NONHOLONOMIC FRAMES FOR FINSLER SPACE WITH SPECIAL (α, β)-METRICS

NARASIMHAMURTHY S.K., MALLIKARJUN Y. KUMBAR., KAVYASHREE A.R.
Department of P.G. Studies and Research in Mathematics,
Kuvempu University, Shankaraghatta - 577451,
Shimoga, Karnataka, INDIA
E-mail:nmurthysk@hotmail.com and mallikarjunykumbar@gmail.com

Abstract: The main aim of this paper is to, first we determine the two Finsler deformations to the special Finsler (α, β)-metrics. Consequently, we obtain the nonholonomic frame for the special (α, β)-metrics, such as $L = \alpha + \beta + \frac{\beta^2}{\alpha}$ (I[st] Approximate Matsumoto metric) and $L = \frac{\beta^2}{\beta - \alpha}$ (Infinite series (α, β)-metric).

Key Words: *Finsler Space, (α, β)-metrics, GL-metric, Nonholonomic Finsler frame.*

1. INTRODUCTION

In 1982, P.R. Holland ([1][2]), studies a unified formalism that uses a nonholonomic frame on space-time arising from consideration of a charged particle moving in an external electromagnetic field. In fact, R.S. Ingarden [3] was first to point out that the Lorentz force law can be written in this case as geodesic equation on a Finsler space called Randers space. The author Beil R.G. ([5][6]), have studied a gauge transformation viewed as a nonholonomic frame on the tangent bundle of a four dimensional base manifold. The geometry that follows from these considerations gives a unified approach to gravitation and gauge symmetries. The above authors used the common Finsler idea to study the existence of a nonholonomic frame on the vertical subbundle *V TM* of the tangent bundle of a base manifold *M*.

In this paper, the fundamental tensor field might be taught as the result of two Finsler deformation. Then we can determine a corresponding frame for each of these two Finsler deformations. Consequently, a nonholonomic frame for a Finsler space with special (α, β)-metrics such as Ist Approximate Matsumoto metric and Infinite series (α, β)-metrics will appear as a product of two Finsler frames formerly determined. This is an extension work of Ioan Bucataru and Radu Miron [10].

Consider $a_{ij}(x)$, the components of a Riemannian metric on the base manifold *M*, $a(x, y) > 0$ and $b(x, y) \geq 0$ two functions on *TM* and $B(x, y) = B_i(x, y)dx^i$ a vertical 1-form on *TM*. Then

$$g_{ij}(x, y) = a(x, y)a_{ij}(x) + b(x, y)B_i(x, y)B_j(x, y) \tag{1.1}$$

is a generalized Lagrange metric, called the *Beil metric* . We say also that the metric tensor g_{ij} is a *Beil deformation* of the Riemannian metric a_{ij}. It has been studied and applied by R.Miron and R.K. Tavakol in General Relativity for $a(x,y) = exp(2\sigma(x,y))$ and b = 0. The case $a(x, y)$ = 1 with various choices of b and B_i was introduced and studied by R.G. Beil for constructing a new unified field theory [6].

2. PRELIMINARIES

An important class of Finsler spaces is the class of Finsler spaces with (α,β)-metrics [11]. The first Finsler spaces with (α,β)-metrics were introduced by the physicist G.Randers in 1940, are called Randers spaces [4]. Recently, R.G. Beil suggested to consider a more general case, the class of Lagrange spaces with (α,β)-metric, which was discussed in [12]. A unified formalism which uses a nonholonomic frame on space time, a sort of plastic deformation, arising from consideration of a charged particle moving in an external electromagnetic field in the background space time viewed as a strained mechanism studied by P.R.Holland [1][2]. If we do not ask for the function L to be homogeneous of order two with respect to the (α,β) variables, then we have a *Lagrange space with* (α,β)*-metric*. Next we look for some different Finsler space with (α,β)-metrics.

Definition 2.1: A Finsler space $F^n = \big(M, F(x,y)\big)$ is called with (α,β)-metric if there exists a 2-homogeneous function L of two variables such that the Finsler metric $F: TM \to R$ is given by,

$$F^2(x,y) = L\big(\alpha(x,y), \beta(x,y)\big),$$

Where $\alpha^2(x,y) = a_{ij}(x)y^i y^j$, α is a Riemannian metric on M, and (2.1)

$\beta(x,y) = b_i(x)y^i$ is a 1-form on M.

Consider $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ the fundamental tensor of the Randers space (M,F). Taking into account the homogeneity of α and F we have the following formulae:

$$p^i = \frac{1}{\alpha}y^i = a^{ij}\frac{\partial \alpha}{\partial y^j} \quad ; \quad p_i = a_{ij}p^j = \frac{\partial \alpha}{\partial y^i} \;;$$

$$l^i = \frac{1}{L}y^i = g^{ij}\frac{\partial L}{\partial y^j} \quad ; \; l_i = g_{ij}l^j = \frac{\partial L}{\partial y^i} = P_i + b_i; \tag{2.2}$$

$$l^i = \frac{1}{L}\,p^i \;; \quad l^i l_i = P^i p_i = 1; \; l^i p_i = \frac{\alpha}{L}; \quad p^i l_i = \frac{L}{\alpha};$$

$$b_iP^i = \frac{\beta}{\alpha} \;; \quad b_il^i = \frac{\beta}{L} \;.$$

With respect to these notations, the metric tensors (a_{ij}) and (g_{ij}) are related by [13],

$$g_{ij}(x,y) = \frac{L}{\alpha}a_{ij} + b_iP_j + P_ib_j + b_ib_j - \frac{\beta}{\alpha}\,p_ip_j \;= \frac{L}{\alpha}\left(a_{ij} - p_ip_j\right) + l_il_j. \qquad (2.3)$$

Theorem 2.1[10]: *For a Finsler space (M,F) consider the matrix with the entries:*

$$Y_j^i = \sqrt{\frac{\alpha}{L}}\left(\delta_j^i - l^il_j + \sqrt{\frac{\alpha}{L}}\;p^ip_j\right) \qquad (2.4)$$

defined on TM. Then $Y_j = Y_j^i\left(\frac{\partial}{\partial y^i}\right)$, $j \in \{1,2,\dots,n\}$ is an nonholonomic frame.

Theorem 2.2 [7]: *With respect to frame the holonomic components of the Finsler metric tensor $\left(a_{\alpha\beta}\right)$ is the Randers metric (g_{ij}),i.e,*

$$g_{ij} = Y_i^\alpha Y_j^\beta a_{\alpha\beta}. \qquad (2.5)$$

Throughout this section we shall rise and lower indices only with the Riemannian metric $a_{ij}(x)$ that is $y_i = a_{ij}y^j$, $b^i = a^{ij}b_j$, and so on. For a Finsler space with (α,β)-metric $F^2(x,y) = L\big(\alpha(x,y),\beta(x,y)\big)$ we have the Finsler invariants [13].

$$\rho_1 = \frac{1}{2\alpha}\frac{\partial L}{\partial \alpha}; \quad \rho_0 = \frac{1}{2}\frac{\partial^2 L}{\partial \beta^2}\;; \quad \rho_{-1} = \frac{1}{2\alpha}\frac{\partial^2 L}{\partial\alpha\partial\beta}\;; \quad \rho_{-2} = \frac{1}{2\alpha^2}\left(\frac{\partial^2 L}{\partial\alpha^2} - \frac{1}{\alpha}\frac{\partial L}{\partial\alpha}\right) \qquad (2.6)$$

where subscripts 1, 0, -1, -2 gives us the degree of homogeneity of these invariants.

For a Finsler space with (α,β)-metric we have,

$$\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0 \qquad (2.7)$$

with respect to the notations we have that the metric tensor g_{ij} of a Finsler space with (α,β)-metric is given by [13].

$$g_{ij}(x,y) = \rho a_{ij}(x) + \rho_0 b_i(x) + \rho_{-1}\big(b_i(x)y_j + b_j(x)y_i\big) + \rho_{-2}y_iy_j. \qquad (2.8)$$

From (2.8) we can see that g_{ij} is the result of two Finsler deformations:

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$$\left.\begin{aligned} &\text{i)} \quad a_{ij} \to h_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j) \\ &\text{ii)} \quad h_{ij} \to g_{ij} = h_{ij} + \frac{1}{\rho_{-2}}(\rho_0\rho_{-2} - \rho_{-1}^2)b_ib_j. \end{aligned}\right\} \quad (2.9)$$

The nonholonomic Finsler frame that corresponding to the 1^{st} deformation (2.9) is according to the theorem (7.9.1) in [10], given by,

$$X_j^i = \sqrt{\rho}\,\delta_j^i - \frac{1}{\beta^2}\left(\sqrt{\rho} + \sqrt{\rho + \frac{\beta^2}{\rho_{-2}}}\right)(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b^i + \rho_{-2}y^j) \quad (2.10)$$

where $B^2 = a_{ij}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b^i + \rho_{-2}y^j) = \rho_{-1}^2b^2 + \beta\rho_{-1}\rho_{-2}$

This metric tensor a_{ij} and h_{ij} are related by,

$$h_{ij} = X_i^kX_j^la_{kl}. \quad (2.11)$$

again the frame that corresponds to the II^{nd} deformation (2.9) given by,

$$Y_j^i = \delta_j^i - \frac{1}{c^2}\left(1 \pm \sqrt{1 + \left(\frac{\rho_{-2}c^2}{\rho_0\rho_{-2} - \rho_{-1}^2}\right)}\right)b^ib_j, \quad (2.12)$$

where $C^2 = h_{ij}b^ib^j = \rho b^2 + \frac{1}{\rho_{-2}}(\rho_{-1}b^2 + \rho_{-2}\beta)^2$.

The metric tensor h_{ij} and g_{ij} are related by the formula;

$$g_{mn} = Y_m^iY_n^jh_{ij}. \quad (2.13)$$

Theorem 2.3: [10] *Let $F^2(x,y) = L(\alpha(x,y),\beta(x,y))$ be the metric function of a Finsler space with (α,β)-metric for which the condition (2.7) is true. Then*

$$V_j^i = X_k^iY_j^k$$

is a nonholonomic Finsler frame with X_k^i and Y_j^k are given by (2.10) and (2.12) respectively.

3. NONHOLONOMIC FRAMES FOR FINSLER SPACE WITH SPECIAL (α,β)-METRICS

In this section we consider two Finlser space with special (α,β)-metrics, such as Ist Approximate Matusmoto metric and Infinite series (α,β)-metric, then we construct nonholonomic Finsler frames.

3.1. NONHOLONOMIC FRAME FOR $L = \left(\alpha + \beta + \frac{\beta^2}{\alpha}\right)^2$:

In the first case, for a Finsler space with the fundamental function $L = \left(\alpha + \beta + \frac{\beta^2}{\alpha}\right)^2$, the Finsler invariants (2.6) are given by:

$$\left.\begin{aligned} \rho_1 &= \frac{(\alpha^2+\alpha\beta+\beta^2)(\alpha^2-\beta^2)}{\alpha^4}, \quad \rho_0 = \frac{3(\alpha^2+2\alpha\beta+2\beta^2)}{\alpha^2}, \\ \rho_{-1} &= \frac{(\alpha^3-3\alpha\beta^2-4\beta^3)}{\alpha^4}, \quad \rho_{-2} = -\frac{\beta(\alpha^3-3\alpha\beta^2-4\beta^3)}{\alpha^6}, \\ B^2 &= \frac{(\alpha^3-3\alpha\beta^2-4\beta^3)^2(b^2\alpha^2-\beta^2)}{\alpha^{10}}. \end{aligned}\right\} \quad (3.1)$$

Using (3.1) in (2.10) we have,

$$X_j^i = \sqrt{\frac{(\alpha^2+\alpha\beta+\beta^2)(\alpha^2-\beta^2)}{\alpha^4}}\,\delta_j^i - \frac{\alpha^4}{b^2\alpha^2-\beta^2}\left\{\sqrt{(\alpha^2+\alpha\beta+\beta^2)(\alpha^2-\beta^2)} \pm \sqrt{(\alpha^2+\alpha\beta+\beta^2)(\alpha^2-\beta^2) - \frac{\alpha^3-3\alpha\beta^2-4\beta^3}{\beta}}\right\}$$

$$\left(b^i - \frac{\beta y^i}{\alpha^2}\right)\left(b_j - \frac{\beta y_j}{\alpha^2}\right); \quad (3.2)$$

Again using (3.1) in (2.12) we have,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\left(1 \pm \sqrt{1 + \frac{\alpha^2\beta C^2}{\alpha^3+3\alpha\beta(\alpha+\beta)+2\beta^3}}\right) b^i b_j; \quad (3.3)$$

where $$C^2 = \frac{(\alpha^6+\alpha\beta+\beta^2)(\alpha^2-\beta^2)b^2}{\alpha^4} - \frac{(\alpha^3-3\alpha\beta^2-4\beta^3)(b^2\alpha^2-\beta^2)^2}{\alpha^6\beta}.$$

Theorem 3.1: *Consider a Finsler space $L = \left(\alpha + \beta + \frac{\beta^2}{\alpha}\right)^2$, for which the condition (2.7) is true. Then*

$$V_j^i = X_k^i Y_j^k$$

is a nonholonomic Finsler frame with X_k^i and Y_j^k are given by (3.2) and (3.3) respectively.

3.2. NONHOLONOMIC FRAME FOR $L = \frac{\beta^4}{(\beta-\alpha)^2}$:

In the second case, for a Finsler space with the fundamental function $L = \frac{\beta^4}{(\beta-\alpha)^2}$, the Finsler invariants (2.6) are given by:

$$\left.\begin{array}{c} \rho_1 = \dfrac{\beta^4}{\alpha(\beta-\alpha)^3}, \quad \rho_0 = \dfrac{\beta^2(\beta^2-4\alpha\beta+6\alpha^2)}{(\alpha-\beta)^4}, \\ \rho_{-1} = \frac{\beta^3(\beta-4\alpha)}{(\alpha-\beta)^4}, \quad \rho_{-2} = \frac{\beta^4(4\alpha-\beta)}{\alpha^3(\alpha-\beta)^4}, \\ B^2 = \dfrac{\beta^6(4\alpha-\beta)^2(b^2\alpha^2-\beta)}{\alpha^4(\alpha-\beta)^8}. \end{array}\right\} \tag{3.4}$$

Using (3.4) in (2.10) we have,

$$X_j^i = \sqrt{\frac{\beta^4}{\alpha(\beta-\alpha)^3}}\,\delta_j^i - \frac{\alpha^2}{b^2\alpha^2-\beta^2}\left\{\sqrt{\frac{\beta^4}{\alpha(\beta-\alpha)^3}} \pm \sqrt{\frac{\beta^2(2\beta^3-5\alpha\beta^2+4b^2\alpha^3-b^2\alpha^2\beta)}{\alpha(\alpha-\beta)^4}}\right\}$$

$$\left(b^i - \frac{\beta y^i}{\alpha^2}\right)\left(b_j - \frac{\beta y_j}{\alpha^2}\right); \tag{3.5}$$

Again using (3.4) in (2.12) we have,

$$Y_j^i = \delta_j^i - \frac{1}{C^2}\left(1 \pm \sqrt{1 + \frac{(\alpha-\beta)^3 C^2}{\alpha^4}}\right) b^i b_j; \tag{3.6}$$

where $$C^2 = \frac{\alpha^2(\alpha-2\beta)b^2}{(\alpha-\beta)^3} - \frac{(\alpha-4\beta)\left(b^2\alpha^2-\beta^2\right)^2}{\beta(\alpha-\beta)^4}.$$

Theorem 3.2: *Consider a Finsler space $L = \frac{\beta^4}{(\beta-\alpha)^2}$, for which the condition (2.7) is true. Then*

$$V_j^i = X_k^i Y_j^k$$

is a nonholonomic Finsler frame with X_k^i and Y_j^k are given by (3.5) and (3.6) respectively.

4. CONCLUSION

Nonholonomic frame relates a semi-Riemannian metric (the Minkowski or the Lorentz metric) with an induced Finsler metric. Antonelli P.L., Bucataru I. ([7][8]), has been determined such a nonholonomic frame for two important classes of Finsler spaces that are dual in the sense of Randers and Kropina spaces [9]. As Randers and Kropina spaces are members of a bigger class of Finsler spaces, namely the Finsler spaces with (α, β)-metric, it appears a natural question: *Does how many Finsler space with (α, β)-metrics have such a nonholonomic frame? The answer is yes, there are many Finsler space with (α, β)-metrics.*

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In this work, we consider the two special Finsler metrics and we determine the nonholonomic Finsler frames. Each of the frames we found here induces a Finsler connection on TM with torsion and no curvature. But, in Finsler geometry, there are many (α, β)-metrics, in future work we can determine the frames for them also.

5. REFERENCES

1) Holland. P.R.: *Electromagnetism, Particles and Anholonomy.* Physics Letters, 91 (6), 275-278 (1982).
2) Holland, P.R.: *Anholonomic deformations in the ether: a significance for the electrodynamic potentials.* In: Hiley, B.J. Peat, F.D. (eds.), *Quantum Implications.* Routledge and Kegan Paul, London and New York, 295-311 (1987).
3) Ingarden, R.S.: *On Physical interpretations of Finsler and Kawaguchi spaces.* Tensor N.S., 46, 354-360 (1987).
4) Randers, G.: *On asymmetric metric in the four space of general rela-tivity*. Phys. Rev., 59, 195-199 (1941).
5) Beil, R.G.: *Comparison of unified field theories.* Tensor N.S., 56, 175-183 (1995).
6) Beil, R.G.: *Equations of Motion from Finsler Geometric Methods.* In: Antonelli, P.L. (ed), *Finslerian Geometries. A meeting of minds.* Kluwer Academic Publisher, FTPH, no. 109, 95-111 (2000).
7) Antonelli, P.L., Bucataru, I.: *On Holland's frame for Randers space and its applications in physics.* In: Kozma, L. (ed), *Steps in Differential Geometry. Proceedings of the Colloquium on Differential Geometry*, Debrecen, Hungary, July 25-30, 2000. Debrecen: Univ. Debrecen, Institute of Mathematics and Informatics, 39-54 (2001).
8) Antonelli, P.L., Bucataru, I.: *Finsler connections in anholonomic geometry of a Kropina space.* Nonlinear Studies, 8 (1), 171-184 (2001).
9) Hrimiuc, D., Shimada, H.: *On the L-duality between Lagrange and Hamilton manifolds.* Nonlinear World, 3, 613-641 (1996).
10) Ioan Bucataru, Radu Miron: *Finsler-Lagrange Geometry*. *Applications to dynamical systems* CEEX ET 3174/2005-2007 & CEEX M III 12595/2007 (2007).
11) Matsumoto, M.: *Theory of Finsler spaces with (α, β)-metric.* Rep. Math. Phys., 31, 43-83 (1992).
12) Bucataru, I.: *Nonholonomic frames on Finsler geometry.* Balkan Journal of Geometry and its Applications, 7 (1), 13-27 (2002).
13) Matsumoto.M: *Foundations of Finsler geometry and special Finsler spaces*, Kaishesha Press, Otsu, Japan, 1986.