a (usually different) function $WV\ f$. Applying both of this operators to f and subtraction, we obtain an operator $[V,W] : C^{\infty}(M) \to C^{\infty}(M)$, called the Lie bracket of V and W , defined by $[V,W]\ f = (VW)f - (WV)f$. This operation is a vector field. The Smooth of vector Field is Lie bracket of any pair of smooth vector fields is a smooth vector field.

Lemma 2.1.21 [Properties of the Lie Bracket]

The Lie bracket satisfies the following identities for all $V,W,X \in (M)$. Bilinearity: $\forall a,b \in R$,

$$[\ aV + bW, X\ ] = a\ [V,X] + b\ [W,X] \quad , \quad [\ X, aV + bW\ ] = a\ [X,V] + b\ [X,W].$$

(a) Ant symmetry $[V,W] = -\ [W,V]$. (b) Jacobi identity $[V,[\ W,X\ ]] + [\ W,[\ X,V]\ ] + [\ X,[V,W]\ ] = 0$. For $f,g \in C^{\infty}(M)$:

(2.3) $$[\ f\ V, g\ W\ ] = f\ g\ [\ V,W\ ] + (\ f\ V\ g)\ W - (\ g\ W\ f)\ V$$

Definition 2.1.22 [Lie Algebra]

A Lie algebra is a real vector space g endowed with a map called the bracket from $g \times g$ *to* g , usually denoted by $(X,Y) \to [X,Y]$, that satisfies the following properties for all $X,Y,Z \in g$:

(a)Bilinearity : For $a,b \in R$, $[a\,X + bY, Z] = a\ [X,Z] + b\ [Y,Z] \quad , \quad [Z, a\,X + bY] = a\ [Z,X] + b\ [Z,Y].$

(b) Ant symmetry: $[X,Y] = -\,[Y,X]$. (c) Jacobi identity: $[\,X,[Y,Z]\,] + [\,Y,[Z,X]] + [Z,[X,Y]\,] = 0$.

Example 2.1.23 [Lie Algebra of Vector Fields]

(a) The space (M) of all smooth vector fields on a smooth manifold M is a Lie algebra under the Lie bracket .

(b) If G is a Lie group, the set of all smooth left-invariant vector field on G is a Lie sub-algebra of (G) and is therefore a Lie algebra.

(c) The vector space $M(n,R)$ of $n \times n$ matrices an $n^2 -$ dimensional Lie algebra under the commentator bracket : $[A,B] = AB - BA$. Bilinearity and ant symmetry are odious from the definition, and the Jacobi identity follows from a straight forward calculation. When we are regarding $M(n,R)$ as a Lie algebra with this bracket, we will denote it by $gl\ (n,R)$.

2.2 Convector Fields

Let V be a finite – dimensional vector space over R and let V^* denote its dual space. Then V^* is the space whose elements are linear functions from V to R , we shall call them Convectors. If $\sigma \in V^*$ then $\sigma : V \to R$ for the any $v \in V$, we denote the value of σ on v by $\sigma(v)$ or by $\langle v,\sigma \rangle$. Addition and multiplication by scalar in V^* are defined by the equations:

(2.4) $$(\sigma_1 + \sigma_2)\ (v) = \sigma_1(v) + \sigma_2(v)\ ,\quad (\alpha\sigma)\,(v) = \alpha\ (\sigma\ (v)\ )$$

Where $v \in V$, $\sigma, \alpha\sigma \in V^*$ and $\alpha \in R$.

Proposition2.2.1 [Convectors]

Let V be a finite- dimensional vector space. If $(E_1,\ldots,E_n)$ is any basis for V ,then the convectors $(\omega^1,\ldots,\omega^n)$ defined by

(2.5) $$\omega^i(E_j) = \delta^i_j = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases},$$

form a basis for V^* ,called the dual basis to (E_j) .Therefore, $\dim V^* = \dim V$.

Definition 2.2.2 [Convectors on Manifolds.]

$AC^r -$ Convector field σ on M , $r \geq 0$, is a function which assigns to each $\beta \in M$ a convector $\sigma_p \in T^*_P(M)$ in such a manner that for any coordinate neighborhood U,ϕ with coordinate frames $E_1,..,E_n$, the functions $\sigma(E_i),\ i = 1,\ldots..,n,$ are of class C^r on U . For convenience, "Convector field" will mean $C^{\infty} -$ convector field.

Remark 2.2.3

It is important to note that a C^r-Convector field σ defines a map $\sigma : \mathcal{X}(M) \to C^r(M)$, which is not only $R-$Linear but even $C^r(M)-$ Linear, More precisely, if $f, g \in C^r(M)$ and X and Y are vector fields on M, then $\sigma\left(f\,X + g\,Y\right) = f\sigma(X) + g\sigma(Y)$. For these functions are equal at each $p \in M$.

Example 2.2.4

If f is a C^∞ function on M, then it defines a $C^\infty-$Convector field, which we shall denote df, by the formula $\langle X_p, df_p \rangle = X_p f$ *or* $df_p(X_p) = X_p f$. For a vector field X on M this gives $d\,f(X) = Xf$, a $C^\infty-$ function on M. This Convector field f, is called the differential of f, and $(df)_p$, its value at P, the differential of f at p. In the case of an open set $U \subset R^n$, we verify that it coincides with the usual notion of differential of a function in advanced calculus, and, in fact, makes it more precise.

In this case the coordinate x^i of a point of U are functions on U and, by our definition, dx^i assigns to each vector X at $p \in U$ a number $X_p\, x^i$, *its ith* component in the natural basis of R^n. In particular $\left\langle \partial/\partial x^j, dx^i \right\rangle = \partial x^i/\partial x^j = S^i_j$ so we see that $dx^1, \ldots\ldots, dx^n$

Is exactly the field of Co frames dual to. $\frac{\partial}{\partial x^1}, \ldots\ldots, \frac{\partial}{\partial x^n}$. Now if f is a $C^\infty-$function on U, then we may express df as a linear combination of $dx^1, \ldots, dx^n$. We know that the coefficients in this combination, that is the components of df, are given by $df\left(\partial/\partial x^i\right) = \partial f/\partial x^i$. Thus, we have $df = \frac{\partial f}{\partial x^1} dx^1 + \ldots + \frac{\partial f}{\partial x^n} dx^n$. Suppose $a \in U$ and $X_a \in T_a(R^n)$. Then X_a has components, say $h^1, \ldots, h^n$ and geometrically X_a is the vector from a to $a+$h. We have $df(X_a) = X_a f = \left(\sum h^i \frac{\partial}{\partial x^i}\right) f = \sum h^i \left(\frac{\partial f}{\partial x^i}\right)_a$;

In particular, $dx^i(X_a) = h^i$, that is, dx^i measures the change in the i-th coordinate of a point as it moves from the initial to the terminal point of X_a. The preceding formula may thus be written.

(2.6)
$$df(X_a) = \left(\frac{\partial f}{\partial x^1}\right)_a dx^1(X_a) + \ldots + \left(\frac{\partial f}{\partial x^n}\right)_a dx^n(X_a).$$

This gives us a very good definition of the differential a function f on $U \subset R^n$; is a field of linear functions which at any point a of the domain of f assigns to each vector X_a a number. Interpreting X_a as the displacement of the n independent variables from a, that is, a as initial point and $a+h$ as terminal point. $d\,f\,(X_a)$ a approximates (linearly) the change in f between these points.

Theorem 2.2.5 [Convector Fields and Mappings]

Let $F : M \to N$ be C^∞ and let σ be a convector field of class C^r on N. Then the formula $F^*(\sigma_{F(p)})\,(X_p) = \sigma_{F(p)}(\,F_*(X_p)\,)$ defines a C^r-convector field on M. (where $F_* : T_p(M) \to T_{F(p)}(N)$ and $F^* : T^*_{F(p)}(N) \to T^*(M)$ are linear mapping, $p \in M$).

Proof:

If σ is the convector field on N, then for any $p \in M$, there is exactly one image point $F(p)$ by definition of mapping. It is thus clear that $F^*(\sigma)$ is defined uniquely at each point of M. Now suppose that for a point $p_o \in M$ we take coordinate neighborhood U, ϕ of p_o and V, Ψ of $F(p_o)$ so chosen that $F(U) \subset V$. If we denote the coordinates on U by $(x^1, \ldots, x^m)$ and those on V by $(y^1, \ldots, y^n)$, then we may suppose the mapping F to be given in local coordinates by $y^i = f^i(x^1, \ldots, x^m)$, $i = 1, \ldots, m$. Let the expression for σ on V in the local co frames be written at $q \in V$ as $\sigma_q = \sum_{i=1}^n \alpha_i(q)\,\tilde{w}^i_q$, where $\tilde{w}^i_q, \ldots, \tilde{w}^n_q$ is the basis of $T^*_q(N)$ dual to the coordinate frames. The functions $\alpha^i(q)$ are of class C^r on V by hypothesis. Using the formula defining F^*, we see that if p is any point on U and $q = F(p)$ its image, then $\left(F^*(\sigma)\right)_p(E_{jp}) = \sigma\,F_{(p)}\left(\,F_*(E_{jp})\,\right) = \Sigma\,\alpha_i(\,F(p)\,)\;\tilde{w}^i_{F(p)}\left(F_*(E_{jp})\,\right)$.

However, we have previously obtained the formula .

(2.7)
$$F_*(E_{J\,P}) = \sum_{k=1}^n \frac{\partial y^k}{\partial x^j} \tilde{E}_{k\,F(P)} \qquad J = 1..m$$

The derivatives being evaluated at $(x^i(p), \ldots, x^m(p)) = \phi(p)$ using $\tilde{w}^i(\tilde{E}_j) = \delta^i_j$, we obtain $\left(F^*(\sigma)\right)_P(E_{jP}) = \sum_{i=1}^n \alpha_i(\,F(P)\,)\left(\frac{\partial y^i}{\partial x^j}\right)_{\phi(P)}$

As p varies over U these expressions give the components of $F^*(\sigma)$ relative to $w^1,\ldots,w^n$ on U, the co frames dual to $E_1,\ldots,E_m$. They are clearly of class C^r at least, and this completes the proof.

Corollary 2.2.6

Using the notation above let $\sigma = \sum_{i=1}^{n} \alpha_i \, \tilde{w}^i$ on V, and let $F^*(\sigma) = \sum_{j=1}^{m} \beta_j w^j$ on U, where ∞_v and β_j are functions on V and U respectively, and $\tilde{w}^i, w^j$ are the coordinate co frames. Then $F^*(\tilde{w}^i) = \sum_{j=1}^{m} \frac{\partial y^i}{\partial x^j} w^j$ and $\beta_j = \sum_{i=1}^{n} \frac{\partial y^i}{\partial x^j} \alpha_i \circ F$. For $i = 1,\ldots\ldots,n$ and $j = 1,\ldots\ldots,m$. The first formulas give the relation of the bases; the second those of the components. If we apply this directly to a map of an open subset of R^m into R^n, these give for $F^*(dy^i)$ the formula $F^*(dy^i) = \sum_{j=1}^{m} \frac{\partial y^i}{\partial x^j} dx^j$, $i = 1,\ldots\ldots,n$

2.3 Tensor Fields

Definition 2.3.1 [A convector Tensor.]

A convector tensor on a vector space V is simply a real valued $\phi(v_1,\ldots,v_r)$ of several vector variables $v_1,\ldots,v_r$ of V, linear in each separately.(i.e. multiline). The number of variables is called the order of the tensor. A tensor field ϕ of order r on a manifold M is an assignment to each point $P \in M$ of a tensor ϕ_P on the vector space $T_P(M)$, which satisfies a suitable regularity condition $C^0, C^r, or C^\infty$ as P varies on M.

Definition 2.3.2 [Tensors on A vector Space.]

A tensor ϕ on V is by definition a multiline map $\phi : \underbrace{V \times \cdots \times V}_{r} \times \underbrace{V^* \times \cdots \times V^*}_{s} \to R$, V^* denoting the dual space to V, r its covariant order, and s its contra variant order.(Assume $r > 0$ or $s > 0$). Thus ϕ assigns to each r-tuple of elements of V and s-tupelo of elements of V^* a real number and if for each k, $1 \le k \le r + s$, we hold every variable except the kth fixed, then ϕ Satisfies the linearity condition $\phi(v_1,\ldots,\alpha\, v_k + \alpha' v_k',\ldots) = \alpha\phi(v_1,\ldots,v_k,\ldots) + \alpha'\phi(v_1',\ldots,v_k',\ldots_1)$. For all $\alpha, \alpha' \in R$ and $v_k, v_k' \in V$ or (V^*) .respectively For a fixed (r,s) we let ${}^r_s(V)$ be the collection of all tensors on V of covariant order r and contra variant order s. We know that as a function from $V \times \ldots \times V \times V^* \times \ldots \times V^*$ to R they may be added and multiplied by scalars elements of R. With this addition and scalar Multiplication $(V)^r_s$ is a vector space, so that if $\phi_1, \phi_2 \in {}^r_s(V)$ and $\alpha_1, \alpha_2 \in R,$ then $\alpha_1\phi_1 + \alpha_2\phi_2$, defined in the way alluded to above, that is, by $(\alpha_1\phi_1 + \alpha_2\phi_2)$ $(v_1, v_2,\ldots) = \alpha_1\phi_1(v_1, v_2,\ldots) + \alpha_2\phi_2(v_1, v_2,\ldots)$,is multiline, and therefore is in $(V)^r_s$. Thus $(V)^r_s$ has a natural vector space structure.

Theorem 2.3.3

With the natural definitions of addition and multiplication by elements of R the set $(V)^r_s$ of all tensors of order (r,s) on V forms a vector space of dimension n^{r+s}.

Definition 2.3.4 [Tensor Fields.]

A C^∞ covariant tensor field of order r on a C^∞ - manifold M is a function ϕ which assigns to each $P \in M$ an element ϕ_P of $(T_P(M))^r$ and which has the additional property that given any $C^\infty -$ Vector fields on an open subset U of M, then $\phi(X_1,\ldots,X_r)$ is a C^∞ function on U, defined by, $\phi(X_1,\ldots,X_r)\ (P) = \phi_P(X_{1P},\ldots,X_{rP})$. We denote by $(M)^r$ the set of all $C^\infty -$ covariant tensor fields of order r on M.

Definition 2.3.4

We shall say that $\phi \in V^r$, $\phi \in V^r$ a vector space, is symmetric if for each $1 \le i, j \le r$, we have : $\phi(v_1,\ldots,v_j,\ldots,v_i,\ldots,v_r) = \phi(v_r,\ldots,v_j,\ldots,v_i,\ldots,v_1)$ .Similarly, if interchanging the (i-th) and (j-th) variables, $1 \le i, j \le r$ Changes the sign, $\phi(v_1,\ldots,v_j,\ldots,v_i,\ldots,v_r) = -\phi(v_r,\ldots,v_j,\ldots,v_i,\ldots,v_1)$,then we say ϕ is skew or anti symmetric or alternating; covariant tensors are often called exterior forms. A tensor field is symmetric (respectively, alternating) if it has this property at each point.

Theorem 2.3.5

Let $F: M \to N$ be a C^∞ map of C^∞ manifolds. Then each C^∞ covariant tensor field ϕ on N determines a C^∞ covariant tensor field $F^*\phi$ on M by the formula $(F^*\phi)_P(X_{1P},\ldots,X_{rP}) = \phi_{F(P)}(F^*(X_{1P}),\ldots,F^*(X_{rP}))$. The map $F^*: {}^r(N) \to {}^r(M)$ so defined is linear and takes symmetric (alternating) tenors to symmetric (alternating) tensors.

Definition 2.3.6 [Alternating Transformations.]

We define two linear transformations on the vector space V^r (a) Summarizing mapping S : $(V)^r \to (V)^r$ Alternating mapping A : $(V)^r \to (V)^r$ By the formula: (S ϕ) $(v_1,\ldots,v_r) = \frac{1}{r!}\sum_\sigma \phi(v_{\sigma(1)},\ldots,v_{\sigma(r)})$ And: (ϕ)$(v_1,\ldots,v_r) = \frac{1}{r!}\sum_\sigma \text{sgn}\ \sigma\phi\ (v_{\sigma(1)},\ldots,v_{\sigma(r)})$, the summation being over all $\sigma \in G_r$, the group of all permutations of r letter. It is immediate that these maps are linear transformations on $(V)^r$ in fact $\phi \to \phi^\sigma$ defined by $\phi^\sigma(v_1,\ldots,v_r) = \phi(v_{\sigma(1)},\ldots,v_{\sigma(r)})$, is such a linear transformations; and any linear combination of linear transformations of a vector space is again a linear transformation.

Properties 2.3.7 [Of summarizing S and A alternating mapping.]

(a) A and S are projections, that is $A^2 = A$ and $S^2 = S$. (b) $V^r = \Lambda^r(V)$ and $S(V^r) = \Sigma V^r$. (c) ϕ is alternating if and only if A $\phi = \phi$; is symmetric if and only if S $\phi = \phi$. (d) If $F: V \to W$ is a linear map then A and S commute with $F^*: {}^r(V) \to {}^r(V)$.All of these statements are easy consequences of the definitions. We check them only for A the verification of S being similar. They are also interrelated so we will not take them in order. First note that if ϕ is alternating, then the definition and $(\text{sgn}\,\sigma)^2 = 1$ imply $\phi(v_1,\ldots,v_r) = \text{sgn}\,\sigma\ \phi\ (v_{\sigma(1)},\ldots,v_{\sigma(r)})$. Since there are $r!$ element of Gr ,summing both sides over all $\sigma \in Gr$, gives $\phi =$ A ϕ. On the other hand if we apply a permutation τ to the variables of $\phi(v_1,\ldots,v_r)$, for an arbitrary $\phi \in (V)$, we Obtain $\phi(v_{\tau(1)},\ldots,v_{\tau(r)}) = \frac{1}{r!}\sum \text{sgn}\,\sigma\phi(v_{\sigma\tau(1)},\ldots,v_{\sigma\tau(r)})$.Now sgn is a homomorphism and sgn $\tau^2 = 1$ so that $\text{sgn}\,\sigma = \text{sgn}\,\sigma\tau\ \text{sgn}\,\tau$. For this equation and since $\sigma\tau$ runs through G_r as σ does, we see that the right side is $\frac{1}{r!}\text{sgn}\ \tau \sum \text{sgn}\sigma\ \tau\phi\ (v_{\sigma\tau(1)},\ldots,v_{\sigma\tau(r)}) = \text{sgn}\,\tau\ \ \phi(v_1,\ldots,v_r)$ .And ϕ is alternating. This show that .If ϕ is alternating, every term in the summation defining ϕ is equal, so $\phi = \phi$ .Thus A is the identity on $\Lambda^r(V)$ and A (${}^r(V) \supset \Lambda^r(V)$).From these facts (a) to (c) all follows for . a Statement (iv) is immediate the definition of F^*, for we have $F^*(v_{\sigma(1)},\ldots,v_{\sigma(r)}) = \phi(F_*(v_{\sigma(1)}),\ldots,F_*(v_{\sigma(r)}))$. Multiplying both sides by $\text{sgn}\,\sigma$ and summing over all σ gives-if we use the linearity of F^* - A $(F^*\phi)\ (V_1,\ldots,V_r)$ on the left and F^* A ϕ $(v_1,\ldots,v_r)$ on the right. Both of these map A and S can be immediately extended to mappings of tensor fields on manifolds with the same properties-by merely applying them at each point and then verifying that both sides of each relation (a) to (d) give C^∞-functions which agree point wise on every r- tuple of C^∞- vector fields.

Theorem 2.3.8

The maps A and S are defined on $(M)^r$ a C^∞- manifold and $(M)^r$ the C^∞- covariant tensor fields of order r , and they satisfy properties there. In these case of (c), $F^*: {}^r(N) \to {}^r(M)$ is the linear map induced by a C^∞ mapping $F: M \to N$.

Definition2.3.9 [Multiplication of Tensors on Vector Space].

Let V be a vector space and $\varphi \in V$ are tensors. The product of φ and ψ , denoted $\varphi \otimes \psi$ is a tensor of order $r+s$ defined by $\varphi \otimes \psi(v_1,..v_r,..v_{r+1},\ldots v_{r+s}) = \varphi(v_1,\ldots,v_r)\psi(v_{r+1},\ldots,v_{r+s})$.
The right hand side is the product of the values of φ and ψ .The product defines a mapping $(\varphi,\psi) \to \varphi \otimes \psi$ of x ${}^r(V) \to {}^{r+s}(V)$.

Theorem 2.3.10

The product ${}^r(V)\, \text{o}\, {}^r(V) \to {}^{r+s}(V)$ just defined is bilinear and associative. If $\omega^1,\ldots,\omega^n$ is a basis of .

(2.8) $$V^* = \varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_k = \frac{(r_1 + r_2 + \ldots + r_k)!}{r_1!\, r_2! \ldots r_k!}$$

then $\{(\omega^{i_1} \otimes \ldots \otimes \omega^{i_r}) / (1 \le i_1,\ldots,i_r \le n)\}$ is a basis of $(V)^r$. Finally $F_*: W \to V$ is linear, then $F^*(\varphi \otimes \psi) = (F^*\varphi) \otimes (F^*\psi)$.

Proof:

Each statement is proved by straightforward computation. To say that $\otimes$ is bilinear means that if α,β are numbers $\varphi_1,\varphi_2,\in (V)^r$ and $\psi\in {}^r(V)$, then $(\alpha\varphi_1+\beta\varphi_2)\otimes\psi=\alpha(\varphi_1\otimes\psi)+\beta(\varphi_2\otimes\psi)$. Similarly for the second variable. This is checked by evaluating each Side on $r+s$ vectors of V ; in fact basis vectors suffice because of linearity Associatively, $(\varphi\otimes\psi)\otimes\theta=\varphi\otimes(\psi\otimes\theta)$, is similarly verified the products on both sides being defined in the natural way. This allows us to drop the parentheses. To see that $\omega^{i_1}\otimes\ldots\otimes\omega^{i_r}$ from a basis it is sufficient to note that if $e_1,\ldots,e_n$ is the basis of V dual to $\omega^1,\ldots,\omega^n$, then the tensor $\Omega^{i_1,\ldots,i_r}$ previously defined is exactly $\omega^{i_1}\otimes\ldots\otimes\omega^{i_r}$ .This follows from the two definitions:

(2.9) $$\Omega^{i_1\ldots i_r}\left(e_{j_1},\ldots,e_{j_r}\right)=\begin{cases}0 & if\left(i_1,\ldots,i_r\right)\neq\left(j_1,\ldots,j_r\right)\\ 1 & if\left(i_1,\ldots,i_r\right)=\left(j_1,\ldots,j_r\right)\end{cases},$$

and $\left(\omega^{i_1}\otimes\ldots\otimes\omega^{i_r}\right)\left(e_{j_1},\ldots,e_{j_r}\right)=\omega^{i_1}\left(e_{j_1}\right)\omega^{i_2}\left(e_{j_2}\right),\ldots,\omega^{i_r}\left(e_{j_r}\right)=\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}..\delta^{i_r}_{j_r}$,which show that both tensors have the same values on any (ordered) set of r basis vectors and are thus equal. Finally, given $F_*:W\to V$, if $w_1,..,w_{r+s}\in W$, then $\left(F^*(\varphi\otimes\psi)\right)\left(w_1,\ldots,w_{r+s}\right)=\varphi\otimes\psi\ (F_*(w_1),\ldots,F_*(w_{r+s}))\ =\ \varphi(F_*(w_1),\ldots,F_*(w_r))\ \psi(F_*(w_{r+1}),\ldots,F_*(w_{r+s}))\ =\left(F^*\varphi\right)\otimes\left(F^*\psi\right)(w_1,\ldots,w_{r+s})$.
Which proves $F^*(\varphi\otimes\psi)=\left(F^*\varphi\right)\otimes\left(F^*\psi\right)$ and completes the proof.

Theorem 2.3.11 [Multiplication of Tensor Field on Manifold]

Let the mapping $(M)^r\times(M)^s\to(M)^{r+S}$ just defined is bilinear and associative. If $\left(\omega^1,\ldots,\omega^n\right)$ is a basis of $(M)^1$, then every element $(M)^r$ is a linear combination with C^∞ coefficients of $\{\left(\omega^{i_1}\otimes\ldots\otimes\omega^{i_r}\right)/\left(1\le i_1,\ldots,i_r\le n\right)\}$. If $F:N\to M$ is a C^∞ mapping, $\varphi\in M$ and $\psi\in {}^s(M)$, then $F^*(\varphi\otimes\psi)=\left(F^*\varphi\right)\left(F^*\psi\right)$, tensor field on N.

Proof:

Since two tensor fields are equal if and only if they are equal at each point, it is only necessary to see that these equations hold at each point, which follows at once from the definitions and the preceding theorem .

Corollary 2.3.12

Each $\varphi\in U^r$ including the restriction to U of any covariant tensor field on M, has a unique expression form $\varphi=\sum_{i_1}\ldots\sum_{i_r}a_{i_1\ldots i_r}\left(\omega^{i_r}\otimes\ldots\otimes\omega^{i_r}\right)$. Where at each point $U, a_{i_1\ldots i_r}=\varphi\left(E_{i_1},\ldots,E_{i_r}\right)$ are the Components of φ in the basis $\left\{\omega^{i_1}\otimes\ldots\otimes\omega^{i_r}\right\}$ and is C^∞ function on U.

2.4 Tangent Space and Cotangent Space

The tangent space $T_p(M)$ is defined as the vector space spanned by the tangents at p to all curves passing through point p in the manifold M . And The cotangent space $T_p^*(M)$ of a manifold, at $p\in M$ is defined as the dual vector space to the tangent space $T_p(M)$. We take the basis vectors $E_i=\partial/\partial x^i$ for $T_p(M)$, and we write the basis vectors for $T_p^*(M)$ as the differential line elements $e^i=dx^i$. Thus the inner product is given by $\langle\partial/\partial x^i, dx^i\rangle=\delta_i^j$.

Definition 2.4.1 [Wedge Product.]

Carton's wedge product, also known as the exterior Product, as the ant symmetric tensor product of cotangent space basis elements $dx\wedge dy=\frac{1}{2}(dx\otimes dy-dy\otimes dx)\ =-dy\wedge dx$. Note that, by definition, $dx\wedge dx=0$. The differential line elements dx and dy are called differential 1-forms or 1-form; thus the wedge product is a rule for construction g 2-forms out of pairs of 1-forms.

Definition 2.4.2

Let $\Lambda^p(x)$ be the set of anti-symmetric p-tensors at a point x .This is a vectors space of dimension $n!/p!(n-p)!$. The $\Lambda^p(x)$ path together to define a bundle over M . $C^\infty(\Lambda^p)$ is the space of smooth p-forms, represented by anti-symmetric tensors $f_{ij\ldots}(x)$, having p indices contracted with the wedge product of p differentials. The elements of $C^\infty(\Lambda^p)$ may then be written explicitly as follows:

$$C^\infty(\Lambda^0)=\{f(x)\}\qquad \dim=1$$
$$C^\infty(\Lambda^1)=\{f_i(x)dx^i\}\qquad \dim=\text{n}$$

$$C^\infty(\Lambda^2) = \{ f_{ij}(x)dx^i \wedge dx^j \} \qquad \dim = n(n-1)/2!$$
$$C^\infty(\Lambda^3) = \{ f_{ijk}(x)dx^i \wedge dx^j \wedge dx^k \} \qquad \dim = n(n-1)(n-2)/3!$$
$$C^\infty(\Lambda^{n-1}) = \{ f_{i_1 \dots i_{n-1}} dx^{i_1} \wedge \dots \wedge dx^{i_{n-1}} \} \qquad \dim = n$$
(2.10)
$$C^\infty(\Lambda^n) = \{ f_{i_1 \dots i_n} dx^{i_1} \wedge \dots \wedge dx^{i_n} \} \qquad \dim = 1.$$

Remark 2.4.3

Let α_p be an element of Λ^p α_p, β_p an element of Λ^q. Then $\alpha_p \wedge \beta_q = (-1)^{pq} \beta_q \wedge \alpha_p$. Hence odd forms ant commute and the wedge product of identical 1-forms will always vanish.

Remark 2.4.4 [Exterior Derivative]

The exterior derivative operation, which takes p-forms into $(p+1)$-forms according to the rule

$$C^\infty(\Lambda^0) \xrightarrow{d} C^\infty(\Lambda^1) \ ; \ d(f(x)) = \frac{\partial f}{\partial x^i} dx^i$$
$$C^\infty(\Lambda^1) \xrightarrow{d} C^\infty(\Lambda^2) \ ; \ d(f_j(x)dx^j) = \frac{\partial f_j}{\partial x^i} dx^i \wedge dx^j$$
(2.11)
$$C^\infty(\Lambda^2) \xrightarrow{d} C^\infty(\Lambda^3) \ ; \ d(f_{jk}(x)dx^j \wedge dx^k) = \frac{\partial f_j}{\partial x^i} dx^i \wedge dx^j \wedge dx^k$$

Here we have taken the convention that the new differential line element is always inserted before any previously existing wedge products.

Property 2.4.5

An important property of exterior derivative is that it gives zero when applied twice: $d \ dw_p = 0$. This identity follows from the equality of mixed partial derivative, as we can see from the following simple example:

$$C^\infty(\Lambda^0) \xrightarrow{d} C^\infty(\Lambda^1) \xrightarrow{d} C^\infty(\Lambda^2) \ , \ df = \partial_j f dx^j , \quad ddf = \partial_i \partial_j f dx^i \wedge dx^j = \frac{1}{2}(\partial_i \partial_j f - \partial_j \partial_i f) dx^i \wedge dx^j = 0.$$

Remark 2.4.6

(a) The rule for differentiating the wedge product of a p-form α_p and a q-form β_q is $d(\alpha_p \wedge \beta_q) = d\alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q$. (b) The exterior derivative anti-commutes with 1-forms.

Examples 2.4.7

Possible p-forms α_p in two-dimensional space are

(2.12)
$$\alpha_0 = f(x,y)$$
$$\alpha_1 = u(x,y)dx + v(x,y)dy$$
$$\alpha_2 = \phi(x,y)dx \wedge dy.$$

The exterior derivative of line element gives the two-Dimensional curl times the area $d(u(x,y)dx + v(x,y)dy) = (\partial_x v - \partial_y u)dx \wedge dy$.

The three-space p-forms α_p are

$$\alpha_0 = f(x)$$
$$\alpha_1 = v_1 dx^1 + v_2 dx^2 + v_3 dx^3$$
$$\alpha_2 = w_1 dx^2 \wedge dx^3 + w_2 dx^3 \wedge dx^1 + w_3 dx^1 \wedge dx^2$$
$$\alpha_3 = \phi(x) dx^1 \wedge dx^2 \wedge dx^3.$$

We see that

$$\alpha_1 \wedge \alpha_2 = (v_1 w_1 + v_2 w_2 + v_3 w_3) dx^1 \wedge dx^2 \wedge dx^3$$
$$d\alpha_1 = (\varepsilon_{ijk} \partial_j v_k) \frac{1}{2} \varepsilon_{ijm} dx^1 \wedge dx^m$$
(2.13)
$$d\alpha_2 = (\partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3) dx^1 \wedge dx^2 \wedge dx^3.$$

(Where ε_{ijk} is the totally anti-symmetric tensor in 3-dimensions).

Definition 2.4.7
An alternating covariant tensor field of order r on M will be called an exterior differential form of degree r (or some time simply , r -form).The set $\Lambda^r(M)$of all such forms is a subspace of $(M)^r$.

Theorem 2.4.8
Let $\Lambda(M)$ denote the vector space over R of all exterior differential forms. Then for $\varphi \in \Lambda^r(M)$ and $\psi \in \Lambda^s(M)$, the formula , $(\varphi \wedge \psi)_P = \varphi_P \wedge \psi_P$ defines an associative product satisfying $\varphi \wedge \psi = (-1)^{rs} \psi \wedge \varphi$. With this product, $\Lambda(M)$is algebra over R . If $f \in C^\infty(M)$, we also have $f(\varphi\psi) = \varphi \wedge (f\psi) = \varphi \wedge (f\psi)$ If $\omega^1,\ldots,\omega^n$ is a field of co frames on M (or an open set U of M), then the set $\left\{ (\omega^{i_1} \wedge \ldots \wedge \omega^{i_r})/ \quad (1 \le i_1 < i_2 < \ldots < i_r \le n) \right\}$is a basis of $\Lambda^r(M)$ *or* $\Lambda(U)$.

Theorem 2.4.9
If $F: M \to N$ is a C^∞ mapping of manifolds, then $F^*: \Lambda(N) \to \Lambda(M)$ is an algebra homomorphism. (We shall call $\Lambda(M)$ the algebra of differential forms or exterior algebra on M).

Definition 2.4.10
An oriented vector space is a vector space plus an equivalence class of allowable bases, choose a basis to determine the orientation those equivalents to it will be called oriented or positively oriented bases or frames. This concept is related to the choice of a basis Ω of $\Lambda^n(V)$.

Lemma 2.4.11
Let $\Omega \ne 0$ be an alternating covariant tensor on V of order, n = dim V and let $e_1,\ldots,e_n$ .be a basis of V . Then for any set of vectors $v_1,\ldots,v_n$, with $v_i = \sum \alpha_i^{\,j} e_j$, , we have , $\Omega(v_1,..v_n) = \det(\alpha_i^j)\ \Omega\ (e_1,\ldots,e_n)$.

Proof:
This lemma says that up to a non vanishing scalar multiple Ω is the determinant of the components of its variables. In particular, If $V = V^n$ is the space on n-tuples and $e_1,\ldots,e_n$ is the canonical basis, then $\Omega(v_1,\ldots,v_n)$ is proportional to the determinant whose rows are $v_1,\ldots,v_n$. The proof is a consequence of the definition of determinant. Given Ω and $v_1,\ldots,v_n$, we use the linearity and ant symmetry of Ω to write $\Omega(v_1,\ldots,v_n) = \sum \alpha_1^{j_1} \ldots \alpha_n^{j_n} \Omega(e_{j_1},\ldots,e_{j_n})$. Since $\Omega(e_{j_1},\ldots,e_{j_n}) = 0$, if two indices are equal, we may write $\Omega\ (v_1,\ldots,v_n) = \sum \operatorname{sgn}\sigma\,(\alpha_1^{\sigma(1)} \ldots \alpha_n^{\sigma(n)}) \Omega\ (e_1,\ldots,e_n) = \det(\alpha_i^j)\ \Omega\ (e_1,\ldots,e_n)$.The last equality uses the standard definition of determinant.

Corollary 2.4.12
Note that if $\Omega \ne 0$, then $v_1,\ldots,v_n$ are linearity independent if and only if $\Omega\ (v_1,\ldots,v_n) \ne 0$. Also note that the formula of the lemma can be construed as a formula for change of component of Ω , there is just one component since $\Lambda^n(V) = 1$, when we change from the basis $e_1,\ldots,e_n$ of V to the basis $v_1,\ldots,v_n$ .These statements are immediate consequences of the formula in the lemma.

Definition 2.4.13
We shall say that M is orient able if is possible to define a C^∞ $n-$form Ω on M which is not zero at any point, in which case M is said to be oriented by the choice of Ω . A manifold M is orient able if and only if it has a covering $\{U_\alpha, \varphi_\alpha\}$ of coherently oriented coordinate neighborhoods.

Theorem 2.4.14
Let M be any C^∞ Manifold and let $\Lambda(M)$ be the algebra of exterior differential forms on M .Then there exists a unique R -linear map $d_M: \Lambda(M) \to \Lambda(M)$ such that (a) If $f \in \Lambda^\circ(M) = C^\infty(M)$, then $d_M f = df$, the differential of f .
(b) If $\theta \in \Lambda^r(M)$ *and* $\sigma \in \Lambda^s(M)$, *then* $d_M(\theta \wedge \sigma) = d_M\theta \wedge \sigma + (-1)^r \theta \wedge d_M \sigma$. (c) $d_M^2 = 0$ .This map will commute with restriction to open sets $U \subset M$, that is, $(d_M\theta)_U = d_U \theta_U$, and map $\Lambda^r(M)$ into $\Lambda^{r+1}(M)$.

2.5 Riemannian Manifold

A bilinear form on a vector space V over R is defined to be a map $\phi : V \times V \to R$ that is linear in each variable separately, that is, for $\alpha, \beta \in R$ and $v, v_1, v_2, w, w_1, w_2 \in V$, $\phi(\alpha v_1 + \beta v_2, w) = \alpha\phi(v_1, w) + \beta\phi(v_2, w)$, $\phi(v, \alpha w_1 + \beta, w_2) = \alpha\phi(v_1, w_1) + \beta\phi(v_1 w_2)$. A similar definition may be made for a map ϕ of a pair of vector space $V \times W$ over R. A bilinear form on V are completely determined by their n^2. Values on basis $e_1, \ldots, e_n$ of V. If $\alpha_{ij} = \phi(e_i, e_j), 1 \le i, j \le n$, are given and $v = \sum \lambda^i e_i$, $w = \sum \mu^j e_j$ are any pair of vectors in V, then bilinearity requires that $\phi\ (v, w) = \Sigma_{i,j=1}^{n} \alpha_{i\ j}\ \lambda^i\ \mu^j$ .A bilinear form, or function is called symmetric if $\phi(v, w) = \phi(w, v)$, and skew–symmetric if $\phi(v, w) = -\phi(w, v)$ asymmetric form is called positive definite if $\phi(v, v) \ge 0$ and if equality holds if and only if $v = 0$; in this case we often call ϕ an inner product on V.

Definition 2.5.1

A field ϕ of C^r – bilinear forms, $r \ge 0$, on a manifold M consists of a function assigning to each point P of M, a bilinear form ϕ_P on $T_P(M)$,that is, a bilinear mapping $\phi_P : T_P(M) \times T_P(M) \to R$, such that for any coordinate neighborhood U, φ, the function $\alpha_{ij} = \phi(E_i, E_j)$, defined by ϕ and the coordinate forms $E_1, \ldots, E_n$,are of class C^r. Unless otherwise stated bilinear forms will be C^∞.To simplify notation we usually write $\phi(X_P, Y_P)$ for $\phi_P(X_P, Y_P)$.

Definition 2.5.2

Suppose $F_* : W \to V$ is a linear map of vector spaces and ϕ is A bilinear form on V .Then the formula $(F^*\phi)(v, w) = \phi(F_*(v), F_*(w))$ defines a linear form $F^*\phi$ on W.

Theorem 2.5.3

Let $F : M \to N$ be a C^∞ map and ϕ a bilinear form of class C^r on N. Then $F^*\phi$ is a C^r – bilinear form on M. If ϕ is symmetric (skew- symmetric), then $F^*\phi$ is symmetric (skew- symmetric).

Proof:

The proof parallels those of theorem (2.6.14) and corollary (2.6.12) and we analogously obtain formulas for the components of $F^*\phi$ in terms of those of ϕ we suppose U, ϕ and V, ψ, are coordinate neighborhoods of P and of $F(P)$ with $F(U) \subset V$ .Using the notation of theorem (2.6.14) and corollary (2.6.12) we may write $\beta_{i\ j}(p) = (F^*\phi)_p(E_{ip}, E_{jp}) = \phi(F_*(E_{ip}), F_*(E_{jp}))$.

Applying as before, we have .

$$\beta_{i\ j}(p) = \sum_{s,t=1}^{n} \frac{\partial y^s}{\partial x^i} \frac{\partial y^t}{\partial x^j}\ \phi\ (\tilde{E}_{sF(p)}, \tilde{E}_{tF(p)}).$$

This gives the formula

$$\beta_{i\ j}(p) = \sum_{s.t=1}^{n} \frac{\partial y^s}{\partial x^i} \frac{\partial y^t}{\partial x^j}\ \alpha_{st}(\ F(p)\ ) \quad , \quad 1 \le i, j \le m, \tag{2.15}$$

for the matrix of components $(\beta_{i\ j})$ of $F^*\phi$ at P in terms of the matrix (α_{st}) of ϕ at $F(p)$. The functions $\beta_{i\ j}$ thus defined are of class C^r at least on U which completes the proof.

Corollary 2.5.4

If F is an immersion and ϕ is a positive definite, symmetric form then $F^*\phi$ is a positive definite, symmetric bilinear form.

Proof:

All that we need to check is that $F^*\phi$ is positive define at each $P \in M$. Let X_P be any vector tangent to M at p. Then

$$F^*\phi\ (X_P, X_P) = \phi\ (F_*(X_P)) \quad , \quad F_*\ (X_P) \ge 0 \tag{2.16}$$

with equality holding only if $F_*(X_P) = 0$. However, since F is an immersion, $F_*(X_P) = 0$ .if and on only if $X_P = 0$.

Definition 2.5.5 [Riemannian Manifold.]

A manifold M on which there is defined a field of symmetric, positive definite, bilinear forms ϕ is called a Riemannian manifold and ϕ the Riemannian metric. We shall assume always that ϕ is of class C^∞.

2.6 Rings Riemannian

a Riemannian manifold , having define vectors and one-form we can define tensor , a tensor of rank (m,n) also called (m,n) tensor , is defined to be scalar function of m one-forms and v vectors that is linear in all of its argument, if follow at once that scalars tensors of rank (0,0) , for example metric tensor scalar product equation $\widetilde{P}(\vec{V})=\langle \widetilde{P},\vec{V}\rangle$ requires a vector and one-form is possible to obtain a scalar from vectors or two one-forms vectors tensor the definition of tensors , any tensor of $(0,2)$ will give a scalar form two vectors and any tensor of rank $(0,2)$ combines two one-forms to given $(0,2)$ tensor field g_x called tensor the g_x^{-1} inverse metric tensor , the metric tensor is a symmetric bilinear scalar function of two vectors that g_x and g_x is returns a scalar called the dot product . $g(\vec{V},\vec{W})=\vec{V}.\vec{W}=\vec{W}.\vec{V}=g(\vec{W},\vec{V})$ .Next we introduce one-form is defined as linear scalar function of vector $\widetilde{P}(\vec{V})$ is also scalar product $\widetilde{P}(\vec{V})=\langle \widetilde{P},\widetilde{V}\rangle$ one-form $\widetilde{p}$ satisfies the following relation.

(2.17) $$\widetilde{P}(a\vec{V}+b\vec{W})=\langle P,a\vec{V}+b\vec{W}\rangle=a\langle \widetilde{P},\vec{V}\rangle+b\langle \widetilde{P},\vec{W}\rangle=a\widetilde{P}(\vec{V})+b\widetilde{P}(\vec{W})$$

and given any two scalars a and b and one-forms $\widetilde{P},\widetilde{Q}$ we define the one-form $a\widetilde{P}+b\widetilde{Q}$ by.

(2.18) $$(a\widetilde{P}+b\widetilde{Q})(\vec{V})=\langle a\widetilde{P}+b\widetilde{Q},\vec{V}\rangle=a\langle \widetilde{P},\vec{V}\rangle+b\langle \widetilde{Q},\vec{V}\rangle=a\widetilde{P}(\vec{V})+b\vec{Q}(\vec{V})$$

and scalar function one-form we may write $\langle \widetilde{P},\vec{V}\rangle=\widetilde{P}(\vec{V})=\vec{V}(\widetilde{P})$.

For example $m=2$, $n=0$ and $T(a\widetilde{P}+b\widetilde{Q},c\widetilde{R}+d\widetilde{S})=acT(\widetilde{P},\widetilde{R})+adT(\widetilde{P},\widetilde{S})+bcT(\widetilde{Q},\widetilde{R})+bdT(\widetilde{Q},\widetilde{S})$ tensor of a given rank form a liner algebra mining that a liner combinations of tensor rank (m,n) is also a tensor rank (m,n) , and tensor product of two vectors A and B given a rank $(2,0)$, $T=\vec{A}\otimes\vec{B}$, $T(\widetilde{P},\widetilde{Q})\equiv\vec{A}(\widetilde{P}).\vec{B}(\widetilde{Q})$ and $\otimes$ to denote the tensor product and non commutative $\vec{A}\otimes\vec{B}\neq\vec{B}\otimes\vec{A}$ and $\vec{B}=c\vec{A}$ for some scalar , we use the symbol $\otimes$ to denote the tensor product of any two tensor e.g $P\otimes T=\widetilde{P}\otimes\widetilde{A}\otimes\widetilde{B}$ is tensor of rank $(2,1)$. The tensor fields in inroad allows one to the tensor algebra $A_R(T_pM)$ the tensor spaces obtained by tensor protects of space R , T_pM and $T^*{}_pM$ using tensor defined on each point $p\in M$ field for example M be n-dimensional manifolds a differentiable tensor $t_p\in A_R(T_pM)$ are same have differentiable components with respect , given by tensor products of bases $\left(\frac{\partial}{\partial x^k}\right)_p\subset T_pM$ $,k=1,....n$ and $(dx^k)_p\subset T^*{}_pM$ induced by all systems on M .

2.7 Riemannian Manifold on Curvature Bounded

Let M be complete Riemannian manifold with sectional curvature bounded below by a constant $-K^2$ Let $u\in USC(M)$ *and* $v\in LSC(M)$ be tow functions satisfying. $\mu_0:=\sup_{x\in M}[\,u\,(x)-v\,(x)\,]\leq+\infty$. Assume that u and v are bounded from above and below respectively and there exists a function $W:[0,\infty]\to[0,\infty]$ satisfying $W(l)\phi\ 0.$ when $l\ \phi\ 0.$ and $W(0+)=0$ such that $u\,(x)-u(y)\leq w\,(d(x,y))$ Then for each $\varepsilon\ \phi\ 0$ there exist $x_\varepsilon,y_\varepsilon\in M$, such that $(p_\varepsilon,X_\varepsilon)\in\bar{J}^{2,+}u\,(x_\varepsilon)$, $(q_\varepsilon,Y_\varepsilon)\in\bar{J}^{2,+}v\,(y_\varepsilon)$ such that $u\,(x)-u\,(y)\geq\mu_0-\varepsilon$ .And such that $d\,(x_\varepsilon,y_\varepsilon)\,\pi\,\varepsilon$, $|\ p_\varepsilon-q_\varepsilon\ \text{o}P_\gamma(l)\,|\,\pi\,\varepsilon$, $X_\varepsilon\leq Y_\varepsilon\ \text{o}\,p_\gamma(l)+\varepsilon\,P_\gamma(l)$

Where $l=d(x_\varepsilon,y_\varepsilon)$ *and* $p_\gamma(l)$ is the parallel transport along the shortest geodesic connecting x_ε *and* y_ε .We divide the proof into tow **parts. [a]:** without loss of generality , we assume that $\mu_0\geq0$. Otherwise we replace u by $u-\mu_0+1$ for each $\alpha\ \phi\ 0$ we take $\hat{x}_\alpha\in M$ such $u\,(\hat{x}_\alpha)-v\,(\hat{x}_\alpha)+w\left(\sqrt{\frac{\mu_0}{\alpha}}\right)\geq\mu_0$.

Part[2]: We apply to $\varphi_\alpha(x,y)=\frac{\alpha}{2}d\,(x,y)^2+\frac{\lambda_\alpha}{2}d\,(\hat{x}_\alpha,x\,)^2+\frac{\lambda_\alpha}{2}d\,(\hat{x}_\alpha,y)^2$. We have for any $\delta\ \phi\ 0$ there exist $X_\alpha\in ST^*_{x\beta}(M)$ *and* $Y_\alpha\in ST^*_{y\alpha}(M)$ such that . $(D_x\varphi_\alpha(x_\alpha,y_\alpha)\ \ ,X_\alpha)\in\hat{J}^{2,+}\ u\,(x_\alpha)$ and $(-D_y\varphi_\alpha(x_\alpha,y_\alpha)\ ,X_\alpha)\in\hat{J}^{2,-}u\ (y_\alpha)$ and the block diagonal matrix satisfies

(2.19) $$-\left(\frac{1}{\delta}+\|\Lambda_\alpha\|\right)I\leq\begin{pmatrix}X_\alpha & 0\\ 0 & -Y_\alpha\end{pmatrix}\leq\Lambda_\alpha+\delta\Lambda_\alpha^2$$

2-7-1 Corollary :[Complete Manifolds with Ricci Curvature Bounded]

Let M be complete Riemannian Manifold with Ricci curvature bounded below by a constant $-(n-1)\,k^2$ and f a C^2 function on M bounded from below then for any $\varepsilon \succ 0$ there exist a point $x_\varepsilon \in M$ such that

(2.20) $$f(x_\varepsilon) \le \inf\ f + \varepsilon, |\ \nabla f\ |(x_\varepsilon) \le \varepsilon, \Delta\ f(x_\varepsilon) \ge -\varepsilon$$

Proof :

Let $u = \inf\ f\ \ , and\ \ v = f\ .w$ can be chose to be a linear function . it is straightforward to verify that all conditions in the theorem are satisfied .

III. DISCRETE LAPLACE – BELTRAMI WITH DISCRETE RIEMANNIAN METRIC

3.1 Laplace-Beltrami operator

Laplace – Beltrami operator plays a fundamental role in Riemannian geometric .In real applications, smooth metric surface is usually represented as triangulated mesh the manifold heat kernel is estimated from the discrete Laplace operator- Discrete Laplace – Beltrami operators on triangulated surface meshes span the entire spectrum of geometry processing applications including mesh parameterization segmentation.

Definition 3.1.1 [Laplace – Beltrami Operator]

Suppose (M, g) is complete Riemannian manifold, g is the Riemannian metric, Δ is Laplace – Beltrami operator . The eigenvalue $\{\lambda_n\}$ and eignfunctions $\{\phi_n\}$ of Δ are $\Delta\phi_n = \lambda_n\phi_n$, where ϕ_n is normalized to be orthonormal in $L^2(M)$, the spectrum is given by $0 = \lambda_0 \le \lambda_1 \le \ldots\ldots.. \le \lambda_n$ and $\lambda_n \to \infty$ then there is heat kernel $K(x, y, t) \in C^\infty(M \times M \times R^+)$ such that $K(x, y, t) = \sum e^{-\lambda_n t}\phi_n(x)\phi_n(y)$ heat kernel reflects all the information of the Riemannian metric.

Theorem 3.1.2

Let $f : (M, g) \to (M_2, g_2)$ diffeomorphism between two Riemannian manifold , If f is an isometric $K(x, y, t) = K_2(Mf(y), t) f(x)$ $\forall x, y \in M, t \ge 0$ Conversely, if f is subjective map and equation holds then f is an isomety.

Definition 3.1.3 : [Polyhedral Surface]

An Euclidean polyhedral surface is a triple (S, T, d) , S: is a closed surface, T: is a triangulation of S , d : is metric on S , whose restriction to each triangle is isometric to on Euclidean triangle.

Definition 3.1.4 : [Cotangent Edge Weight]

Suppose $\lfloor V_i, V_j \rfloor$ is boundary edge of M and $\lfloor V_i, V_j \rfloor \in \partial M$, Then $\lfloor V_i, V_j \rfloor$ is associated with one triangle $\lfloor V_i, V_j . V_k \rfloor$ the against $\lfloor V_i, V_j \rfloor$ at the vertex V_k is α then the weight of $\lfloor V_i, V_j \rfloor$ is given by $W_{ij} = \frac{1}{2}\cot\alpha$, otherwise if $\lfloor V_i, V_j \rfloor$ is an interior edge the two angles are α, β then the weight is $W_{ij} = \frac{1}{2}(\cot\alpha + \cot\beta)$.

Definition 3.1.5 : [Discrete Heat Kernel]

The discrete heat kernel is defined as, $K(t) = \phi \exp\ (-\wedge C\ \ )\ \phi^T$

Definition3.1.6

Suppose two Euclidean polyhedral surfaces (S, T, d_1) and (S, T, d_2) are give $L_1 = L_2$ if and d_1 and d_2 differ by a scaling .

Suppose two Euclidean polyhedral , surface (S, T, d_1) and (S, T, d_2) are given $K_1(t) = K_2(t)$ $\forall t \ge 0$, if d_1 and d_2 differ by a scaling .

Proof :

Therefore the discrete Laplace metric and the discrete heat kernel mutually determine each other. We fix the connectivity of polyhedral surface (S,T) .Suppose the edge set of (S,T) is sorted as $E=(e_1,e_2,.............e_m)$ where $m=|E|$, the face set as F and a triangle $[V_i,V_j.V_k]\in F$ as $\{i,j,k\}\in F$. We denote an Euclidean polyhedral metric $d\equiv(d_1,d_2,........,d_m)$ where $d:E\to R^+$ is the edge length function $d_i:d(e_i)$ is the length of edge e_i is $E_d(2)=\{(d_1,d_2d_3)\,|d_i+d_j|\geq d_k\}$, Be the space of all Euclidean triangles parameterized by the edge where $\{i,j,k\}$ is a cyclic permutation of $\{1,2,3\}$. In this work for convene , we use $u=(u_1,u_2,........u_m)$ .To represent the metric , where $u_k=\frac{1}{2}d_k^2$.

Definition 3.1.5 : [Energy]

An Energy $E:\Omega_u\to R$ is defined as $E(u_1,u_2........u_m)=\int_{(1,1,....1)}^{(u_1,u_2,.....,u_m)}\sum_{k+1}^{m}W_k(u)d\mu_k$. Where $W_k(u)$ the cotangent weight on the edge e_k determined by the metric μ .

Lemma 3.1.6

Suppose $\Omega\subset R^n$, is an open convex domain in R^n , $E:\Omega\to R$ is a strictly convex function with positive definite Hessian matrix then $\nabla E:\Omega\to R^n$ is a smooth embedding we show that Ω_u is a convex domain in R^m, the energy E is convex . According to gradient of energy. $\nabla E(d):\Omega\to R^m$, $\nabla E=(u_1,u_2,.........,u_m)\to(w_1,w_2,........w_m)$ is an embedding Namely the metric determined by the edge weight unique up to a sealing .

Lemma 3.1.7

Suppose an Euclidean triangle is with angles (ϕ_i,ϕ_j,ϕ_k) and edge lengths (d_i,d_j,d_k) Angles are treated as function of the edge lengths $\phi_i(d_i,d_jd_k)$ then $\frac{\partial\phi_i}{\partial d_i}=\frac{d_i}{2A}$, $\frac{\partial\phi_j}{\partial d_j}= -\frac{d}{2A}\cos\phi_k$. Where A is the area of the triangle.

Lemma 3.1.8

In an Euclidean triangle , let $u_i=\frac{1}{2}d_i^2$ and $u_j=\frac{1}{2}d_j^2$, then $\frac{\partial\cot\phi_i}{\partial u_j}=\frac{\partial\cot\phi_j}{\partial u_i}$

Corollary 3.1.9

The differential form $W=\cot\phi_i du_i+\cot\phi_j du_j+\cot\phi_k du_k$. Is a closed 1-form.

Corollary3.1.10 [Open Surfaces]

The mapping on an Euclidean polyhedral surface with boundaries $\nabla E:\Omega_u\to R^m$, $(u_1,u_2,.........,u_m)\to(w_1,w_2,........w_m)$ is smooth embedding , it can proven using double covering technique.

3.2 A Liouville Type Theorem for Complete Riemannian Manifolds

First we consider the most popular maximum principle, let U be an connected set in an m-dimensional Euclidean space R^m and $\{x^j\}$ a Euclidean coordinate . We denote by L a differential operator defined by .

(3.1) $$L=\Sigma a^{ij}\frac{\partial^2}{\partial x^i\partial x^j}+\Sigma\, b^j\frac{\partial}{\partial x^j}$$

Where a^{ij} and b^j are smooth function on U for any indices . When the matrix a^{ij} is positive definite and symmetric , it is called a second order elliptic differential operator . We assume that L is an elliptic differential operator . The maximum principle is explained as follows.

Defections3.2.1 [Maximum Harmonic on Riemannian Geometry]

For a smooth function f on U if it satisfies $Lf\geq 0$, and if there exists a point in U at which it attains the maximum , namely , if there exists a point x_0 in U at which $f(x_0)\geq f(x)$, for any point x in M then the function f is constant. In Riemannian Geometry . this property is reformed as follows . Let (M,g) be a Riemannian manifold with the Riemannian metric g , then we denote by Δ the Laplacian associated with the Riemannian metric g a function f is said to sub harmonic or harmonic if satisfies $\Delta f\geq 0$ *or* $\Delta f=0$

Defection 3.2.2 [Maximum Principles Riemannian]

For a sub harmonic function on f on Riemannian manifold M if there exist a pints in M at which attains the this property is to give a certain condition for a sub harmonic function to be constant , when we give attention to the fact relative t these maximum principles.

Definition 3.2.3Liouvile's

(a)Let f be a sub harmonic function on R^n, if it is bounded then it is constant.(b) Let f be a harmonic functions on R^n , $m \geq 3$. If it is bounded then it is constant . We are interested in Riemannian analogues of Liouvile,s theorem compared with these Last tow theorems we give attention to the fact that there is an essential difference between base manifold . In fact one is compact and the other is complete and an compact , we consider have a family of Riemannian manifold (M,g) at the global situations it suffices to consider a bout the family of complete Riemannian manifold of course , the subclass of compact Riemannian manifolds. (M,g): is complete Riemannian manifold since a compact Riemannian manifold .

Theorem 3.2.4 [Complete Riemannian Manifold]

A let M be complete Riemannian manifold whose Ricci curvature is bounded from below , if C^2- nonnegative function f satisfies Where Δ denotes the Laplacian on M , then f vanishes identically, the purpose of this theorem is t prove the following (Leadville Type) theorem in a complete Riemannian manifolds similar to theorem in a complete Riemannian manifold similar to give anther proof of (Nishikawas theorem) . In this note main theorem is as follows

Theorem 3.2.5 [Riemannian Manifold whose Ricci is Bounded]

Let M be a complete Riemannian manifold whose Ricci curvature is bounded from blew , if C^2- nonnegative function f satisfies $\Delta\ f \geq C_0\ f\ n$

Where C_0 is any positive constant and n is any real number greater f vanishes identically .

Theorem 3.2.6 [Ricci Riemannian Manifold]

Let M an n-dimensional Riemannian manifold whose Ricci curvature is bonded from below on M , Let G be a C^2- functions bounded from below on M , then for any $\varepsilon \geq 0$, there exists a point p such that

$$\left|\nabla G(P)\right| \leq \varepsilon \quad , \quad \Delta G(P) - \varepsilon \ and \ \inf G + \varepsilon \geq G(p) \tag{3.2}$$

Proof :

In this section we prove the theorem stated in introduction first all in order prove theorem , then our theorem is directly obtained as a corollary of this property and hence Nishikawas theorem is also a direct consequence of this (Nishikawas one)

Theorem 3.2.7 [Manifold and Ricci Curvature]

Let M be a complete Riemannian manifold whose Ricci Curvature is bounded from blew , Let F be any formula of the variable F with constant coefficients such that $F(f) = (C_0 f^n + C_1 f^{n-1} + \dots\dots\dots \dots + C_k f^{n-k}) + C_{k+1}$ Where $n \geq 1$, $1 \geq n-k \geq 0$ and $C_0 \geq C_{k+1}$ if a C^2- nonnegative function f satisfies .

$$\Delta\ f \geq F\,(f) \tag{3.3}$$

Then we have Where f_1 denotes the super mum f the given function f .

Proof :

From the assumption there exists a positive number a which satisfies $C_{k+1} \leq a^n C_0$ For the constant a given above the function $G(f)$ with respect to 1-variable f is defined by $(f+a)^{\frac{1-n}{2}}$, n is the maximal degree of the f , then it is easily seen that G is the C^2-function so that it is bounded from appositive by the constant $a^{\frac{1-n}{2}}$ and bounded from below by 0 , By the simple calculating we have

$$\nabla G = -\frac{n-1}{2} G^{\frac{n+1}{n-1}} \nabla f \tag{3.4}$$

Hence we get by using the above equation $\frac{1-n}{2}\ G\ \frac{2n}{n-1}\ \Delta\ f = G\Delta G - \frac{n+1}{n-1}\left|\nabla G\right|^2$ Since the Ricci curvature is bounded from below by the assumption and the function G defined above satisfies the condition that it is bounded from below , we can apply the theorem (32.6) to the function G. Given any positive number ε there exist a point P at which it satisfies (3.2) and (3.3) , (3.4) the following relationship at P .

$$\frac{1-n}{2} G(P)^{\frac{2n}{n-1}} \Delta(f) \geq -\varepsilon G(P) - \frac{n+1}{n-1}\varepsilon^2$$

Can be derived , where $G(P)$ denotes $G(f\varphi)$ thus for any convergent sequence e $G_0 = \inf G$, by taking a sub sequence , if necessary because the sequence is bounded and therefore each term $G(P_m)$ of the sequence satisfies equation we have $G(P_m) \to G_0 = \inf G$ and the assumption $n \geq 1$.An the other hand it follows from (5.2) we have

$$\frac{1-n}{2} G(P_m)^{\frac{2n}{n-1}} \Delta(P_m) \geq -\varepsilon_m G(P_m) - \frac{n+1}{n-1}\varepsilon_m{}^2$$

And the right side of the a above inequality converges to zero because the function G is bounded by choosing the constant a it satisfies $C_{k+1} a^{-n} \leq C_0$, A accordingly there is a positive number δ such that $\frac{1-n}{2} C_{k+1} a^{-n} \leq \delta \leq \frac{n+1}{2} C_0$, C_0 is the constant coefficient of the maximal degree of function F so for a given such that $a\delta \geq 0$, we can take a sufficiently large integer m such that

(3.5) $$\frac{1-n}{2} G(P_m)^{\frac{2n}{n-1}} F(f(P_m)) \geq -\delta$$

Where we have used the assumption equation (3.2) of the theorem (3.2.6) and equation (3.4) so this inequality together with the definition of $G(P_m)$ Yield $F(f(P_m)) \leq \frac{2\delta}{n-1}(f+a)(P_m)^n$

Remark 3.2.8

Suppose that a nonnegative function f satisfies the condition we can directly yield $\nabla f^{\,n-1} = (n-1) f^{\,n-2} \nabla f$,

(3.6) $$\Delta f^{\,n-1} = (n-1)(n-2) f^{n-3} \nabla(f \nabla f) + (n-1) f^{n-2} \Delta f$$

we define a function h by $f^{\,n-1}$, if $n \geq 2$ then it satisfies $\Delta h \geq (n-1) C_0 h^2$ Thus concerning the theorem in the case $n \geq 2$ the condition (2.7) is equivalent $1 \leq n \leq 2$ where C_1 is a positive constant

3.2 [Geometric Maximum Principle on Riemannian Manifolds]

We now fix our sign conventions on the imbedding invariants of smooth hyper surfaces in Riemannian manifold (M,g). It will be convenient to assume that our hyper surfaces are the boundaries of open sets. An this a always true Locally it is not a restriction by ∇ let $D \subset M$ be connected open set and let $N \subset \partial D$, be part of all ∂D is smooth , let n be the outward pointing unit normal along N then the second fundamental form of N is symmetric bilinear form defined on the tangent space to N by $h^N(X,Y) = \langle \nabla_x, \nabla_n \rangle$. The mean curvature of N is then $H^N = \frac{1}{n-1} trace \,|_{g_N}$ and $h^N = \frac{1}{n-1} \sum_{i=1}^{n-1} h^N(e_i, e_i)$ where $(e_1, e_2, \ldots\ldots, e_{n-1})$ is local orthogonal from for $T(N)$ this is the sign convention so that for the boundary S^{n-1} of the unit ball β^n in R^n the second fundamental form $h^N = -g \,|_{S^N}$ is negative definite the mean curvature is $H^{s^{n-1}} = -1$.

3.2.1 Definition :[Hypersuface on Curvature $\geq H_0$]

Let U be an open set in the Riemannian manifold (M,g) then .(a) ∂U has mean curvature $\geq H_0$ in the sense of contact hypersurfaces iff for all $q \in \partial U$ and $\varepsilon \geq 0$ there is an open set D of M with $\bar{D} \subseteq \bar{U}$ and $q \in \partial D$ near q is a C hypersuface of M and at point q, $H_q^{\partial D} \geq H_0 - \varepsilon$.(b) ∂U has mean curvature $\geq H_0$ in the sense contact hypersurface is constant $C_k \geq 0$ so that for all $q \in k$ and $\varepsilon \geq 0$ there is open set D of M with $\bar{D} \subseteq \bar{U}$ and $q \in \partial D$ the of ∂D near q, $H_q^{\partial D} \geq H_0 - \varepsilon$ and also $H_q^{\partial D} \geq -C_{Kg} \,|_{\partial D}$.

3.2.3 Theorem

Let (M,g) be a Riemannian manifold $U_0, U_1 \subset M$ open sets and let H_0 be a constant , assume that .(a) $U_0 \cap U_1 = 0$ (b) ∂U_0 has mean curvature $\geq -H_0$ in the sense of contact hyper surfaces. (c) ∂U_1 has mean curvature $\geq H_0$ in the sense of contact hypersurfaces with a one sided Hessian bound .4. there is a point $P \in \bar{U}_0 \cap \bar{U}$ and a neighborhood N of P that has coordinates $(x^1, x^2, \ldots\ldots x^n)$ cantered at P so that for some $r \geq 0$ the image of these coordinates is the box $(x^1, x^2, \ldots\ldots x^n) = |x^i| \leq r$ and there are Lipschitz continues and there are Lipschitz continuous function $U_0, U_1 : \{ (x^1, x^2, \ldots\ldots\ldots, x^{n-1} : |x^i| \leq r) \}$, $(-r, r)$ so that $U_0 \cap N$ are given by

(3.7) $$U_0, N = \{ (x^1, x^2, \ldots\ldots, x^n) : x^n \geq U_0 (x^1, x^2, \ldots\ldots\ldots, x^{n-1}) \} \quad , \quad U_1, N = \{ (x^1, x^2, \ldots\ldots, x^n) : x^n \geq U_1 (x^1, x^2, \ldots\ldots\ldots, x^{n-1}) \}$$

This implies $U_0 \equiv U_1$ and U_0 is smooth function , therefore $\partial U_0 \ \cap \ N = \ \partial U_1 \cap N$ is a smooth embedded hyper surface with constant mean curvature H_0 (with respect to the outward normal to U_1).

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IV. DIFFERENTIABLE RIEMANNIAN METRIC

5.1 A differentiable Structures on Topological

A differentiable structures is topological is a manifold it an open covering U_α where each set U_α is homeomorphism, via some homeomorphism h_α to an open subset of Euclidean space R^n , let M be a topological space , a chart in M consists of an open subset $U \subset M$ and a homeomorphism h of U onto an open subset of R^m , a C^r atlas on M is a collection (U_α, h_α) of charts such that the U_α cover M and h_B, h_α^{-1} the differentiable vector fields on a differentiable manifold M . Let X and Y be a differentiable vector field on a differentiable manifolds M then there exists a unique vector field Z such that such that , for all $f \in D, Zf = (XY - YX)\ f$ if that $p \in M$ and let $x: U \to M$ be a parameterization at p and

$$\left(X = \sum_i a_i \frac{\partial}{\partial x_i} \right), \left(Y = \sum_j a_j \frac{\partial}{\partial y_j} \right)$$

(5.1) $$\left(XYf = X\ (\sum_i b_j \frac{\partial f}{\partial x_i}) \right)\ , \left(YXf = Y (\sum_j a_i \frac{\partial f}{\partial x_j} \right)$$

Therefore Z is given in the parameterization x by Z . $Z\ f = (X\ Y\ f - Y\ X\ f)$, $\sum_{i,j} (a_i \frac{\partial b_j}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j})$. Are differentiable this a regular surface is intersect from one to other can be made in a differentiable manner the defect of the definition of regular surface is its dependence on R^3 . A differentiable manifold is locally homeomorphism to R^n the fundamental theorem on existence , uniqueness and dependence on initial conditions of ordinary differential equations which is a local theorem extends naturally to differentiable manifolds .For familiar with differential equations can assume the statement below which is all that we need for example X be a differentiable on a differentiable manifold M and $p \in M$ then there exist a neighborhood $p \in M$ and $U_p \subset M$ an inter $(-\delta, \delta)$, $\delta \geq 0$, and a differentiable mapping $\varphi: (-\delta, \delta) \times U \to M$ such that curve $t \to \varphi(t, q)$ and $\varphi(0, q) = q$ a curve $\alpha: (-\delta, \delta) \to M$ which satisfies the conditions $\alpha^{-1}(t) = X\ (\alpha(t))$ and $\alpha\ (0) = q$ is called a trajectory of the field X that passes through q for $t = 0$.

Definition 5.1.1

A differentiable manifold of dimension N is a set M and a family of injective mapping $x_\alpha \subset R^n \to M$ of open sets $u_\alpha \in R^n$ into M such that (a) $u_\alpha x_\alpha(u_\alpha) = M$ (b) for any α, β with $x_\alpha(u_\alpha) \cap x_\beta(u_\beta)$ (c) the family (u_α, x_α) is maximal relative to conditions (a),(b) the pair (u_α, x_α) or the mapping x_α with $p \in x_\alpha(u_\alpha)$ is called a parameterization , or system of coordinates of M , $u_\alpha x_\alpha(u_\alpha) = M$ the coordinate charts (U, φ) where U are coordinate neighborhoods or charts , and φ are coordinate homeomorphisms transitions are between different choices of coordinates are called transitions maps .

$$\varphi_{i,j} : (\varphi_j\ o \varphi_i^{-1})$$

Which are anise homeomorphisms by definition , we usually write $x = \varphi(p), \varphi: U \to V \subset R^n$ collection U and $p = \varphi^{-1}(x), \varphi^{-1}: V \to U \subset M$ for coordinate charts with is $M = \cup U_i$ called an atlas for M of topological manifolds. A topological manifold M for which the transition maps $\varphi_{i,j} = (\varphi_j\ o \varphi_i)$ for all pairs φ_i, φ_j in the atlas are homeomorphisms is called a differentiable , or smooth manifold , the transition maps are mapping between open subset of R^m , homeomorphisms between open subsets of R^m are C^∞ maps whose inverses are also C^∞ maps , for two charts U_i and U_j the transitions mapping

(5.2) $$\varphi_{i,j} = (\varphi_j\ o \varphi_i^{-1}) : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

And as such are homeomorphisms between these open of R^m .

Example 5.1.2

the differentiability $(\varphi''\ o \varphi^{-1})$ is achieved the mapping $(\varphi''\ o (\tilde{\varphi})^{-1})$ and $(\tilde{\varphi}\ o \varphi^{-1})$ which are homeomorphisms since $(A \approx A'')$ by assumption this establishes the equivalence $(A \approx A'')$, for example let N and M be smooth manifolds n and m respecpectively , and let $f: N \to M$ be smooth mapping in local coordinates $f' = (\psi\ o\ f\ o \varphi^{-1}) : \varphi(U) \to \psi(V)$ with respects charts (U, φ) and

(V,ψ) , the rank of f at $p\in N$ is defined as the rank of f' at $\varphi(p)$ i.e $rk(f)_p = rk(J\ f')_{\varphi(p)}$ is the Jacobean of f at p this definition is independent of the chosen chart , the commutative diagram in that $f'' = (\psi' o\psi^{-1}) o \tilde{f}\, o(\varphi' o\varphi^{-1})^{-1}$ Since $(\psi' o\psi^{-1})$ and $(\varphi' o\varphi^{-1})$ are homeomorphisms it easily follows that which show that our notion of rank is well defined $(J\ f'')_{xj} = J(\psi' o\psi^{-1})_{yi}\ J\ f'(\varphi' o\varphi^{-1})^{-1}$, if a map has constant rank for all $p\in N$ we simply write $rk(f)$, these are called constant rank mapping.

Definition 5.1.3

The product two manifolds M_1 and M_2 be two C^k -manifolds of dimension n_1 and n_2 respectively the topological space $M_1\times M_2$ are arbitral unions of sets of the form $U\times V$ where U is open in M_1 and V is open in M_2 , can be given the structure C^k manifolds of dimension $n_1.n_2$ by defining charts as follows for any charts M_1 on (V_j,ψ_j) on M_2 we declare that $(U_i\times V_j\ ,\ \varphi_i\times\psi_j)$ is chart on $M_1\times M_2$ where $\varphi_i\times\psi_j : U_i\times V_j \to R^{(n_1+n_2)}$ is defined so that $\varphi_i\times\psi_j(p,q) = (\varphi_i(p)\ ,\ \psi_j(q))$ for all $(p,q)\in U_i\times V_j$. A given a C^k n-atlas, A on M for any other chart (U,φ) we say that (U,φ) is compatible with the atlas A if every map $(\varphi_i\, o\varphi^{-1})$ and $(\varphi\, o\varphi_i^{-1})$ is C^k whenever $U\cap U_i \neq 0$ the two atlases A and $\tilde{A}$ is compatible if every chart of one is compatible with other atlas.

Definition 5.1.4

A sub manifolds of others of R^n for instance S^2 is sub manifolds of R^3 it can be obtained as the image of map into R^3 or as the level set of function with domain R^3 we shall examine both methods below first to develop the basic concepts of the theory of Riemannian sub manifolds and then to use these concepts to derive a equantitive interpretation of curvature tensor , some basic definitions and terminology concerning sub manifolds, we define a tensor field called the second fundamental form which measures the way a sub manifold curves with the ambient manifold , for example X be a sub manifold of Y of $\pi : E\to X$ and $g: E_1\to Y$ be two vector brindled and assume that E is compressible , let $f: E\to Y$ and $g: E_1\to Y$ be two tubular neighborhoods of X in Y then there exists a C^{p-1}.

Definition 5.1.5

Let M_1 and M_2 be differentiable manifolds a mapping $\varphi: M_1\to M_2$ is a differentiable if it is differentiable , objective and its inverse φ^{-1} is diffeomorphism if it is differentiable φ is said to be a local diffeomorphism at $p\in M$ if there exist neighborhoods U of p and V of $\varphi(p)$ such that $\varphi: U\to V$ is a diffeomorphism , the notion of diffeomorphism is the natural idea of equivalence between differentiable manifolds , its an immediate consequence of the chain rule that if $\varphi: M_1\to M_2$ is a diffeomorphism then $d\varphi: T_pM_1\to T_{\varphi(p)}M_2$. Is an isomorphism for all $\varphi: M_1\to M_2$ in particular , the dimensions of M_1 and M_2 are equal a local converse to this fact is the following $d\varphi: T_pM_1\to T_{\varphi(p)}M_2$ is an isomorphism then φ is a local diffeomorphism at p from an immediate application of inverse function in R^n , for example be given a manifold structure again A mapping $f^{-1}: M\to N$ in this case the manifolds N and M are said to be homeomorphism , using charts (U,φ) and (V,ψ) for N and M respectively we can give a coordinate expression $\tilde{f}: M\to N$

Example 5.1.6

Let M_1^{-1} and M_2^{-1} be differentiable manifolds and let $\varphi: M_1\to M_2$ be differentiable mapping for every $p\in M_1$ and for each $v\in T_pM_1$ choose a differentiable curve $\alpha: (-\varepsilon,\varepsilon)\to M_1$ with $\alpha(M) = p$ and $\alpha'(0) = v$ take $\alpha\, o\beta = \beta$ the mapping $d\varphi_p : T_\varphi(p)M_2$ by given by $d\varphi(v) = \beta'(M)$ is line of α and $\varphi: M_1^{-1}\to M_2^{-1}$ be a differentiable mapping and at $p\in M_1$ be such $d\varphi: T_pM_1\to T_\varphi M_2$ is an isomorphism then φ is a local homeomorphism

Theorem 5.1.7

Let G be lie group of matrices and suppose that log defines a coordinate chart the near the identity element of G , identify the tangent space $g = T_1G$ at the identity element with a linear subspace of matrices , via the log and then a lie algebra with $[B_1, B_2] = B_1B_2 - B_2B_1$ the space g is called the lie algebra of G .

Proof:

It suffices to show that for every two matrices $B_1, B_2 \in g$ the $[B_1, B_2]$ is also an element of g as $[B_1, B_2]$ is clearly anti commutative and the (Jacobs identity) holds for $A(t)=(B_1 t)_{\exp}(B_2 t)_{\exp}(-B_1 t)_{\exp}(-B_2 t)_{\exp}$. Define for $[t]\le \varepsilon$ with sufficiently small ε a path $A(T)$ in G such that $A(O)=I$ using for each factor the local formula

(5.3) $$(Bt)_{\exp}=I+Bt+1/2B^2t^2+O(t^2) \qquad A(t)=I+[B_1,B_2]\ t^2+O(t), t\to 0$$

hence $B(t)=\log A(t)=[B_1,B_2]t^2+O(t^2)$ exp $B(t)=A(t)$ hold for any sufficiently that lie bracket $[B_1,B_2]\in g$ on algebra is an infinitesimal version of the commutation $(g_1,g_1)(g_1^{-1},g_2^{-1})$ in the corresponding (lie group).

Theorem 5.1.8

The tangent bundle TM has a canonical differentiable structure making it into a smooth 2N-dimensional manifold , where N=dim. The charts identify any $U_p \in U(T_pM)\subseteq (TM)$ for an coordinate neighborhood $U\subseteq M$, with $U\times R^n$ that is hausdorff and second countable is called (The manifold of tangent vectors)

Definition 5.1.9

A smooth vectors fields on manifolds M is map $X: M\to TM$ such that . (a) $X(P)\in T_pM$ for every G (b) in every chart X is expressed as $a_i\ (\partial/\partial x_i)$ with coefficients $a_i(x)$ smooth functions of the local coordinates x_i .

5.2 Differentiable manifolds tangent space $T_p(M)$

In this sub section is defined tangent space to level surface γ be a curve is in R^n , $\gamma: t\to \left(\gamma^1(t),\gamma^2(t),\ldots,\gamma^n(t)\right)$ a curve can be described as vector valued function converse a vector valued function given curve , the tangent line at the point $T_p(M)$ $\frac{d\gamma}{dt}(t)=\left(\frac{d\gamma^1}{dt}t_0,\ldots,\frac{d\gamma^n}{dt}t_0\right)$ we many k bout smooth curves that is curves with all continuous higher derivatives cons the level surface $f\left(x^1,x^2,\ldots,x^n\right)=c$ of a differentiable function f where x^i to $i-th$ coordinate the gradient vector of f at point $P=x^1(P),x^2(P),\ldots,x^n(P)$ is $\nabla f=\left(\frac{\partial f}{\partial x^1},\ldots,\frac{\partial f}{\partial x^n}\right)$ is given a vector $u=(u^1,\ldots,u^n)$ the direction derivative $D_u f=\nabla f\cdot \bar{u}=\frac{\partial f}{\partial x^1}u^1+\ldots+\frac{\partial f}{\partial x^n}u^n$, the point P on level surface $f\left(x^1,x^2,\ldots,x^n\right)$ the tangent is given by equation.

(5.6) $$\frac{\partial f}{\partial x^1}(P)(x^1-x^1)(P)+\ldots+\frac{\partial f}{\partial x^n}(P)(x^n-x^n)(P)=0$$

For the geometric views the tangent space shout consist of all tangent to smooth curves the point P , assume that is curve through $t=t_0$ is the level surface $f\left(x^1,x^2,\ldots,x^n\right)=c$, $f\left(\gamma^1(t),\gamma^2(t),\ldots,\gamma^n(t)\right)=c$ by taking derivatives on both $\frac{\partial f}{\partial x^1}(P)\ (\gamma'(t_0)+\ldots+\frac{\partial f}{\partial x^n}(P)\gamma^n(t))=0$ and so the tangent line of γ is really normal orthogonal to ∇f , where γ runs over all possible curves on the level surface through the point P . The surface M be a C^∞ manifold of dimension n with $k\ge 1$ the most intuitive to define tangent vectors is to use curves , $p\in M$ be any point on M and let $\gamma:]-\varepsilon,\varepsilon[\to M$ be a C^1 curve passing through p that is with $\gamma(M)=p$ unfortunately if M is not embedded in any R^N the derivative $\gamma'(M)$ does not make sense ,however for any chart (U,φ) at p the map $(\varphi o\gamma)$ at a C^1 curve in R^n and tangent vector $v=(\varphi o v)'(M)$ is will defined the trouble is that different curves the same v given a smooth mapping $f: N\to M$ we can define how tangent vectors in T_pN are mapped to tangent vectors in T_qM with (U,φ) choose charts $q=f(p)$ for $p\in N$ and (V,ψ) for $q\in M$ we define the tangent map or flash-forward of f as a given tangent vector $X_p=[\gamma]\in T_pN$ and $d\ f_*: T_pM$, $f_*([\gamma]=[f o\gamma])$.

A tangent vector at a point p in a manifold M is a derivation at p , just as for R^n the tangent at point p form a vector space $T_p(M)$ called the tangent space of M at p , we also write $T_p(M)$ a differential of map $f: N\to M$ be a C^∞ map between two manifolds at each point $p\in N$ the map F induce a linear map of tangent space called its differential at p , $F_*: T_pN\to T_{F(p)}N$ as follows it $X_p\in T_pN$ then $F_*(X_p)$ is the tangent vector in $T_{F(p)}M$ defined .

(5.7) $$\left(F_*(X_p)\right)f=X_p\left(f oF\right)\in R \quad , \quad f\in C^\infty(M)$$

The tangent vectors given any C^{∞}-manifold M of dimension n with $k \geq 1$ for any $p \in M$,tangent vector to M at p is any equivalence class of C^{1}-curves through p on M modulo the equivalence relation defined in the set of all tangent vectors at p is denoted by T_pM we will show that T_pM is a vector space of dimension n of M . The tangent space T_pM is defined as the vector space spanned by the tangents at p to all curves passing through point p in the manifold M , and the cotangent T_p^*M of a manifold at $p \in M$ is defined as the dual vector space to the tangent space T_pM , we take the basis vectors $E_i = \left(\frac{\partial}{\partial x^i}\right)$ for T_pM and we write the basis vectors T_p^*M as the differential line elements $e^i = dx^i$ thus the inner product is given by $\langle \partial / \partial x, dx^i \rangle = \delta_i^j$.

Theorem 5.2.1 tangent bundle TM

The tangent bundle TM has a canonical differentiable structure making it into a smooth 2N-dimensional manifold , where N=dim. The charts identify any $U_p \in U(T_pM) \subseteq (TM)$ for an coordinate neighborhood $U \subseteq M$, with $U \times R^n$ that is hausdorff and second countable is called (The manifold of tangent vectors)

Definition 5.2.2

A smooth vectors fields on manifolds M is map $X : M \to TM$ such that (a) $X(P) \in T_pM$ for every G (b) in every chart X is expressed as $a_i\,(\partial / \partial x_i)$ with coefficients $a_i(x)$ smooth functions of the local coordinates x_i .

Theorem 5.2.3 Isomorphic T_pM

Suppose that on a smooth manifold M of dimension n there exist n vector fields $\left(x^{(1)}, x^{(2)}, \ldots, x^{(n)}\right)$ for a basis of T_pM at every point p of M , then T_pM is isomorphic to $M \times R^n$ m here isomorphic means that TM and $M \times R^n$ are homeomorphism as smooth manifolds and for every $p \in M$, the homeomorphism restricts to between the tangent space T_pM and vector space $\{P_i\} \times R^n$.

Proof:

define $\pi : \overset{\rho}{a} \in T_pM \subset TM$ on other hand , for any $M \times R^n$ for some $a_i \in R$ now define

$\Phi : \overset{\rho}{a} \in TM \to \left[\pi(\overset{\rho}{s}) : (a_1, \ldots, a_n) \in M \times R^n\right]$ is it clear from the construction and the hypotheses of theorem that Φ and Φ^{-1} are smooth using an arbitrary chart $\varphi : U \subseteq M \to R^n$ and corresponding chart.

(5.8) $$\varphi T : \pi^{-1}(U) \subseteq TM \to R^n \times R^m$$

V. CONCLUSION

The paper study Riemannian differenterentiable manifolds is a generalization of curves and surfaces , locally Euclidean E^n in every point has a neighbored is called a chart homeomorphic , so that many concepts from R^n as differentiability manifolds. We give the basic definitions, theorems and properties of smooth topological be comes a manifold is to exhibit a collection of C^{∞} is compatible charts . The tangent , cotangent vector space manifold of dimension k with $k \geq 1$ the most intuitive method to define tangent vectors to use curves , tangent space $T_p M$ and tangent space at some point $p \in M$ the cotangent $T_p^* M$ is defines as dual vector space of $p \in M$.

APPENDIX

Appendixes, if needed, appear before the acknowledgment.

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The preferred spelling of the word "acknowledgment" in American English is without an "e" after the "g." Use the singular heading even if you have many acknowledgments.

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