# MATRIX MODELS, STRING WORLD SHEET DUALITY, AND OTHERS

## A MODEL AD CONSEQUENTIAM A GESTALT GESAMTKUNSTWERK

Dr. K.N.PRASANNA KUMAR
PROF.B.S.KIRANAGI
PROF.C.S. BAGEWADI

Publishing Partner:
**IJSRP Inc.**
**ijsrp.org**

ISSN 2250-3153

# Preface

The observed particles can be classified into two main classes, according to whether they are affected by nuclear ("strong") forces: Hadrons are, leptons aren't. There is a second, independent way of dividing up the particles, according to spin or statistics: Bosons can occupy the same space, and have integral spin (0, 1...), while fermions can't, and have half-integral spin (1/2, 3/2...). On a less technical level, fermions can be thought of as "matter", while bosons are the "energy" that mediates the interactions between the fermions. For example, an atom consists of a nucleus made up of baryons (the fermionic hadrons), namely the nucleons (protons and neutrons), and also electrons (a kind of fermionic lepton); the nucleons are held together by mesons (the bosonic hadrons), mostly pions, while the electrons are held in to the nucleus by photons (a kind of bosonic lepton). Hadrons can be treated as made up of yet other particles, which haven't been observed freely: Three quarks (fermions) are held together by gluons (bosons) to form a baryon, while two quarks (really a quark and an antiquark) plus gluons make a meson. The reason why hadrons are treated this way is that, unlike the (known) leptons, they resemble atoms in that they appear in related forms that differ only by being "excited" to different energy levels. However, unlike atoms, which fall apart if you hit them hard enough (very hard if you want to break the nucleus), quarks and gluons have never been broken off of hadrons (except as parts of newly created hadrons). The interpretation is then that the potential well describing the force between the quarks, unlike that for electrons in atoms (or bodies in the Earth's gravitational pull), rises infinitely high on the sides, so the quarks can never escape no matter how fast or far they travel. This quark-gluon theory of hadrons is called "(quantum) chromodynamics" (QCD). Nuclei are usually thought of as bound states of nucleons, but they can probably be described better as bounds states of many quarks and gluons, which at high temperature can form a "quark-gluon plasma". For example, an isolated neutron decays, but inside a helium nucleus it is stable: The helium nucleus is thus more of a particle than the neutron is. Details of the observed "energy levels" of hadrons show they have the same form as those for the vibrational modes of a string. The QCD interpretation is that gluons, because they interact not only with quarks but also with themselves (unlike photons), tend to condense into tubes of flux, which act like strings. The string is also a relatively simple model, both calculation ally and interpretation ally, and so is a useful approximation to any finite-size generalization of particles. In particular, the simplest string models automatically have a graviton (the boson responsible for gravity), and solve some of the problems found when attempting to describe the graviton as just a particle. Consequently string theory is now the most popular method for describing quantum gravity. Another consequence of string theory, although it is also a feature of some particle theories, is that it unifies all the known particles by treating them as different varieties of the same particle. In particular, string theory implies super symmetry, which relates bosons to fermions.

**Acknowledgements:** It is to be stated in unmistakable and unequivocal terms, that the literary expatiation, predication integrity, Introductory remarks, character constitution, ontological consonance primordial exactitude, Acolytish representation, atrophied asseveration, essential predications, consummate abstractions, rational representation, conferential extrinsicness, manifestation of histories, standard remarks, professed developments(Google Search) interfacial interference and syncopated justices, are taken from various sources. Such as, Wikipedia, author's Home Page, ask a Physicist Column, Abstracts of articles of various authors, papers of various authors, Web Graphs, Google search photographs, Google search results and other sources which included literally dialectic deliberation, polemical argumentation, Conjugatory confatalia.

We have to state that I have put all concerted efforts, sustained struggles, and protracted endeavour to mention each and every source at the cross reference or at the reference list at the end. In the eventuality of any act of omission or commission it is to be stated that such an eventuality has occurred attributable to inadvertence and in deliberation and I beg professedly, profusely, assiduously, and avidly with all fervour and can dour the persons concerned. I am not presenting any panacea for all the ills despite the penance done for there for, and it is attributable and ascribable to the fact that many highly esteemed and eminent persons allowed us to piggy ride on their backs I have been able to write summarily and expressly this paper. Explanation and deliberation of concatenation equations are done in the next paper. Towards the end of consummation, consolidations, concretization, reinforcement, revitalization, rejuvenation, resurrection, and consubstantiation of this mammoth project, singlehandedly I have gone through millions of pages and drafted and typed myself, and if by chance there are any repetition, I make a sincere entreat, earnest beseech, and fervent appeal and obsequesial consecration to kindly pardon me on that score.

**Following delineation and dissemination of information with respect to the subject matter captioned is formatted and consolidated, consubstantiate, concertized and reinforced based on Web pages, Web Graph, Encyclopaedia and other sources.**

In Pb+Pb collisions, we see similar features, depending on the momentum of the particles we look at, but the situation becomes more complicated as we throw new effects into the mix: jet quenching, hydrodynamics, and fluctuations. These are the things that hold keys to the physics we are interested in. In the next post we will explore further how the picture is changed when we correlate particles from Pb+Pb collisions.

Following are the parameters taken in to consideration:

1. Matrix models
2. String world sheet duality
3. R^4 corrections to heterotic M-theory
4. Twistor transform of all tree amplitudes in SYM theory
5. Ghost-free string effective actions
6. Nonunitary conformal field theories
7. Perturbative string S-matrix

ISSN 2250-3153


8. Supermatrix models for M-theory
9. Fractal space signatures in quantum physics and cosmology—I. Space, time, matter, fields and gravitation
10. Logarithmic conformal field theories
11. Brane constructions, conifold and M-theory
12. Topological quantum order
13. Two-dimensional dilaton gravity black hole solution
14. Nonlinear dynamic behavior OF PARTICLES
15. Nonabelian noncommutative gauge theory
16. Noncommutative extra dimensions
17. Heterotic M-theory in five dimensions
18. Complexity theory interpretation of high energy particle physics

# Copyright and Trademarks

# Authors

**Dr. K. N.Prasanna Kumar** has three doctorates one each in Mathematics, Economics, Political Science. Thesis was based on Mathematical Modeling. He was recently awarded Dlitt. for his work on 'Mathematical Models in Political Science'--- Department of studies in Mathematics, Kuvempu University, Shimoga, Karnataka, India.

**Prof. B.S Kiranagi** is the Former Chairman of the Department of Studies in Mathematics, Manasa Gangotri and present Professor Emeritus of UGC in the Department. Professor Kiranagi has guided over 25 students and he has received many encomiums and laurels for his contribution to Co homology Groups and Mathematical Sciences. Known for his prolific writing, and one of the senior most Professors of the country, he has over 150 publications to his credit. A prolific writer and a prodigious thinker, he has to his credit several books on Lie Groups, Co Homology Groups, and other mathematical application topics, and excellent publication history.-- UGC Emeritus Professor (Department of studies in Mathematics), Manasagangotri, University of Mysore, Karnataka, India

**Prof. C.S. Bagewadi** is the present Chairman of Department of Mathematics and Department of Studies in Computer Science and has guided over 25 students. He has published articles in both national and international journals. Professor Bagewadi specializes in Differential Geometry and its wide-ranging ramifications. He has to his credit more than 159 research papers. Several Books on Differential Geometry, Differential Equations are co-authored by him--- Chairman, Department of studies in Mathematics and Computer science, Jnanasahyadri Kuvempu University, Shankarghatta, Shimoga district, Karnataka, India.

**Corresponding Author:**
Dr. K. N.Prasanna Kumar
drknpkumar@gmail.com

ISSN 2250-3153

# Table of Content

## Module Numbered One

## Gravitational Collapse Of A Giant Molecular Cloud, ( GMC Breaks Into Smaller And Smaller Pieces With Each Of These Fragments, The Collapsing Gas Releases Gravitational Potential Energy As Heat, The Increase Of Which (Temperature And Pressure Increase,) A Fragment Condenses Into A Rotating Sphere Of Superhot Gas ( A Protostar):

**NOTATION :**

$G_{13}$ : Category One Of Gravitational Collapse Of A Giant Molecular Cloud, ( GMC Breaks Into Smaller And Smaller Pieces With Each Of These Fragments, The Collapsing Gas **Releases** Gravitational Potential Energy As Heat, The Increase Of Which (Temperature And Pressure Increase,)

$G_{14}$ : Category Two Of Gravitational Collapse Of A Giant Molecular Cloud, ( GMC Breaks Into Smaller And Smaller Pieces With Each Of These Fragments, The Collapsing Gas **Releases** Gravitational Potential Energy As Heat, The Increase Of Which (Temperature And Pressure Increase,)

$G_{15}$ : Category Three Of Gravitational Collapse Of A Giant Molecular Cloud, ( GMC Breaks Into Smaller And Smaller Pieces With Each Of These Fragments, The Collapsing Gas **Releases** Gravitational Potential Energy As Heat, The Increase Of Which (Temperature And Pressure Increase,)

$\mathrm{T}_{13}$ : Category One Of Rotating Sphere Of Superhot Gas ( A Protostar)**:**

$T_{14}$ : Category Two Of Rotating Sphere Of Superhot Gas ( A Protostar)**:**

$\mathrm{T}_{15}$ :Category Three Of Rotating Sphere Of Superhot Gas ( A Protostar)**:**

**ArXiv: 1209.5778 [pdf, ps, other]: T: Can R-parity violation hide vanilla supersymmetry at the LHC: Masaki Asano, Krzysztof Rolbiecki, and Kazuki Sakurai**

Current experimental constraints on a large parameter space in supersymmetric models **rely on** the large missing energy signature. This is usually **provided by** the lightest neutralino which stability is ensured by the R-parity. However, if the R-parity is violated, the lightest neutralino **decays** into the standard model particles and the missing energy cut is not efficient anymore. In particular, the UDD type R-parity violation **induces** the neutralino decay to three quarks which potentially **leads to** the most difficult signal to be searched at hadron colliders. In this paper authors study the constraints on the R-parity violating supersymmetric **model using a** same-sign dilepton and multijet signatures. We show that the gluino and squarks lighter than a TeV are already **excluded i**n constrained minimal supersymmetric standard model with R-parity violation if their masses are approximately equal. Authors also analyze constraints in a simplified model with R-parity violation. R-parity violation **changes** some of the observables typically used to distinguish a supersymmetric signal from standard model backgrounds.

## Matrix Models

## String World Sheet Duality

## MODULE NUMBERED TWO:

$\mathrm{G}_{16}$ : CATEGORY ONE OF Matrix models

The term matrix model may refer to one of several concepts: In theoretical physics, a matrix model is a system (usually a quantum mechanical system) with matrix-valued physical quantities. See, for example, Lax pair. The "old" matrix models are relevant for string theory in two spacetime dimensions. The "new" matrix model is a synonym for Matrix theory. Matrix population models are used to model wildlife and human population dynamics. The Matrix Model of substance abuse treatment was a model developed by the Matrix

ISSN 2250-3153

Institute in the 1980s to treat cocaine and methamphetamine addiction. It is a concept from Algebraic logic. The term "Matrix model" is used to describe Matrix management where each worker in an organization has two managers, one "functional": managing their expertise, the other "executive":

$G_{17}$ : CATEGORY TWO OF Matrix models

Population models are used in population ecology to model the dynamics of wildlife or human populations. Matrix population models are a specific type of population model that uses matrix algebra. Matrix algebra, in turn, is simply a form of algebraic shorthand for summarizing a larger number of often repetitious and tedious algebraic computations.

All populations can be modeled by one simple equation:

$$N_{t+1} = N_t + B - D + I - E,$$

where:

Nt+1 = abundance at time t+1

Nt = abundance at time t

B = number of births within the population between Nt and Nt+1

D = number of deaths within the population between Nt and Nt+1

I = number of individuals immigrating into the population between Nt and Nt+1

E = number of individuals emigrating from the population between Nt and Nt+1

This equation is called a BIDE model (Birth, Immigration, Death, Emigration model).

Although BIDE models are conceptually simple, reliable estimates of the 5 variables contained therein (N, B, D, I and E) are often difficult to obtain. Usually a researcher attempts to estimate current abundance, Nt, often using some form of mark and recapture technique. Estimates of B might be obtained via a ratio of immatures to adults soon after the breeding season, Ri. Number of deaths can be obtained by estimating annual survival probability, usually via mark and recapture methods, then multipling present abundance and survival rate. Often, immigration and emigration are ignored because they are so difficult to estimate. For added simplicity it may help to think of time t as the end of the breeding season in year t and to imagine that one is studying a species that has only one discrete breeding season per year.

The BIDE model can then be expressed as:

$$N_{t+1} = N_{t,a} \times S_a + N_{t,a} \times R_i \times S_i$$

where:

Nt,a = number of adult females at time t

Nt,i = number of immature females at time t

Sa = annual survival of adult females from time t to time t+1

Si = annual survival of immature females from time t to time t+1

Ri = ratio of surviving young females at the end of the breeding season per breeding female

In matrix notation this model can be expressed as:

$$\begin{pmatrix} N_{t+l_i} \\ N_{t+l_a} \end{pmatrix} = \begin{pmatrix} S_i R_i & S_a R_i \\ S_i & S_a \end{pmatrix} \begin{pmatrix} N_{t_i} \\ N_{t_a} \end{pmatrix} \quad .$$

Suppose that you are studying a species with a maximum lifespan of 4 years. The following is an age-based Leslie matrix for this species. Each row in the first and third matrices corresponds to animals within a given age range (0–1 years, 1–2 years and 2–3 years). In a Leslie matrix the top row of the middle matrix consists of age-specific fertilities: F1, F2and F3. Note, that F1 = Si×Ri in the matrix above. Since this species does not live to be 4 years old the matrix does not contain an S3 term.

$$\begin{pmatrix} N_{t+l_1} \\ N_{t+l_2} \\ N_{t+l_3} \end{pmatrix} = \begin{pmatrix} F_1 & F_2 & F_3 \\ S_1 & 0 & 0 \\ 0 & S_2 & 0 \end{pmatrix} \begin{pmatrix} N_{t_1} \\ N_{t_2} \\ N_{t_3} \end{pmatrix} \quad .$$

These models can give rise to interesting cyclical or seemingly chaotic patterns in abundance over time when fertility rates are high.

The terms Fi and Si can be constants or they can be functions of environment, such as habitat or population size. Randomness can also be incorporated into the environmental component.

$G_{18}$ : CATEGORY THREE OF Matrix models

Matrix string theory is a set of equations that describe superstring theory in a non-perturbative framework. Type IIA string theory can be shown to be equivalent to a maximally supersymmetric two-dimensional gauge theory, the gauge group of which is U(N) for a large value of N. This Matrix string theory was first proposed by Luboš Motl in 1997 and later independently in a more complete paper by Robbert Dijkgraaf, Erik Verlinde, and Herman Verlinde. Another matrix string theory equivalent to Type IIB string theory was constructed in 1996 by Ishibashi, Kawai, Kitazawa and Tsuchiya. This version is known as the IKKT matrix model. M(atrix) Theory (also known as BFSS-Matrix theory) is a fundamental formulation of M-theory as a random matrix model. Matrix string theory is related to M (atrix) theory in the same sense that superstring theory is related to M-theory. M(atrix) theory is written in terms of interacting zero-dimensional Dirichlet branes in infinite momentum frame. It was proposed by Banks, Fischler, Shenker, and Susskind in 1996

$T_{16}$ :CATEGORY ONE OF STRING WORLD SHEET DUALITY

S. P. de Alwis puts up a argument that the scaling limit used recently to derive matrix models, and a certain analyticity assumption, are invoked to argue that the agreement between some matrix model calculations and supergravity is a consequence of string world sheet duality. Authors Takuya Okuda, Hirosi Ooguri (Caltech) here generalize the worldsheet derivation of the topological open/closed string duality given in hep-th/0205297 to cases when there are different types of D branes on the open string side. We use the mirror Landau-Ginzburg description to clarify the correspondence between D branes on the open string side and C phases on the closed string side. We also discuss the duality from the point of view of the B model. A.A. Tseytlin suggest a formulation of string theory in which the string coordinatex and its “dualx”~x are treated on an equal footing. **As a result**, the duality symmetry of torus compactifications can be realised as symmetry of a world sheet action.xandx~xare similar to “non-commuting” phase space type coordinates. The corresponding “double” volume factor is duality symmetric. We discuss 2D models with interactions involving bothxand~x emphasizing the issue of 2D Lorentz invariance. String vacua correspond to models which have both conformal and 2D Lorentz symmetry at the quantum level

$T_{17}$ : CATEGORY TWO OF STRING WORLD SHEET DUALITY

Matti Pitkänen consummates and consolidates the generalization of AdS5 duality of N=4 SYMs to TGD framework is highly suggestive and states that string world sheets and partonic 2-surfaces play a dual role in the construction of M-matrices. In the following I give an argument providing a "proof" of this duality and also demonstrating that for singular string world sheets and partonic 2-surfaces perturbative description of generalized Feynman diagrams is especially simple since string effectively reduces to point like particles. Let us agree that string world sheets and partonic 2-surfaces refer to 2-surfaces in the slicing of space-time region defined by Hermitian structure or Hamilton-Jacobi structure. Let us also agree that singular string world sheets and partonic 2-surfaces are surfaces at which the effective metric defined by the anticommutators of the modified gamma matrices degenerates to effectively 2-D one. Braid strands at wormhole throats in turn would be loci at which the induced metric of the string world sheet transforms from Euclidian to Minkowskian as the signature of induced metric changes from Euclidian to Minkowskian. AdS5 duality suggest that string world sheets are in the same role as string world sheets of 10-D space connecting branes in AdS5 duality for N=4 SYM. What is important is that there should exist a duality meaning two manners to calculate the amplitudes. What the duality could mean now? Also in TGD framework the first manner would be string model like description using string world sheets. The second one would be a generalization of conformal QFT at light-like 3-surfaces (allowing generalized conformal symmetry) defining the lines of generalized Feynman diagram. The correlation functions to be calculated would have points at the intersections of partonic 2-surfaces and string world sheets and would represent braid ends. General Coordinate Invariance (GCI) implies that physics should be codable by 3-surfaces. Light-like 3-surfaces define 3-surfaces of this kind and same applies to space-like 3-surfaces. There are also preferred 3-surfaces of this kind.

The orbits of 2-D wormhole throats at which 4-metric degenerates to 3-dimensional one define preferred light-like 3-surfaces. Also the space-like 3-surfaces at the ends of space-time surface at light-like boundaries of causal diamonds (CDs) define preferred space-like 3-surfaces. Both light-like and space-like 3-surfaces should code for the same physics and therefore their intersections defining partonic 2-surfaces plus the 4-D tangent space data at them should be enough to code for physics. This is strong form of GCI implying effective 2-dimensionality. As a special case one obtains singular string world sheets at which the effective metric reduces to 2-dimensional and singular partonic 2-surfaces defining the wormhole throats. For these 2-surfaces situation could be especially simple mathematically. The guess inspired by strong GCI is that string world sheet -partonic 2-surface duality holds true. The functional integrals over the deformations of 2 kinds of 2-surfaces should give the same result so that functional integration over either kinds of 2-surfaces should be enough. Note that the members of a given pair in the slicing intersect at discrete set of points and these points define braid ends carrying fermion number.

Discretization and braid picture follow automatically. Scattering amplitudes in the twistorial approach could be thus calculated by using any pair in the slicing - or only either member of the pair if the analog of AdS5 duality holds true as argued. The possibility to choose any pair in the slicing means general coordinate invariance as symmetry of the Kähler metric of WCW and of the entire theory suggested already early: Kähler functions for difference choices in the slicing would differ by a real part of holomorphic function and give rise to same Kähler metric of "world of classical worlds" (WCW). For a general pair one obtains functional integral over deformations of space-time surface inducing deformations of 2-surfaces with only other kind 2-surface contributing to amplitude. This means the analog of stringy QFT: Minkowskian or Euclidian string theory depending on choice.

For singular string world sheets and partonic 2-surfaces an enormous simplification results. The propagators for fermions and correlation functions for deformations reduce to 1-D instead of being 2-D: the propagation takes place only along the light-like lines at which the string world sheets with Euclidian signature (inside CP2 like regions) change to those with Minkowskian signature of induced metric. The local reduction of space-time dimension would be very real for particles moving along sub-manifolds at which higher dimensional space-time has reduced metric dimension: they cannot get out from lower-D sub-manifold. This is like ending down to 1-D black hole interior and one would obtain the analog of ordinary Feynman diagrammatics.

This kind of Feynman diagrammatics involving only braid strands is what I have indeed ended up earlier so that it seems that I can trust good intuition combined with a sloppy mathematics sometimes works. These singular lines represent orbits of point like particles carrying fermion number at the orbits of wormhole throats. Furthermore, in this representation the expansions coming from string world sheets and partonic 2-surfaces are identical automatically. This follows from the fact that only the light-like lines connecting points common to singular string world sheets and singular partonic 2-surfaces appear as propagator lines! The TGD analog of AdS5 duality of N=4 SUSYs would be trivially true as an identity in this special case, and the good guess is that it is true also generally. One could indeed use integral over either string world sheets or partonic 2-sheets to deduce the amplitudes. What is important to notice that singularities of Feynman diagrams crucial for the Grassmannian approach of Nima and others would correspond at space-time level 2-D singularities of the effective metric defined by the modified gamma matrices defined as contractions of canonical momentum currents for Kähler action with ordinary gamma matrices of the imbedding space and therefore directly reflecting classical dynamics.

$T_{18}$ : CATEGORY THREE OF STRING WORLD SHEET DUALITY

**Tseytlin, A**

A suggest a formulation of string theory in which the string coordinate x and its ``dual'' x~ are treated on an equal footing. As a result, the duality symmetry of torus compactifications can be realised as symmetry of a world sheet action. X and x~ are similar to ``non-commuting'' phase space type coordinates. The corresponding ``double'' volume factor is duality symmetric. We discuss 2D models with interactions involving both x and x~ emphasizing the issue of 2D Lorentz invariance. String vacua correspond to models which have both conformal and 2D Lorentz symmetry at the quantum level.

The simplest case to imagine is a single string traveling in a flat spacetime in d dimensions. As the string moves around in spacetime, it sweeps out a surface in spacetime called the string worldsheet, a two-dimensional surface with one dimension of space (s) and one dimension of time (t). There are many ways to examine this string theory. One way is to expand the string coordinates Xa(s,t) into oscillator modes and demand spacetime Lorentz invariance and the absence of negative norm states. A different way to examine the string theory is through the field theory defined on the worldsheet, which is described by the action

$$S = \frac{1}{4\pi\alpha'}\int d\sigma d\tau\sqrt{h}\,\left(h^{mn}\partial_m X^a \partial_n X^b \eta_{ab} + \alpha' R_{(2)}\Phi\right)$$

Where hmn is the metric on the worldsheet, R(2) is the curvature of the worldsheet, and F is a scalar field called the dilaton. The consistency condition for string theory when described in this manner is that the field theory on the worldsheet satisfies the condition for scale invariance, also known as conformal invariance.

The set of functions that describe the scaling properties of quantum fields are called the beta functions. String worldsheet physics is invariant under a change in scale if the beta function bF for the dilaton field F vanishes, which happens when d=26 for bosonic strings.

$$\beta^{\Phi} = 0 \rightarrow d = 26$$

(For superstring theories, conformal invariance is replaced by superconformal invariance, and the required spacetime dimension is 10.)

The space-time oscillation spectrum satisfies Lorentz invariance in 26 dimensions, so that these string oscillations on the worldsheet can be classified by the spacetime properties of mass and spin, just like elementary particles. A theory based on open strings has massless oscillations that are Lorentz vectors, with spin 1. A closed string theory is like a product of two open string theories, with an oscillation mode that travels in spacetime as a two index symmetric tensor, with spin 2.

This mode with spin 2 propagates like as small fluctuation in the gravitational field propagates according to general relativity. This string oscillation mode should then be the graviton, the particle that mediates the gravitational force. The presence of this spin 2 oscillation mode was the first clue that string theory was not a theory of strong interactions, but a potential quantum theory of gravity. In string theory, if we start with flat spacetime, we see gravitons in the spectrum, and therefore we deduce that gravity must exist. But if gravity exists, then spacetime must be curved and not flat. How do the Einstein equations for the curvature of spacetime come out of string theory?

If a closed string is traveling in a curved spacetime with metric field gab(X) , then the string worldsheet theory looks like

$$S_P = \frac{1}{4\pi\alpha'}\int d\sigma d\tau\sqrt{h}\left(h^{mn}\partial_m X^a \partial_n X^b g_{ab}(X) + \alpha' R_{(2)}\Phi\right)$$

The spacetime metric gab(X) enters the two-dimensional theory on the string worldsheet as a matrix of nonlinear couplings between the Xa(s,t). Once again, the goal of conformal invariance is met by demanding that the beta functions vanish. When the string coordinates are expanded in a perturbation series in the string scale a', the terms in the beta functions that are the lowest order in a' contain terms proportional to the Ricci curvature Rab of the spacetime metric field gab(x) and second derivatives of the scalar field F(x). The vanishing of the beta functions ends up being equivalent to satisfying the Einstein equation for a spacetime with a scalar field

$$\beta^g = 0 \rightarrow R_{ab} + 2D_a D_b \Phi = 0 + O(\alpha')$$
$$\beta^\Phi = 0 \rightarrow d = 26, \quad D^a D_a \Phi - 2D^a \Phi D_a \Phi = 0 + O(\alpha')$$

at distance scales large compared to the string scale. Notice this means that our understanding of spacetime from perturbative string theory will always be incomplete, except in some special circumstances described below. Strings and black holes present a different scenario. Black holes are solutions to the Einstein equation; therefore string theories that contain gravity also predict the existence of black holes. But string theories give rise to more interesting symmetries and types of matter than are commonly assumed in ordinary Einstein relativity. In particular, electric/magnetic duality in string theory has led to the discovery of many new types of black holes with combinations of electric and magnetic charge, coupled to both scalar and axion fields. Also, string theory has motivated an understanding of black holes in higher dimensions, and of black extended objects such as strings and branes.

Some of these new stringy extreme black hole solutions possess unbroken supersymmetries at the event horizon, so that the physics at the horizon is protected from higher order perturbative corrections by virtue of supersymmetric nonrenormalization theorems. These types of black holes have been important for understanding the origin of black hole entropy in string theory Note that string theory does not predict that the Einstein equations are obeyed exactly. Perturbative string theory adds an infinite series of corrections to the Einstein equation

$$R_{ab} - \frac{1}{2} g_{ab} R + O(\alpha' R^2) + O(\alpha'^2 R^4) + \ldots = 0$$

So our understanding of spacetime in perturbative string theory is only valid as long as spacetime curvature is small compared to the string scale. However, when these correction terms become large, there is no spacetime geometry that is guaranteed to describe the result. Only under very strict symmetry conditions, such as unbroken supersymmetry, are there known exact solutions to the spacetime geometry in string theory. This is a hint that perhaps spacetime geometry is not something fundamental in string theory, but something that emerges in the theory at large distance scales or weak coupling. This is an idea with enormous philosophical implications.

**R^4 Corrections To Heterotic M-Theory And**

**Twistor Transform Of All Tree Amplitudes In $\mathcal{N}=4$ Sym Theory**

## Module Numbered Three:

$G_{20}$ : CATEGORY ONE OF **R^4 CORRECTIONS TO HETEROTIC M-THEORY**

Lilia Anguelova, Diana Vaman study R^4 corrections in heterotic M-theory. We derive to order kappa^{4/3} the induced modification to the Kahler potential of the universal moduli and its implications for the soft supersymmetry breaking terms. The soft scalar field masses still remain small for breaking in the T-modulus direction. We investigate the deformations of the background geometry due to the R^4 term. The warp-factor deformation of the background M_4 x CY (3) x S^1/Z_2 can no longer be integrated to a fully non-linear solution, unlike when neglecting higher derivative corrections. We find explicit solutions to order kappa^{4/3} and, in particular, find the expected shift of the Calabi-Yau volume by a constant proportional to the Euler number. We also study the effect induced by the R^4 terms on the de Sitter vacua found previously by balancing two non-perturbative contributions to the superpotential, namely open membrane instantons and gaugino condensation. To order kappa^{4/3} all induced corrections are proportional to the Euler number of the Calabi-Yau three-fold.

Theoretical physics, M-theory is an extension of string theory in which 11 dimensions are identified. Because the dimensionality exceeds that of superstring theories in 10 dimensions, proponents believe that the 11-dimensional theory unites all five string theories (and supersedes them). Though a full description of the theory is not known, the low-entropy dynamics are known to be supergravity interacting with 2- and 5-dimensional membranes. This idea is the unique supersymmetric theory in eleven dimensions, with its low-entropy matter content and interactions fully determined, and can be obtained as the strong coupling limit of type IIA string theory because a new dimension of space emerges as the coupling constant increases.

Drawing on the work of a number of string theorists (including Ashoke Sen, Chris Hull, Paul Townsend, Michael Duff and John Schwarz), Edward Witten of the Institute for Advanced Study, suggested its existence at a conference at USC in 1995, and used M-theory to explain a number of previously observed dualities, initiating a flurry of new research in string theory called the second superstring revolution. In the early 1990s, it was shown that the various superstring theories were related by dualities which allow the description of an object in one super string theory to be related to the description of a different object in another super string theory.

These relationships imply that each of the super string theories is a different aspect of a single underlying theory, proposed by Witten, and named "M-theory". Originally the letter M in M-theory was taken from membrane, a construct designed to generalize the strings of string theory. However, as Witten was more skeptical about membranes than his colleagues, he opted for "M-theory" rather than "Membrane theory". Witten has since stated that the different interpretations of the M can be a matter of taste for the user, such as magic, mystery, and mother theory. M-theory (and string theory) has been criticized for lacking predictive power or being untestable. Further work continues to find mathematical constructs that join various surrounding theories. However, the tangible success of M-theory can be questioned, given its current incompleteness and limited predictive power.

$G_{21}$ : CATEGORY TWO OF **R^4 CORRECTIONS TO HETEROTIC M-THEORY SASAKIAN GEOMETRY**

There are other dualities between the other string theories. The heterotic SO (32) and the heterotic E8×E8 theories are also related by T-duality; the heterotic SO(32) description of a circle of radius R is exactly the same as the heterotic E8×E8 description of a circle of radius 1/R. This implies that there are really only three superstring theories, which might be called (for discussion) the Type I theory, the Type II theory, and the heterotic theory. There are still more dualities, however. The Type I string theory is related to the heterotic SO(32) theory by S-duality; this means that the Type I description of weakly interacting particles can also be seen as the heterotic SO(32) description of very strongly interacting particles. This identification is somewhat more subtle, in that it identifies only extreme limits of the respective theories. String theorists have found strong evidence that the two theories are really the same, even away from the extremely strong and extremely weak limits, but they do not yet have a proof strong enough to satisfy mathematicians. However, it has become clear that the two theories are related in some fashion; they appear as different limits of a single underlying theory. Given the above commonalities there appear to be only two string theories: the heterotic string theory (which is also the type I string theory) and the type II theory. There are relations between these two theories as well, and these relations are in fact strong enough to allow them to be identified. This last step is best explained first in a certain limit. In order to describe our world, strings must be extremely tiny objects. So when one studies string theory at low energies, it becomes difficult to see that strings are extended objects — they become effectively zero-dimensional (point like). Consequently, the quantum theory describing the low energy limit is a theory that describes the dynamics of these points moving in spacetime, rather than strings. Such theories are called quantum field theories. However, since string theory also describes gravitational interactions, one expects the low-energy theory to describe particles moving in gravitational backgrounds.

Finally, since superstring string theories are supersymmetric for supersymmetry is needed for consistency, one expects to see supersymmetry appearing in the low-energy approximation. These three facts imply that the low-energy approximation to a superstring theory is a supergravity theory. In Supergravity theories, theories were classified by Werner Nahm in the 1970s. In 10 dimensions, there are only two supergravity theories, which are denoted Type IIA and Type IIB. This similar denomination is not a coincidence; the Type IIA string theory has the Type IIA supergravity theory as its low-energy limit and the Type IIB string theory gives rise to Type IIB supergravity. The heterotic SO (32) and heterotic E8×E8 string theories also reduce to Type IIA and Type IIB supergravity in the low-energy limit. This suggests that there may indeed be a relation between the heterotic/Type I theories and the Type II theories. In 1994, Edward Witten outlined the following relationship: The Type IIA supergravity (corresponding to the heterotic SO(32) and Type IIA string theories) can be obtained by dimensional reduction from the single unique eleven-dimensional supergravity theory.

This means that if one studied supergravity on an eleven-dimensional spacetime that looks like the product of a ten-dimensional spacetime with another very small one-dimensional manifold, one gets the Type IIA supergravity theory. (And the Type IIB supergravity theory can be obtained by using T-duality.) However, eleven-dimensional supergravity is not consistent on its own — it does not make sense at extremely high energy, and likely requires some form of completion. It seems plausible, then, that there is some quantum theory — which Witten dubbed M-theory — in eleven-dimensions which gives rise at low energies to eleven-dimensional supergravity, and is related to ten-dimensional string theory by dimensional reduction. Dimensional reduction to a circle yields the Type IIA string theory, and dimensional reduction to a line segment yields the heterotic SO (32) string theory.M-theory would implement the notion that all of the different string theories are different special cases. Lilia Anguelova, Diana Vaman put up a deliberative confabulation that R4 corrections in heterotic M-theory. We derive to order κ4/3 the induced modification to the Kähler potential of the universal moduli and its implications for the soft supersymmetry breaking terms. The soft scalar field masses still remain small for breaking in the T-modulus direction.

We investigate the deformations of the background geometry due to the R4 term. The warp-factor deformation of the background M4×CY(3)×S1/Z2 can no longer be integrated to a fully non-linear solution, unlike when neglecting higher derivative corrections. We find explicit solutions to order κ4/3 and, in particular, find the expected shift of the Calabi–Yau volume by a constant proportional to the Euler number. We also study the effect induced by the R4 terms on the de Sitter vacua found previously by balancing two non-perturbative contributions to the superpotential, namely open membrane instantons and gaugino condensation. To order κ4/3 all induced corrections are proportional to the Euler number of the Calabi–Yau three-fold.

$G_{22}$ : CATEGORY THREE **R^4 CORRECTIONS TO HETEROTIC M-THEORY SASAKIAN GEOMETRY**

André Lukas1 and Kellog S. Stelle studied the constraints on five-dimensional $\mathcal{N}$ = 1 heterotic M-theory imposed by a consistent anomaly-free coupling of bulk and boundary theory. This requires analyzing the cancellation of triangle gauge anomalies on the four-dimensional orbifold planes due to anomaly inflow from the bulk. We find that the semi-simple part of the orbifold gauge groups and certain U(1) symmetries have to be free of quantum anomalies. In addition there can be several anomalous U(1) symmetries on each orbifold plane whose anomalies are cancelled by a non-trivial variation of the bulk vector fields. The mixed U(1) non-abelian anomaly is universal and there is at most one U(1) symmetry with such an anomaly on each plane. In an alternative approach, we also analyze the coupling of five-dimensional gauged supergravity to orbifold gauge theories. We find a somewhat generalized structure of anomaly cancellation in this case which allows, for example, non-universal mixed U(1) gauge anomalies. Anomaly cancellation from the perspective of four-dimensional $\mathcal{N} = 1$ effective actions obtained from E8 × E8heterotic string- or M-theory by reduction on a Calabi-Yau three-fold is studied as well. The results are consistent with the ones found for five-dimensional heterotic M-theory. Finally, we consider some related issues of phenomenological interest such as model building with anomalous U(1) symmetries, Fayet-Illiopoulos terms and threshold corrections to gauge kinetic functions.

Standard Models from Heterotic M-theory has been studied by Ron Donagi (UPenn), Burt A. Ovrut (UPenn), Tony Pantev (UPenn), Daniel Waldram (Princeton University and CERN) to rest a case for a class of N=1 supersymmetric models of particle physics, derived directly from heterotic M-theory, that contain three families of chiral quarks and leptons coupled to the gauge group $SU(3)_C\times SU(2)_{L}\times U(1)_{Y}$. These models are a fundamental form of ``brane-world'' theories, with an observable and hidden sector each confined, after compactification on a Calabi-Yau threefold, to a BPS threebrane separated by a five-dimensional bulk space with size of the order of the intermediate scale. The requirement of three families, coupled to the fundamental conditions of anomaly freedom and supersymmetry, constrains these models to contain additional fivebranes wrapped around holomorphic curves in the Calabi-Yau threefold. These five branes ``live'' in the bulk space and represent new, non-perturbative aspects of these particle physics vacua.

We discuss, in detail, the relevant mathematical structure of a class of torus-fibered Calabi-Yau threefolds with non-trivial first homotopy groups and construct holomorphic vector bundles over such threefolds, which, by including Wilson lines, break the gauge symmetry to the standard model gauge group. Rules for constructing phenomenological particle physics models in this context are presented and we give a number of explicit examples. André Coimbra, Charles Strickland-Constable, Daniel Waldram formulated ten-dimensional type II supergravity as a generalized geometrical analogue of Einstein gravity, defined by an O(9, 1) × O(1, 9) ⊂ O(10, 10) × R $^{+}$ structure on the generalized tangent space. Using the notion of generalized connection and torsion, they introduce the analogue of the Levi-Civita connection, and derive the corresponding tensorial measures of generalized curvature.

It **<u>is shown that</u>** leading order in the fermion fields, these structures allow one to rewrite the action, equations of motion and supersymmetry variations in a simple, manifestly Spin(9, 1) × Spin(1, 9)-covariant form. The same formalism also describes d-dimensional compactifications to flat space. tefanos Katmadas (Paris) investigated the duality covariant black holes in N=2 supergravity under extremal under rotating black holes in N=2 supergravity coupled to vector multiplets. For the case of symmetric scalar manifolds we use group theory techniques on the coset resulting under dimensional reduction to three dimensions. We thus obtain a first order system for the scalars describing the composite non-BPS class that is characterised by a so called very small charge vector. Lifting back to four dimensions, we present the general solution for the single center class of black holes for arbitrary asymptotic moduli in a manifestly duality covariant form. We briefly comment on the multi center case.

$T_{20}$ : CATEGORY ONE OF **TWISTOR TRANSFORM OF ALL TREE AMPLITUDES IN $\mathcal{N} = 4$ SYM THEORY**

G.P. Korchemsky, E. Sokatchev perform a ontological univocity and disjunctive syllogism of the twistor (half-Fourier) transform of all tree n-particle super amplitudes in N=4 SYM and show that it has a transparent geometric interpretation. We find that the N^kMHV amplitude is supported on a set of (2k+1) intersecting lines in twistor space and demonstrate that the corresponding line moduli form a light like (2k+1)-gon in moduli space. This polygon is triangulated into two kinds of light like triangles lying in different planes. A formulation of simple graphical rules for constructing the triangulated polygons, from which the analytic expressions of the N^kMHV amplitudes follow directly, both in twistor and in momentum space. We also discuss the ordinary and dual conformal properties and the cancellation of spurious singularities in twistor space. Tim Adamo and Lionel Mason give a adjunct formulation of a twistor-string formulation for all tree amplitudes of Einstein (super-)gravities for $\mathcal{N}=0$ and 4.

Formulae are given with and without cosmological constant and with various possibilities for the gauging. The formulae are justified by use of Maldacena's observation that conformal gravity tree amplitudes with Einstein wavefunctions and non-zero cosmological constant will correctly give the Einstein tree amplitudes. This justifies the construction of Einstein gravity amplitudes at $\mathcal{N}=0$ from twistor-string theory and is extended to $\mathcal{N}=4$ by requiring the standard relation between the MHV-degree and the degree of the rational curve for Yang–Mills; this systematically excludes the spurious conformal supergravity gravity contributions. For comparison, BCFW recursion is used to obtain twistor-string-like formulae at degree 0 and 1 (anti-MHV and MHV) for amplitudes with $\mathcal{N}=8$ supersymmetry with and without cosmological constant. Louise Dolan, Peter Goddard conduct a study of The gluon tree amplitudes of open twistor string theory, defined as contour integrals over the ACCK link variables, are shown to satisfy the BCFW relations, thus confirming that they coincide with the corresponding amplitudes in gauge field theory. In this approach, the integration contours are specified as encircling the zeros of certain constraint functions that force the appropriate relation between the link variables and the twistor string world-sheet variables. To do this, methods for calculating the tree amplitudes using link variables are developed further including diagrammatic methods for organizing and performing the calculations.

## $T_{21}$ :CATEGORY TWO OF **TWISTOR TRANSFORM OF ALL TREE AMPLITUDES IN $\mathcal{N}=4$ SYM THEORY**

David Skinner provided a indubitable proof of t the leading trace part of the genus zero twistor-string path integral obeys the BCFW recursion relation. This is the first complete proof that the twistor-string correctly computes all tree amplitudes in maximally supersymmetric Yang-Mills theory. The recursion has a beautiful geometric interpretation in twistor space that closely rejects the structure of BCFW recursion in momentum space, both on the one hand as a relation purely among tree amplitudes with shifted external momenta, and on the other as a relation between tree amplitudes and leading singularities of higher loop amplitudes. The proof works purely at the level of the string path integral and is intimately related to the recursive structure of boundary divisors in the moduli space of stable maps to CP3. In what probably forms the bastion and stylobate of hypostatized signification and proper frontier to the study of the subject matter Lionel Mason, David Skinner provide Twistor ideas have led to a number of recent advances in our understanding of scattering amplitudes. Much of this work has been indirect, determining the twistor space support of scattering amplitudes by examining the amplitudes in momentum space. In this paper, we construct the actual twistor scattering amplitudes themselves.

We show that the recursion relations of Britto, Cachazo, Feng and Witten have a natural twistor formulation that, together with the three-point seed amplitudes, allows us to recursively construct general tree amplitudes in twistor space. We obtain explicit formulae for n-particle MHV and NMHV super-amplitudes, their CPT conjugates (whose representations are distinct in our chiral framework), and the eight particle N2MHV super-amplitude. We also give simple closed form formulae for the N=8 supergravity recursion and the MHV and MHV−−−−− amplitudes. This gives a formulation of scattering amplitudes in maximally supersymmetric theories in which superconformal symmetry and its breaking is manifest. For N k MHV, the amplitudes are given by 2n − 4 integrals in the form of Hilbert transforms of a product of n − k − 2 purely geometric, super conformally invariant twistor delta functions, dressed by certain sign operators. These sign operators subtly violate conformal invariance, even for tree-level amplitudes in N=4 super Yang-Mills, and we trace their origin to a topological property of split signature space-time. We develop the twistor transform to relate our work to the ambidextrous twistor diagram approach of Hodges and of Arkani-Hamed, Cachazo, Cheung and Kaplan.

## $T_{22}$ : CATEGORY THREE OF **TWISTOR TRANSFORM OF ALL TREE AMPLITUDES IN $\mathcal{N}=4$ SYM THEORY**

G. P. Korchemsky, E. Sokatchev investigated the thromboses and benedictory parts of the twistor (half-Fourier) transform of all tree n-particle super amplitudes in N=4 SYM and show that it has a transparent geometric interpretation. We find that the N^kMHV amplitude is supported on a set of (2k+1) intersecting lines in twistor space and demonstrate that the corresponding line moduli form a light like (2k+1)-gon in moduli space. This polygon is triangulated into two kinds of light like triangles lying in different planes. We formulate simple graphical rules for constructing the triangulated polygons, from which the analytic expressions of the N^kMHV amplitudes follow directly, both in twistor and in momentum space. We also discuss the ordinary and dual conformal properties and the cancellation of spurious singularities in twistor space.In this note is also studied we study spin chain operators in the N=6 Chern-Simons-matter theory recently proposed by Aharony, Bergman, Jafferis and Maldacena to be dual to type IIA string theory in AdS4xCP3. We study the two-loop dilatation operator in the gauge theory, and compare to the Penrose limit on the string theory side. Construction of the 5 parameter generating solution of N=8 BPS regular supergravity black holes as a five parameter solution of the N=2 STU model is the leit motif of the model. Our solution has a simpler form with respect to previous constructions already appeared in the literature and moreover, through the embedding [SL(2)]^3\subset SU(3,3)\subset E_{(7)7} .Further investigated conceptually related to the subject matter captioned are investigate in detail the correspondence between E10 and Romans' massive deformation of type IIA super gravity.

We analyse the dynamics of a non-linear sigma model for a spinning particle on the coset space E10/K(E10) and show that it reproduces the dynamics of the bosonic as well as the fermionic sector of the massive IIA th... action for the D5-brane Pietro Frè, Leonardo Modesto Massive Type IIA Supergravity Correspondence "is another note worthy paper that studied (Henneaux, Ella Jamsin,Axel Kleinschmidt,Daniel Persson )in a sententious and impactful study of detail the correspondence between E10 and Romans' massive deformation of type IIA supergravity. We analyse the dynamics of a non-linear sigma model for a spinning particle on the coset space E10/K(E10) and show that it reproduces the dynamics of the bosonic as well as the fermionic sector of the massive IIA theory, within the standard truncation.

The mass deformation parameter corresponds to a generator of E10 outside the realm of the generators entering the usual D=11 analysis, and is naturally included without any deformation of the coset model for E10/K(E10). Our analysis thus provides a dynamical unification of the massless and massive versions of type IIA supergravity inside E10. We discuss a number of additional and general features of relevance in the analysis of any deformed supergravity in the correspondence to Kac-Moody algebras, including recently studied deformations where the trombone symmetry is gauged. Iib Supergravity And E10 analysed the subtleties and nuances of the geodesic E10/K(E10) sigma-model in a level decomposition w.r.t. the A8xA1 subalgebra of E10, adapted to the bosonic sector of type IIB supergravity, whose SL(2,R) symmetry is identified with the A1 factor. The bosonic supergravity equations of motion, when restricted to zeroth and first order spatial gradients, are also interrelated.

Further studies also included study the fermionic extension of the E10/K(E10) coset model and its relation to eleven-dimensional supergravity. Finite-dimensional spinor representations of the compact subgroup K (E10) of E(10,R) are studied and the supergravity equations are rewritten using the resulting algebraic variables. The canonical bosonic and fermionic constraints. Hidden Symmetries And The Fermionic Sector Of Eleven Dimensional Supergravity investigated the hidden symmetries of the fermionic sector of D=11 supergravity, and the role of K(E10) as a generalized `R-symmetry'. We find a consistent model of a massless spinning particle on an E10/K(E10) coset manifold whose dynamics can be mapped onto the fermionic and bosonic dynamics of D=11 supergravity in the near space-like singular particles and regions. This is a very trend setting study which throws light on many aspects

**Ghost-Free String Effective Actions And Nonunitary Conformal Field Theories**

## Module Numbered Four:

$G_{24}$ : CATEGORY ONE OF **GHOST-FREE STRING EFFECTIVE ACTIONS**

Faddeev–Popov ghosts (also called ghost fields) are additional fields which are introduced into gauge quantum field theories to maintain the consistency of the path integral formulation. They are named after Ludvig Faddeev and Victor Popov. There is also a more general meaning of the word "ghost" in theoretical physics, which is discussed below (see general ghosts in theoretical physics). The necessity for Faddeev–Popov ghosts follows from the requirement that in the path integral formulation, quantum field theories should yield unambiguous, non-singular solutions. This is not possible when a gauge symmetry is present since there is no procedure for selecting any one solution from a range of physically equivalent solutions, all related by a gauge transformation.

The problem stems from the path integrals over counting field configurations related by gauge symmetries, since those correspond to the same physical state; the measure of the path integrals contains a factor which does not allow obtaining various results directly from the original action using the regular methods (e.g., Feynman diagrams). It is possible, however, to modify the action, such that the regular methods will be applicable by adding some additional fields, which break the gauge symmetry, which are called the ghost fields. This technique is called the Faddeev–Popov procedure (see also BRST quantization). The ghost fields are a computational tool in that they do not correspond to any real particles in external states: they only appear as virtual particles in Feynman diagrams – or as the absence of some gauge configurations. However they are necessary to preserve unitarity. The exact form or formulation of ghosts is dependent on the particular gauge chosen, although the same physical results are obtained with all the gauges. The Feynman-'t Hooft gauge is usually the simplest gauge for this purpose, and is assumed for the rest of this article. The Faddeev–Popov ghosts violate the spin-statistics relation, which is another reason why they are often regarded as "non-physical" particles. For example, in Yang–Mills theories (such as quantum chromodynamics) the ghosts are complex scalar fields (spin 0), but they anti-commute (like fermions).

In general, anti-commuting ghosts are associated with fermionic symmetries, while commuting ghosts are associated with bosonic symmetries. Gauge fields and associated ghost fields Every gauge field has an associated ghost, and where the gauge field acquires a mass via the Higgs mechanism, the associated ghost field acquires the same mass (in the Feynman-'t Hooft gauge only, not true for other gauges). In Feynman diagrams the ghosts appear as closed loops wholly composed of 3-vertices, attached to the rest of the diagram via a gauge particle at each 3-vertex. Their contribution to the S-matrix is exactly cancelled (in the Feynman-'t Hooft gauge) by a contribution from a similar loop of gauge particles with only 3-vertex couplings or gauge attachments to the rest of the diagram. (A loop of gauge particles not wholly composed of 3-vertex couplings is not cancelled by ghosts.) The opposite sign of the contribution of the ghost and gauge loops is due to them having opposite fermionic/bosonic natures. (Closed fermion loops have an extra −1 associated with them; bosonic loops don't.)

$G_{25}$ : CATEGORY TWO OF **GHOST-FREE STRING EFFECTIVE ACTIONS**

Lovelock's theory of gravity (often referred to as Lovelock gravity) is a generalization of Einstein's theory of general relativity introduced by David Lovelock in 1971. It is the most general metric theory of gravity yielding conserved second order equations of motion in arbitrary number of spacetime dimensions $D$. In this sense, Lovelock's theory is the natural generalization of Einstein's General Relativity to higher dimensions. In dimension three and four ($D = 3, 4$), Lovelock's theory coincides with Einstein's theory, but in higher dimension both theories are different. In fact, for $D > 4$ Einstein gravity can be thought of as a particular case of Lovelock gravity since the Einstein–Hilbert action is one of several terms that constitute the Lovelock action. Jack, I.; Jones, D. R. T. Explored the σ-model β-functions and ghost-free string effective actions and the relationship between the effective action for a bosonic string propagating in a massless background and the corresponding σ-model, and show that a simple criterion proposed by Mavromatos and Miramontes leads to a manifestly ghost-free form for the O(α'2) action. Lawrence proves that prove that four-dimensional Friedmann-Robertson-Walker solutions to ghost-free superstring and heterotic string effective actions are stable when compactified on a torus. We also consider the case of more general compactifications, and argue that the requirement of stability may have a role in determining the geometry of the compactification. Effect of removing ghosts from string actions by field redefinitions is also undertaken for execution.. Nick E. Mavromatos and Eleftherios Papantonopoulos enucleated and expatiated upon the resultant orientationality of string-inspired effective actions, representing the low-energy bulk dynamics of brane/string theories, the higher-curvature ghost-free Gauss-Bonnet combination is obtained by local field redefinitions which leave the (perturbative) string amplitudes invariant. They show that such redefinitions lead to surface terms which induce curvature on the brane world boundary of the bulk spacetime. (Induced curvature in brane worlds by surface terms in string effective actions with higher-curvature corrections)

$G_{26}$ : CATEGORY THREE OF **GHOST-FREE STRING EFFECTIVE ACTIONS**

MARCO LITTERIO LUCA AMENDOLA study the classical instability of Einstein gauss Bonnet Theory The energy spectrum and stability of the effective theory resulting from the Einstein-Gauss-Bonnet gravity theory with compactified internal space are investigated. The internal space can evolve in its volume and/or shape, giving rise to a system of scalar fields in the external space-time. The resulting scalar-tensor theory of gravity has physically unacceptable properties. First of all, the scalar fields' energy is indefinite and unbounded from below, and thereby the gravitational and scalar fields form a self-exciting system. In contradistinction to the case of multidimensional Einstein gravity, this inherent instability of the effective theory cannot be removed by field redefinitions in the process of dimensional reduction (e.g. by a conformal rescaling of the metric in four dimensions, as is done in the former case). To get a viable effective gravity theory one should discard either the geometric scalar fields or the Gauss-Bonnet term from the Lagrangian of the multidimensional theory. It is argued that it is the Gauss-Bonnet term that should be discarded.

String field theory (SFT) is formalism in string theory in which the dynamics of relativistic strings is reformulated in the language ofquantum field theory. This is accomplished at the level of perturbation theory by finding a collection of vertices for joining and splitting strings, as well as string propagators, that give a Feynman diagram-like expansion for string scattering amplitudes. In most string field theories, this expansion is encoded by a classical action found by second-quantizing the free string and adding interaction terms. As is usually the case in second quantization, a classical field configuration of the second-quantized theory is given by a wave function in the original theory. In the case of string field theory, this implies that a classical configuration, usually called the string field, is given by an element of the free string Fock space. The principal advantages of the formalism are that it allows the computation of off-shell amplitudes and, when a classical action is available, gives non-perturbative information that cannot be seen directly from the standard genus expansion of string scattering. In particular, following the work of Ashoke Sen, it has been useful in the study of tachyon condensation on unstable D-branes. It has also had applications to topological string theory, non-commutative geometry, and strings in low dimensions.

String field theories come in a number of varieties depending on which type of string is second quantized: Open string field theories describe the scattering of open strings, closed string field theories describe closed strings, while open-closed string field theories include both open and closed strings. In addition, depending on the method used to fix the worldsheet diffeomorphisms and conformal transformations in the original free string theory, the resulting string field theories can be very different. Using light cone gauge, yields light-cone string field theories whereas using BRST quantization, one finds covariant string field theories. There are also hybrid string field theories, known ascovariantized light-cone string field theories which use elements of both light-cone and BRST gauge-fixed string field theories. A final form of string field theory, known as background independent open string field theory, takes a very different form; instead of second quantizing the worldsheet string theory, it second quantizes the space of two-dimensional quantum field theories.

$\mathrm{T}_{24}$ :CATEGORY ONE OF **NONUNITARY CONFORMAL FIELD THEORIES**

T. Gannon comments on non unitary conformal field theories in a candid and succinct manner. As is well-known, nonunitary RCFTs are distinguished from unitary ones in a number of ways, two of which are that the vacuum 0 doesn't have minimal conformal weight, and that the vacuum column of the modular S matrix isn't positive. However there is another primary field, call it o, which has minimal weight and has positive S column. We find that often there is a precise and useful relationship, which we call the Galois shuffle, between primary o and the vacuum; among other things this can explain why (like the vacuum) its multiplicity in the full RCFT should be 1. As examples he considers the minimal WSU(N) models. We conclude with some comments on fractional level admissible representations of affine algebras. As an immediate consequence of our analysis, we get the classification of an infinite family of nonunitary WSU(3) minimal models in the bulk. A supersymmetric approach is resorted to in the analysis of the Non-unitary Conformal Field Theory and Logarithmic Operators for Disordered Systems by Z. Maassarani, D. Serban supersymmetric approach to Gaussian disordered systems like the random bond Ising model and Dirac model with random mass and random potential has been not often studied except in certain old papers. . These models appeared in particular in the study of the integer quantum Hall transition. The supersymmetric approach reveals an osp (2/2) _1 affine symmetry at the pure critical point.

A similar symmetry should hold at other fixed points. We apply methods of conformal field theory to determine the conformal weights at all levels. These weights can generically be negative because of non-unitarity. Constraints such as locality allow us to quantize the level k and the conformal dimensions. This provides a class of (possibly disordered) critical points in two spatial dimensions. Solving the Knizhnik-Zamolodchikov equations we obtain a set of four-point functions which exhibit a logarithmic dependence. These functions are related to logarithmic operators. We show how all such features have a natural setting in the superalgebra approach as long as Gaussian disorder is concerned. Hall effect has been obtained from Unitary states in Modified Coulomb gas construction of quantum Hall states from non-unitary conformal field theories M.V. Milovanović, Th. Jolicœur, I. Vidanović Some fractional quantum Hall states observed in experiments may be described by first-quantized wavefunctions with special clustering properties like the Moore-Read Pfaffian for filling factor nu = 5/2.

This wavefunction has been constructed by constructing correlation functions of a two-dimensional conformal field theory (CFT) involving a free boson and a Majorana fermion. By considering other CFTs many other clustered states have been proposed as candidate FQH states under appropriate circumstances. It is believed that the underlying CFT should be unitary if one wants to describe an incompressible i.e. gapped liquid state. We show that by changing the way one derives the wavefunction from its parent CFT it is possible to obtain an incompressible candidate state when starting from a non-unitary parent. The construction mimics a global change of parameters in the phase space of the electron system. We explicit our construction in the case of the so-called Gaffnian states (a state for filling factor 2/5) and also for the Haldane-Rezayi states (a spin-singlet state at filling 1/2).

## $T_{25}$ :CATEGORY TWO OF **NONUNITARY CONFORMAL FIELD THEORIES**

The wave functions of the Haldane-Rezayi paired Hall state have been previously described by a non-unitary conformal field theory with central charge c=-2. Moreover, a relation with the c=1 unitary Weyl fermion has been suggested. We construct the complete unitary theory and show that it consistently describes the edge excitations of the Haldane-Rezayi state. Actually, we show that the unitary (c=1) and non-unitary (c=-2) theories are related by a local map between the two sets of fields and by a suitable change of conjugation. The unitary theory of the Haldane-Rezayi state is found to be the same as that of the 331 paired Hall state. Furthermore, the analysis of modular invariant partition functions shows that no alternative unitary descriptions are possible for the Haldane-Rezayi state within the class of rational conformal field theories with abelian current algebra. Finally, the known c= 3/2 conformal theory of the Pfaffian state is also obtained from the 331 theory by a reduction of degrees of freedom which can be physically realized in the double-layer Hall systems. on-unitary conformalfieldtheory and logarithmic operators for disordered systems has been the subject matter of authors like Maassarani, D. Serban wherein a supersymmetric approach has been utilized and applied for the Integer Hall Quantum Hall effect. Supersymmetric approach to Gaussian disordered systems like the random bond Ising model and Dirac model with random mass and random potential. These models appeared in particular in the study of the integer quantum Hall transition. The supersymmetric approach reveals an $osp(^2/_2)_1$ affine symmetry at the pure critical point. A similar symmetry should not hold at other fixed points. We apply methods of conformalfieldtheory to determine the conformal weights at all levels. These weights can generically be negative because of non-unitarity. Constraints such as locality allow us to quantize the level kand the conformal dimensions.

This provides a class of (possibly disordered) critical points in two spatial dimensions. Solving the Knizhnik-Zamolodchikov equations we obtain a set of four-point functions which exhibit a logarithmic dependence. These functions are related to logarithmic operators. We show how all such features have a natural setting in the superalgebra approach as long as Gaussian disorder is concerned. Partitions functions studies and extension of definitions have been done nby authors like MICHAEL A.I. FLOHR who extend the definitions of characters and partition functions to the case of conformal field theories which contain operators with logarithmic correlation functions. As an example we consider the theories with central charge c=cp,1=13−6(p+p−1), the “border” of the discrete minimal series. We show that there is a slightly generalized form of the property of rationality for such logarithmic theories. In particular, we obtain a classification of theories with c=cp,1 which is similar to the A-D-E classification of c=1 models.

Potential connections to number theory: Again has been done by Moore with a fringe analysis of both attractor mechanism and supersymmetry. One feels that more depthy issues could have been raised for lesser researchers like the me. The attractor mechanism of supersymmetric black holes singles out Calabi-Yau varieties with relations to complex multiplication. Conformal field theories, rational conformal field theories, especially applications to the theory of anyons and nonabelions.Topological field theories, and applications to invariants of manifolds. String field theory has been studied by Greg Moore. Manohar points to anomalous Inequivalence of phenomenological theories and odd dimensions. Some of the interesting issues have been exploited by Greg Moore and the credit goes to him for the lying the foundations of some of the progressive and cathedral work on the subject matter in question Schematic representationalitiesof and functionalities such as he theory of branes and generalized abelian gauge theories in supergravity. This involves interesting topological issues related to generalized differential cohomology theories, especially K-theory. There are also interesting relations to the theory of self-dual fields, anomaly cancellation, and noncommutative geometry. Effective low energy supergravity theories in string compactification and the computation of nonperturbative stringy effects in effective supergravities.D-branes on Calabi-Yau manifolds and BPS state counting. Relations to Borcherds products, automorphic forms, black-hole entropy, and wall-crossing, applications of the theory of automorphic forms to conformal field theory, string compactification, black hole entropy counting, and the AdS/CFT correspondence. Potential connections to number theory. For example - I pointed out in 1998 that the attractor mechanism.

$T_{26}$ : CATEGORY THREE OF **NONUNITARY CONFORMAL FIELD THEORIES**

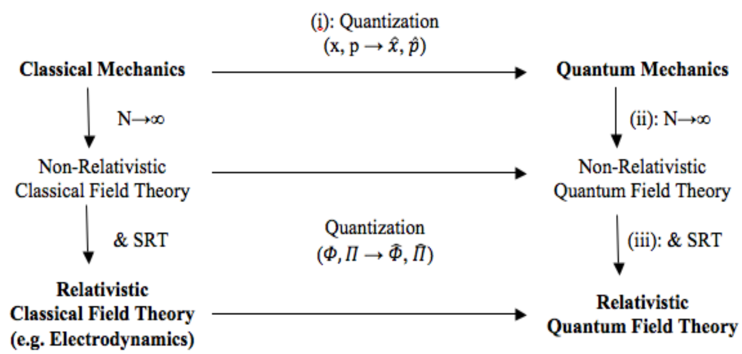
In contrast to many other physical theories there is no canonical definition of what QFT is more concisely and sententiously defined and expatiated and enucleated with the Socratic subjectivity, discourse relativity, and Leibnezian rationality is in Stanford Encyclopedia,. We draw exclusively from this monumental and mammoth documents the visualization of the conformal field theories and Quantum Field theories which should clarify in unmistakable and uncertain terms some of the problems and apprehensions that arise in the field, instead one can formulate a number of totally different explications, all of which have their merits and limits. One reason for this diversity is the fact that QFT has grown successively in a very complex way. Another reason is that the interpretation of QFT is particularly obscure, so that even the spectrum of options is not clear. Possibly the best and most comprehensive understanding of QFT is gained by dwelling on its relation to other physical theories, foremost with respect to QM, but also with respect to classical electrodynamics, Special Relativity Theory (SRT) and Solid State Physics or more generally Statistical Physics. However, the connection between QFT and these theories is also complex and cannot be neatly described step by step. If one thinks of QM as the modern theory of one particle (or, perhaps, a very few particles), one can then think of QFT as an extension of QM for analysis of systems with many particles—and therefore with a large number of degrees of freedom. In this respect going from QM to QFT is not inevitable but rather beneficial for pragmatic reasons. However, a general threshold is crossed when it comes to fields, like the electromagnetic field, which are not merely difficult but impossible to deal with in the frame of QM. Thus the transition from QM to QFT allows treatment of both particles and fields within a uniform theoretical framework. (As an aside, focusing on the number of particles, or degrees of freedom respectively, explains why the famous renormalization group methods can be applied in QFT as well as in Statistical Physics. The reason is simply that both disciplines study systems with a large or an infinite number of degrees of freedom, either because one deals with fields, as does QFT, or because one studies the thermodynamic limit, a very useful artifice in Statistical Physics.)

Moreover, issues regarding the number of particles under consideration yield yet another reason why we need to extend QM. Neither QM nor its immediate relativistic extension with the Klein-Gordon and Dirac equations can describe systems with a variable number of particles. However, obviously this is essential for a theory that is supposed to describe scattering processes, where particles of one kind are destroyed while others are created. One gets a very different kind of access to what QFT is when focusing on its relation to QM and SRT. One can say that QFT results from the successful reconciliation of QM and SRT. In order to understand the initial problem one has to realize that QM is not only in a potential conflict with SRT, more exactly: the locality postulate of SRT, because of the famous EPR correlations of entangled quantum systems. There is also a manifest contradiction between QM and SRT on the level of the dynamics. The

Schrödinger equation, i.e. the fundamental law for the temporal evolution of the quantum mechanical state function, cannot possibly obey the relativistic requirement that all physical laws of nature be invariant under Lorentz transformations. The Klein-Gordon and Dirac equations, resulting from the search for relativistic analogues of the Schrödinger equation in the 1920s, do respect the requirement of Lorentz invariance. Nevertheless, ultimately they are not satisfactory because they do not permit a description of fields in a principled quantum-mechanical way.

Fortunately, for various phenomena it is legitimate to neglect the postulates of SRT, namely when the relevant velocities are small in relation to the speed of light and when the kinetic energies of the particles are small compared to their mass energies mc2. And this is the reason why non-relativistic QM, although it cannot be the correct theory in the end, has its empirical successes. But it can never be the appropriate framework for electromagnetic phenomena because electrodynamics, which prominently encompasses a description of the behavior of light, is already relativistically invariant and therefore incompatible with QM. Scattering experiments are another context in which QM fails. Since the involved particles are often accelerated almost up to the speed of light, relativistic effects can no longer be neglected. For that reason scattering experiments can only be correctly grasped by QFT. Unfortunately, the catchy characterization of QFT as the successful merging of QM and SRT has its limits. On the one hand, as already mentioned above, there also is a relativistic QM, with the Klein-Gordon- and the Dirac-equation among their most famous results. On the other hand, and this may come as a surprise, it is possible to formulate a non-relativistic version of QFT (see Bain 2011).

The nature of QFT thus cannot simply be that it reconciles QM with the requirement of relativistic invariance. Consequently, for a discriminating criterion it is more appropriate to say that only QFT, and not QM, allows describing systems with an infinite number of degrees of freedom, i.e. fields (and systems in the thermodynamic limit). According to this line of reasoning, QM would be the modern (as opposed to classical) theory of particles and QFT the modern theory of particles and fields. Unfortunately however, and this shall be the last turn, even this gloss is not untarnished. There is a widely discussed no-go theorem by Malament (1996) with the following proposed interpretation: Even the quantum mechanics of one single particle can only be consonant with the locality principle of special relativity theory in the framework of a field theory, such as QFT. Hence ultimately, the characterization of QFT, on the one hand, as the quantum physical description of systems with an infinite number of degrees of freedom, and on the other hand, as the only way of reconciling QM with special relativity theory, are intimately connected with one another.




The diagram depicts the relations between different theories, where Non-Relativistic Quantum Field Theory is not a historical theory but rather an ex post construction that is illuminating for conceptual purposes. Theoretically, [(i), (ii), (iii)], [(ii), (i), (iii)] and [(ii), (iii), (i)] are three possible ways to get from Classical Mechanics to Relativistic Quantum Field Theory. But note that this is meant as a conceptual decomposition; history didn't go all these steps separately. On the one hand, by good luck, so to say, classical electrodynamics is relativistically invariant already, so that its successful quantization leads directly to Relativistic Quantum Field Theory. On the other hand, some would argue (e.g. Malament 1996) that the only way to reconcile QM and SRT is in terms of a field theory, so that (ii) and (iii) would coincide. Note that the steps (i), (ii) and (iii), i.e. quantization, transition to an infinite number of degrees of freedom, and reconciliation with SRT, are all ontologically relevant. In other words, by these steps the nature of the physical entities the theories talk about may change fundamentally. See Huggett 2003 for an alternative three-dimensional "map of theories".

Further Reading on QFT and Philosophy of QFT. Mandl and Shaw (2010), Peskin and Schroeder (1995), Weinberg (1995) and Weinberg (1996) are standard textbooks on QFT. Teller (1995) and Auyang (1995) are the first systematic monographs on the philosophy of QFT. The anthologies Brown and Harré (1988), Cao (1999) and Kuhlmann et al. (2002) are anthologies with contributions by physicists and philosophers (of physics), where the last anthology has a focus on ontological issues. The literature on the philosophy of QFT has increased significantly in the last decade. Besides a number of separate papers there are two new monographs, Cao (2010) and Kuhlmann (2010), and one special issue (May 2011) of Studies in History and Philosophy of Modern Physics. Bain (2011), Huggett (2000) and Ruetsche (2002) provide article length discussions on a number of issues in the philosophy of QFT. Some authors study the role of low momentum transfer (soft) interactions between high-transverse momentum heavy particles and beam remnants (spectators) in hadronic collisions.

Such final-state interactions are power suppressed for single-particle inclusive cross sections whenever that particle is accompanied by a recoiling high-p_T partner whose momentum is not fixed. An example is the single-top inclusive cross section in top pair production. Final-state soft interactions in multi-particle inclusive cross sections, including transverse momentum distributions, however, produce leading power corrections in the absence of hard recoiling radiation. Nonperturbative corrections due to scattering from spectators are generically suppressed by powers of \Lambda/p'_T, where \Lambda is a hadronic scale, and p'_T is the largest transverse momentum of radiation recoiling against the particles whose momenta are observed. An important way to check quantum chrome-dynamics is to test its novel predictions — especially effects unique to local gauge theory. In this talk I will discuss a number of unusual or unexpected aspects of QGD.

These include: "null zone* phenomena — zeroes in the cross section for photon emission specific to gauge theories; "color transparency" phenomena—the small value of interaction cross sections for specific components of hadronic wavefunctions; "formation zone" phenomena — the suppression of inelastic interactions at high energies in targets of fixed length; and "intrinsic charm" — the unusual kinematical effect of virtual heavy quark components in the wavefunctions of ordinary hadrons. I will also discuss progress in proving the standard factorization ansati for high momentum trans ir inclusive processes. Factorization for the DreHYan process and the absent of color correlations — now verified to two loops in perturbation theory — is itself a novel aspect of local gauge theory. Interactions: (1) absorption cross sections, (2) single-particle inclusive spectra, and (3) two-particle ...hadronic interactions are very long. ..... of the multiparticle final states corresponding to ...... lead to the breakdown of the model((arXiv:1209.5798 [pdf, ps, other]: Final state interactions **in** single- and multi-particle inclusive cross sections for hadronic collisions- Alexander Mitov, George Sterman)

**Final State Interactions In Single- And Multi-Particle Inclusive Cross Sections And Hadronic Collisions**

## MODULE NUMBERED FIVE:

$G_{28}$ : CATEGORY ONE OF **FINAL STATE INTERACTIONS IN SINGLE- AND MULTI-PARTICLE INCLUSIVE CROSS SECTIONS**

High-pT hadrons produced in hard collisions and detected inclusively bear peculiar features: (i) they originate from jets whose initial virtuality and energy are of the same order; (ii) such jets are rare and have a very biased energy sharing among the particles, namely, the detected hadron carries the main fraction of the jet energy. The former feature leads to an extremely intensive gluon radiation and energy dissipation at the early stage of hadronization, either in vacuum or in a medium. As a result, a leading hadron must be produced on a short length scale. Evaluation within a model of perturbative fragmentation confirm the shortness of the production length. This result is at variance with the unjustified assumption of long production length, made within the popular energy loss scenario. Thus we conclude that the main reason of suppression of high-pT hadrons in heavy ion collisions is the controlled by color transparency attenuation of a high-pT dipole propagating through the hot medium. Adjusting a single parameter, the transport coefficients, we describe quite well the data from LHC and RHIC for the suppression factor RAA as function of pT , collision energy and centrality. We observe that the complementary effect of initial state interaction causes a flattening and even fall of RAA at large pT. The azimuthally anisotropy of hadron production, calculated with no further adjustment, also agrees well with data at different energies and centralities. Recent NA49 results on multiparticle distributions and fluctuations, **shows f**luctuations in the wave functions of hadrons in inelastic interactions with nuclei are (like Color transparency and cross-section fluctuations in hadronic collisions ... particles made of quarks, antiquarks and gluons in a net color singlet state).Particles tend to be **produced** early in the collision; they probe the primordial ... jets and **leads to** the suppression of high p⊥ hadrons in the final state. ..... 4.1 p+p and p+¯p NSD inclusive cross sections versus √s..... a parton-hadron scattering along with the **breakdown** .9 Quenching of high-pT hadrons: Energy Loss vs Color Transparency: B. Z. Kopeliovich, J. Nemchik, I. K. Potashnikova, and Iv´an Schmidt)

$G_{29}$ : CATEGORY TWO OF **FINAL STATE INTERACTIONS IN SINGLE- AND MULTI-PARTICLE INCLUSIVE CROSS SECTIONS**

Particle production in relativistic heavy ion collisions is a basic and important topic for both ... Multiple mechanisms are thought to **be involved** for intermediate-pT hadron ...and final-state interactions between hadrons **produced i**n relativistic heavy ion.... 3.9 Invariant Cross Section . ... 3.10 Weak Decay Feed-down. Hardons is rather **insensitive t**o the mechanism of multiparticle production. ... The n–n cross section is **very small** compared to the transverse area of the nucleus, and..... the deuterium both break up, and that ii) the proton interacts with the nucleus while the ....the final-state single inclusive hadron spectrum is maximal in semi-peripheral. multiparticle production **induced by** very-high- energy cosmic ... bundle of electromagnetic and hadronic particles ... the last decade, accelerator physics in the 10"- ... (1') (a) Single slope of primary energy spec- .... an invariance of normalized inclusive cross sec- .\Ultra-High Energy Cosmic Rays: a Window to Post-Inflationary Reheating Epoch of the Universe? V.A. Kuzmin, V.A. Rubakov (INR, Moscow) We conjecture that the highest energy cosmic rays beyond the Greisen-Zatsepin-Kuzmin cut-off may **provide a** unique window into the very early epoch of the Universe, namely, that of reheating after inflation, provided these cosmic rays are due to decays of parent superheavy long-living X-particles.

These particles may **constitute a** considerable fraction of cold dark matter in the Universe. We argue that the unconventionally long lifetime of the super heavy particles, which should be in the range of $10^ {10} - 10^{22} $ years, might require novel particle physics mechanisms of their decays, such as instantons. We propose a toy model i**llustrating** the instanton scenario.

Generic expected features of ultra-high energy extensive air showers in our scenario are similar to those of other top-down scenarios. However, some properties of the upper part of the cosmic ray spectrum **make the** instanton scenario distinguishable, at least in principle, from other ones

$G_{30}$ :CATEGORY THREE OF **FINAL STATE INTERACTIONS IN SINGLE- AND MULTI-PARTICLE INCLUSIVE CROSS SECTIONS**

The predictions for the Higgs mass in extensions of the Minimal Supersymmetric Standard Model are discussed. We propose a simple theory where the Higgs mass is modified at tree-level and one can achieve a mass around 125 GeV without assuming heavy stops or large left-right mixing in the stop sector. All the parameters in the theory can be perturbative up to the grand unified scale, and one predicts the existence of new colored fields at the TeV scale. We refer to this model as Adjoint MSSM. We discuss the main phenomenological aspects of this scenario and the possible signatures at the Large Hadron Collider. The Affleck-Dine mechanism, which is one of the most attractive candidates for the baryogenesis in supersymmetric theories, often predicts the existence of baryonic Q balls in the early universe. In this scenario, there is a possibility to explain the observed baryon-to-dark matter ratio because Q balls decay into supersymmetric particles as well as into quarks.

If the gravitino mass is small compared to the typical interaction energy, the longitudinal component of the gravitino behaves like the massless goldstino. We numerically calculate the goldstino production rates from Q balls in the leading semi-classical approximation without using large radius limit or effective coupling. We also calculate the quark production rates from Q balls in the Yukawa theory with a massive fermion. In deriving the decay rate we also take into account the scalar field configuration of the Q ball. These results are applied to a realistic model in the gauge-mediated supersymmetry breaking and yield the branching ratio of the Q ball decay into the gravitino. We obtain the branching ratio much smaller than the one estimated in the previous analysis. ( ArXiv: 1209.5781 [pdf, ps, other]: Q ball Decay Rates into Gravitinos and Quarks: Masahiro Kawasaki, Masaki Yamada)Neutrinos **from** propagation of ultrahigh energy protons.Authors present a calculation of the **production of** neutrinos during propagation of ultrahigh energy cosmic rays, as may be produced in astrophysical sources. Photo production interactions are modeled with the event generator SOPHIA **that represents** very well the experimentally measured particle production cross sections at accelerator energies. We give the fluxes expected from different assumptions on cosmic ray source distributions, cosmic ray injection spectra, cosmological evolution of the sources and different cosmologies, and compare them to the Waxman-Bahcall limit on source neutrinos. We estimate rates for detection of neutrino induced showers in a km3 water detector. The ratio of the local high energy neutrino flux to the ultrahigh energy cosmic ray flux is a crucial parameter in distinguishing between astrophysical and cosmological (top-down) scenarios of the ultrahigh energy cosmic ray origin

$T_{28}$ : CATEGORY ONE OF **HADRONIC COLLISIONS**

Extensive Monte Carlo calculation on γ-ray families was carried out under appropriate model parameters which are currently used in high-energy cosmic-ray phenomenology. Characteristics of γ-ray families are systematically investigated by the comparison of calculated results with experimental data obtained at mountain altitudes. The discussion is devoted mainly to examining the validity of Feynman scaling in the fragmentation region of multiplemeson production. It is concluded that the experimental data cannot be reproduced under the assumption of the scaling law if primary cosmic rays are dominated by protons. Other possibilities on primary composition and increase of interaction cross section are also examined. These assumptions are consistent with experimental data only when we introduce intense dominance of heavy primaries in the high-energy region and very strong increase of the interaction cross section (say σ∝E00.06) simultaneously. Otherwise, the breakdown of Feynman scaling in the fragmentation region and the existence of azimuthally asymmetry in production mechanism are strongly suggested by high-energy cosmic-ray interactions (E0≳1015 eV). (Breakdown of Feynman scaling in high-energy cosmic-ray interactions: M. Shibata ) Primary cosmic rays above energies of about 100 TeV are investigated by observations of extensive air showers (EAS) using large area ground based detector installations for registering various components of the EAS cascade development. By such indirect studies of the primary cosmic rays a steepening of the power-law spectrum at around 3–5 PeV, known as the knee, has been identified. At higher energies around 5 EeV there appears a further change of the spectral index towards a flattening of the spectrum, called the ankle.

The energy region above ca 50 EeV, where a cut-off of the cosmic ray spectrum (Greisen–Zatsepin–Kuz'min (GZK) cut-off) is theoretically predicted, is of particular current interest and provides an astrophysical enigma, since obviously trans-GZK events have been observed. Any explanation of these features of the cosmic ray spectrum needs sufficiently detailed knowledge of the shape of the spectrum and of the variation of the mass composition of cosmic rays. In these paper different experimental approaches deducing mass and energy sensitive information from the EAS experiments and their results are discussed. The experiments involve measurements of secondary particle distributions at various observation levels and of muons by deep underground detectors, as well as measurements of air Cherenkov light and, in particular at higher energies, of air fluorescence light emitted during the EAS development. Recently, methods for analysing multi-dimensional EAS parameter distributions have been favoured. They take into account correlations of different EAS parameters and, in particular by non-parametric techniques, also the influence of the intrinsic fluctuation of the air shower development. This paper illustrates the application of such methods in a coherent view of recent results. The advanced analysing methods are corroborated by hybrid experimental set-ups registering a larger set of different EAS observables simultaneously in an event-by-event mode. In addition such approaches provide the possibility to test the consistency of the hadronic interaction models and Monte Carlo procedures used as reference for the analyses. The physical and astrophysical implications of the current findings in various energy regions are briefly discussed and prospects of future experiments are presented.( Energy spectrum and mass composition of high-energy cosmic rays: Andreas Haungs, Heinigerd Rebel and Markus Roth)

$T_{29}$ :CATEGORY TWO OF **HADRONIC COLLISIONS**

The azimuthally anisotropies of the collective transverse flow of charged hadrons are investigated in a wide range of heavy-ion collision energies within the microscopic parton-hadron-string dynamics (PHSD) transport approach which incorporates explicit partonic degrees of freedom in terms of strongly interacting quasiparticles (quarks and gluons) in line with an equation of state from lattice QCD as well as the dynamical hadronization and hadronic collision dynamics in the final reaction phase. The experimentally observed increase of the elliptic flow $v_2$ of charged hadrons with collision energy is successfully described in terms of the PHSD approach. The PHSD scaling properties of various collective observables are confronted with experimental data as well as with hydrodynamic predictions. The analysis of higher-order harmonics $v_3$ and $v_4$ in the azimuthal angular distribution shows a similar tendency of growing deviations between partonic and purely hadronic models with increasing collision energy. This demonstrates that the excitation functions of azimuthal anisotropies reflect the increasing role of quark-gluon degrees of freedom in the early phase of relativistic heavy-ion collisions. Furthermore, the specific variation of the ratio $v_4/(v_2)^2$ with respect to bombarding energy, centrality and transverse momentum is found to provide valuable information on the underlying dynamics Azimuthal anisotropies for Au+Au collisions in the parton-hadron transient energy range: V. P. Konchakovski, E. L. Bratkovskaya, W. Cassing, V. D. Toneev, S. A. Voloshin, V. Voronyuk

$T_{30}$ :CATEGORY THREE OF **HADRONIC COLLISIONS** Authors study the physics potential of future long-baseline neutrino oscillation experiments at large $\theta_{13}$, focusing **especially on** systematic uncertainties. We discuss super beams, \bbeams, and neutrino factories, and for the first time compare these experiments on an equal footing with respect to systematic errors. We explicitly simulate near detectors for all experiments, we use the same implementation of systematic uncertainties for all experiments, and we fully correlate the uncertainties among detectors, oscillation channels, and beam polarizations as appropriate. As our primary performance indicator, we use the achievable precision in the measurement of the CP violating phase $\deltacp$. We find that a neutrino factory is the only instrument that can measure $\deltacp$ with a precision similar to that of its quark sector counterpart. All neutrino beams operating at peak energies $\gtrsim 2$ GeV are quite robust with respect to systematic uncertainties, whereas especially \bbeams and \thk suffer from large cross section uncertainties in the quasi-elastic regime, combined with their inability to measure the appearance signal cross sections at the near detector. A noteworthy exception is the combination of a $\gamma=100$ \bbeam with an \spl-based superbeam, in which all relevant cross sections can be measured in a self-consistent way. This provides a performance, second only to the neutrino factory. For other superbeam experiments such as \lbno and the setups studied in the context of the \lbne reconfiguration effort, statistics turns out to be the bottleneck. In almost all cases, the near detector is not critical to control systematic since the combined fit of appearance and disappearance data already constrains the impact of systematic to be small provided that the three active flavor oscillation frameworks is valid.( Systematic uncertainties in long-baseline neutrino oscillations for large $\theta_{13}$: Pilar Coloma, Patrick Huber, Joachim Kopp, Walter Winter)

**Perturbative String S-Matrix And Supermatrix Models For M-Theory**

**Module Numbered Six:**

$G_{32}$ : CATEGORY ONE OF **PERTURBATIVE STRING S-MATRIX**

Miranda C.N. Cheng, Robbert Dijkgraaf, Cumrun Vafa study Non-Perturbative Topological Strings And Conformal Blocks ny a thesis on the non-perturbative completion of a class of closed topological string theories in terms of building blocks of dual open strings. In the specific case where the open string is given by a matrix model these blocks correspond to a choice of integration contour. We then apply this definition to the AGT setup where the dual matrix model has logarithmic potential and is conjecturally equivalent to Liouville conformal field theory. By studying the natural contours of these matrix integrals and their monodromy properties, with a proposition of a precise map between topological string blocks and Liouville conformal blocks. Remarkably, this description makes use of the light-cone diagrams of closed string field theory, where the critical points of the matrix potential correspond to string interaction points. Group Theoretic approach to computation has been talked by Group theoretic approach to the perturbative string S-matrix is given by A. Neveu, P. West. Here the authors give signature computation for string scattering. . From duality, unitarity and a generic overlap property, we determine entirely the N-string amplitude, including the integration measure, and its gauge properties. The techniques do not use any oscillator algebra, but the computation is reduced to a straightforward exercise in conformal group theory.

This can be applied to fermionic trees and multiloop diagrams, but in this paper it is demonstrated on the open bosonic tree. Integrability of a family of models with U (1)×SU(N) symmetry has been studied by Benjamin Bassoa, Adam Rej They admit fermionic and bosonic formulations related through bosonization and subsequent T-duality. The fermionic theory is just the CPN−1 sigma model coupled to a self-interacting massless fermion, while the bosonic one defines a one-parameter deformation of the O(2N) sigma model. For N=2 the latter model is equivalent to the integrable deformation of the O(4) sigma model discovered by Wiegmann. At higher values of N we find that integrability is more sporadic and requires a fine-tuning of the parameters of the theory. A special case of our study is the N=4 model, which was found to describe the AdS4×CP3 string theory in the Alday–Maldacena decoupling limit.

In this case we propose a set of asymptotic Bethe ansatz equations for the energy spectrum.Radha Balakrishnan etal. Identify two distinct low-energy sectors in the classical isotropic antiferromagnetic Heisenberg spin-S chain, in the continuum limit. We show that two types of rotation generators arise for the field in each sector. Using these, the Lagrangian for sector I is shown to be that of the nonlinear sigma model. Sector II has a null Lagrangian. Its Hamiltonian density is just the Pontryagin term. Exact solutions are found in the form of magnons and processing pulses in I and moving kinks in II. The kink has 'spin' S. Sector I has a higher minimum energy than II. Thermodynamics of Fateev's models in the presence of external fields has been investigated by Davide Controzzi, Alexei M. Tsvelik Bethe Ansatz equations for a one-parameter quantum field theory recently introduced by V.A. Fateev. The presence of chemical potentials produces a kink condensate that modifies the excitation spectrum. For some combinations of the chemical potentials an additional gapless mode appears. Various energy scales emerge in the problem. An effective field theory, describing the low energy excitations, is also introduced. S-matrix theory was a proposal for replacing local quantum field theory as the basic principle of elementary particle physics. It avoided the notion of space and time by replacing it with abstract mathematical properties of the S-matrix. In S-matrix theory, the S-matrix relates the infinite past to the infinite future in one step, without being decomposable into intermediate steps corresponding to time-slices.

This program was very influential in the 1960s, because it was a plausible substitute for quantum field theory, which was plagued with the zero interaction phenomenon at strong coupling. Applied to the strong interaction, it led to the development of string theory. S-matrix theory was largely abandoned by physicists in the 1970s, as quantum chromodynamics was recognized to solve the problems of strong interactions within the framework of field theory. But in the guise of string theory, S-matrix theory is still the best accepted approach to the problem of quantum gravity. The S-matrix theory is related to the holographic principle and the AdS/CFT correspondence by a flat space limit. The analog of the S-matrix relations in AdS space are the boundary conformal theory. The most lasting legacy of the theory is string theory. Other notable achievements are the Froissart bound, and the prediction of the Pomeron. Boot strap Model also called analyticity of the Second Kind, or the bootstrap principle. The principle is that all strongly interacting particles lie on Regge trajectories.

This was considered the definitive sign that all the hadrons are composite particles, but within S-matrix theory, they are not thought of as being made up of elementary constituents. The Regge theory hypotheses allowed for the construction of string theories, based on bootstrap principles. The additional assumption was the narrow resonance approximation, which started with stable particles on Regge trajectories, and added interaction loop by loop in a perturbation series. String theory was given a Feynman path-integral interpretation a little while later. The path integral in this case is the analog of a sum over particle paths, not of a sum over field configurations. Feynman's original path integral formulation of field theory also had little need for local fields, since Feynman derived the propagators and interaction rules largely using Lorentz invariance and unitarity.

$G_{33}$ : CATEGORY TWO OF **PERTURBATIVE STRING S-MATRIX**

Mohammad Garoussi etal, applies their investigation to the collateral of the S-matrix elements of the gauge invariant operators corresponding to on-shell closed strings, in open string field theory. In particular, we calculate the tree level S-matrix element of two ${\it arbitrary}$ closed strings, and the S-matrix element of one closed string and two open strings. By mapping the world-sheet of these amplitudes to the upper half $z$-plane, and by evaluating explicitly the correlators in the ghost part, we show that these S-matrix elements are ${\it exactly}$ identical to the corresponding disk level S-matrix elements in perturbative string theory. There are indications that the model could be extended to Rosh by Deformation radius and other aspectionalities. With nomadic singularities and counteractualisation of primitive ground of states, the question of how to do physics when the notion of space and time breaks down was a major worry of Heisenberg in the 1940s.

He thought space and time already **breaks down** at nuclear scales, because the proton isn't point like. His resolution to the problem of doing space less timeless physics was to use Wheeler's S-matrix as the fundamental variable for physics, to do a pure S-matrix theory. The justification for S-matrix theory is that while we don't know how space and time work in the microscopic theory, we know that the macroscopic theory has certain symmetries at large macroscopic scales, away from the problematic regime. We have translation invariance and rotational/Lorentz invariance. This ***allows*** **one** to define asymptotic particle states on macro-scales, which **are defined** as eigenstates of the Hamiltonian (limiting eigenstates, in the sense of plane-waves), **and define** their scattering. The point is that the S-matrix in-states are always well defined as boosted stable particles in the infinite past, regardless of how horribly spacetime break down during the intermediate stages, and the out-states are similarly well defined. So there is no problem of principle is giving the S-matrix at all energies and all momenta for all collections of particles, without ever directly mentioning space and time in-between. The basic idea for constructing theories **without** space and time, and the answer to your third and fourth question**, is to shift** from a space-time theory to an S-matrix theory. Does String theory say that spacetime is not fundamental but should be considered an emergent phenomenon? Not in the usual description of string theory. Spacetime enters the theory by stating that a string has coordinates in spacetime, so in this sense it is a fundamental property of the theory. Some string theories **can be described** in an alternative way, by a field theory (not a string theory) in which there is one fewer spacetime dimension. This is called holography. In this alternative description, you can say that this 'extra' dimension **is emergent because** it is not part of the spacetime of the holographic field theory.

Chew advocated this point of view for the strong interaction, and in the late 1950s and early 1960s, this was the dominant approach to hadronic physics. Stanley Mandelstam defined extra relations on S-matrix theories which allowed you to make restrictive conditions on the S-matrix from the eigenstates of the Hamiltonian (limiting eigenstates, in the sense of plane-waves), and define their scattering. The point is that the S-matrix in-states are always well defined as boosted stable particles in the infinite past, regardless of how horribly spacetime break down during the intermediate stages, and the out-states are similarly well defined. So there is no problem of principle is giving the S-matrix at all energies and all momenta for all collections of particles, without ever directly mentioning space and time in-between. The basic idea for constructing theories without space and time, and the answer to your third and fourth question, is to shift from a space-time theory to an S-matrix theory. Chew advocated this point of view for the strong interaction, and in the late 1950s and early 1960s, this was the dominant approach to hadronic physics. Stanley Mandelstam defined extra relations on S-matrix theories which allowed you to make restrictive conditions on the S-matrix from the spectrum of the theory, the allowed particles.

This was the way string theory was discovered by Venziano and others in the 1968-1974 era. This is not emphasized in most modern treatments, because most of the development of string theory from this starting point was devoted to working backwards--- now that you have an S-matrix theory, you try to reconstruct the space and time again to as great an extent as possible. In string theory, as it was originally formulated, you didn't have space and time variables (on our space and time), you only had an S-matrix for the collision of various particles, which could be reconstructed order by order in a string expansion. The string expansion did have an infinite tower of resonances, and this allowed Veneziano and others (including Fubini, Ramond, Nambu, Susskind, and Neilson) to identify an internal space, which eventually turned into the string worldsheet. Mandelstam and collaborators showed how to formulate this as a field theory in light-front coordinates, which reconstructed all but two of the space-time coordinates that the string is moving in explicitly, so that the theory was almost formulated in space and time completely. It is also true that the reconstruction of space-time constantly hit a brick wall.

For example, one approach that tried to produce a full space-time description is string field theory, and in the end, it is defined on a loop space, and the way in which you check that it works is by reproducing the string scattering expansion order by order, you don't have a non-perturbative way to calculate string field theory to all orders (at least not in a way that is just as murky about how much space-time is left as in any other approach). Further, the string fields don't obey microcausality (because they are on loops), and the theory doesn't clearly make sense non-perturbatively. So the S-matrix character of the theory never really went away, and in the 1990s, it became clear why. The string theory has a property of holographic duality, which means that the spacetime near a black hole is reconstructed from the boundary data. The precise formulation of holography in Matrix theory and AdS/CFT showed that one can give many alternate formulations of string theory, which all have the property that the spacetime in the bulk is reconstructed from boundary space-time (in the case of matrix theory, you reconstruct it by a large N limit of a one-dimensional quantum mechanical system).

The flat-space limit of AdS/CFT boundary theory is the S-matrix theory of a flat space theory, so the result was the same--- the "boundary" theory for flat space becomes normal flat space in and out states, which define the Hilbert space, while in AdS space, these in and out states are sufficiently rich (because of the hyperbolic braching nature of AdS) that you can define a full field theory worth of states on the boundary, and the S-matrix theory turns into a unitary quantum field theory of special conformal culminating in Polyakov's formulation of the perturbation theory of strings as a path-integral over string trajectories in a normal space-time with background fields. Friedan showed that the asymptotic behavior of these background fields is what you would expect from classical massless theories evolving in space-time, with supersymmetry. The result of this work was that people tended to renounce the idea of S-matrix theory, saying that strings are just space-time moving linearly extended objects. So string theory in the 1980s was pretty conservative with regard to how space-time is to be considered, since it seemed that almost all the space-time is still there. This wasn't completely true, though. None of the approaches actually reconstructed the full space-time at all distance scales, they simply used space-time as intermediate variables in the calculation, where you sum over world-sheets. The world-sheet sum is not so sensitive to small-scale things in the spacetime, so that people were able to do discrete large matrix reductions of certain low-dimensional string theory, where the full string theory emerged from string-bits. It is also true that the reconstruction of space-time constantly hit a brick wall.

For example, one approach that tried to produce a full space-time description is string field theory, and in the end, it is defined on a loop space, and the way in which you check that it works is by reproducing the string scattering expansion order by order, you don't have a non-perturbative way to calculate string field theory to all orders (at least not in a way that is just as murky about how much space-time is left as in any other approach). Further, the string fields don't obey microcausality (because they are on loops), and the theory doesn't clearly make sense non-perturbatively. So the S-matrix character of the theory never really went away, and in the 1990s, it became clear why. The string theory has a property of holographic duality, which means that the spacetime near a black hole is reconstructed from the boundary data. Asymptotic boundary conditions. The question of deSitter boundary conditions is most interesting, both because it is unsolved, and because the universe we live in was deSitter in the past, during inflation, and looks like it will be deSitter in far future.

The question of what the appropriate boundary theory for deSitter spaces requires a new idea, and there are many proposals, although none is fully persuasive today. The answer to your 4th and 5th question is not so simple, because S-matrix theory is so difficult to reconcile with intuition. Within an S-matrix theory, there is still a Hilbert space, defined by asymptotic in and out states, and in principle you are supposed to imagine every state of the world as a superposition of incoming particles which would produce this state "now" (even though the "now" is not completely well defined, the superposition of the incoming states is well defined). Then a measurement is a projection of the "in" state in response to an observation, which can be viewed as a selection of which branch of ridiculously complicated macroscopic "in" state superposition we have found ourselves to be in. The same is true in AdS/CFT. Any state of the interior of an AdS quantum gravity theory should be thought of as a state of the boundary field theory, and then the measurement phenomenon is just the same as in measurements in ordinary quantum field theory, or in ordinary quantum mechanics. The only difference with ordinary quantum mechanics is that the classical limit is removed from intuition even more by an extra layer of abstraction--- shoving every state to an asymptotic "in" and "out", or to the boundary CFT. But if you are comfortable with the process of measurement in ordinary time-coordinate quantum mechanics, you shouldn't be less comfortable with it in S-matrix theory. The only question is how comfortable you should be with it in ordinary quantum mechanics

$G_{34}$ : CATEGORY THREE OF **PERTURBATIVE STRING S-MATRIX**

Neveu, A.; West, P resorted to group Theoretic approach to the perturbative String S matrix .A new approach to the computation of string scattering is given. From duality, unitarity and a generic overlap property, we determine entirely the N-string amplitude, including the integration measure, and its gauge properties. The techniques do not use any oscillator algebra, but the computation is reduced to a straightforward exercise in conformal group theory. This can be applied to fermionic trees and multiloop diagrams, but in this paper it is demonstrated on the open bosonic tree. Hirotaka Irie (NCTS) People believe that non-perturbative effects can give a non-trivial potential uplift String theory leads to fractional superstring theory. Matrix models give Multi Cut matrix models M Theory has four characteristics

1. This lives in 11 dimensional spacetime (1Dim higher).
2. The fundamental DOF is Membrane (M2)
3. The low energy effective theory is 11D SUGRA.
4. Strong coupling dual theory of IIA string theory

Quantum Field theory with the Lorentz-covariant S-matrix was a great step toward combining quantum mechanics with relativity (**Following are the excerpts and points taken from enlightening artile by Y.M.Kim, University of Mary land**)

Quantum Field theory with the Lorentz-covariant S-matrix was a great step toward combining quantum mechanics with relativity. String Theory has the same purpose unless one invents a new quantum mechanics and/or a relativity which are drastically different from those known to us (with a new Einstein and/or a new Heisenberg). What is the difference between strings and quantum fields? Quantum mechanics is based on wave-particle duality. When a particles moves, we should **describe it** also as a wave propagating in a given direction. If a particle is confined within a specified region, like an electron in the hydrogen atom, we need a standing wave with appropriate boundary conditions. It is not difficult to make plane waves **compatible with** Lorentz covariance. This is the reason why it was possible to construct the present form of quantum field theory with the scattering matrix and Feynman diagrams. How about bound states with standing waves? If a standing wave consists of two running waves moving in opposite directions, we have to deal with boundary conditions. Consider, for example, a one-dimensional particle confined between two infinite walls. How do those walls appear to **an observer in a** different Lorentz frame?

The answer to this question is very simple. We do not know. The present form of quantum field theory leads to the Lorentz-covariant scattering matrix or the S matrix, from which we can calculate scattering amplitudes and cross sections. In addition, it leads to a very powerful tool of Feynman diagrams. Quantum electrodynamics has been very successful in producing some of numbers which can be measured experimentally. For other scattering processes, it is now the rule to draw Feynman diagrams first. Those diagrams sometimes produce correct numbers and sometimes not. If not, our excuse is that we cannot calculate all the terms in perturbation series of the S matrix. Instead of relying solely on the perturbation expansion, we can choose selected sets of Feynman diagrams. We call these procedures models. There have been some successful models. For instance, the Lee model is a soluble model. The Bogoliubov transformation in superconductivity is also a soluble model. Both of these models can be reduced to a problem of coupled oscillators.

Toward the end of 1950s, the S matrix became the central issue in physics. The question was whether it is possible to derive the results which can be observed in high-energy laboratories, from analytic properties of the S matrix. If you are sufficiently old, you will remember the words dispersion relations, Regge poles, N over D methods, strip approximation, bootstrap dynamics. The philosophy behind all these ideas was that the

S matrix is the starting point for everything in physics. Within this framework of physics, in 1964, Roger Dashen and Steven Frautschi calculated the neutron-proton mass difference from the belief that the mass difference comes from an electromagnetic pert diagrams. We call these procedures models. There have been some successful models. For instance, the Lee model is a soluble model. The Bogoliubov transformation in superconductivity is also a soluble model. Both of these models can be reduced to a problem of coupled oscillators. Toward the end of 1950s, the S matrix became the central issue in physics. The question was whether it is possible to derive the results which can be observed in high-energy laboratories, from analytic properties of the S matrix. If you are sufficiently old, you will remember the words dispersion relations, Regge poles, N over D methods, strip approximation, bootstrap dynamics. The philosophy behind all these ideas was that the S matrix is the starting point for everything in physics. Within this framework of physics, in 1964, Roger Dashen and Steven Frautschi calculated the neutron-proton mass difference from the belief that the mass difference comes from an electromagnetic perturbation. Indeed, it was once believed to be a history making calculation.

$T_{32}$ : CATEGORY ONE OF **SUPERMATRIX MODELS FOR M-THEORY**

Scattering matrix (or S-matrix) **relates the** initial state and the final state of a physical system undergoing a scattering process. It is used in mechanics, scattering and quantum field theory. More formally, the S-matrix is defined as the unitary matrix connecting asymptotic particle states in the Hilbert space of physical states (scattering channels). While the S-matrix may be defined for any background (spacetime) that is asymptotically solvable and has no horizons, it has a simple form in the case of the Minkowski space. In this special case, the Hilbert space is a space of irreducible unitary representations of the inhomogeneous Lorentz group; the S-matrix is the evolution operator between time equal to minus infinity, and time equal to and infinity. It is defined only in the limit of zero energy density (or infinite particle separation distance). It can be shown that if a quantum field theory in Minkowski space has a mass gap, the state in the asymptotic past and in the asymptotic future are both described by Fock spaces Maxime Bagnoud, Luca Carlevaro, Adel Bilal Studied Supermatrix models for Mtheory. Quintessentially, it was based on osp (1/32,R)They derive supersymmetry transformations for all fields that appear in 11- and 12-dimensional realizations and give the associated SUSY algebras. We study the background-independent osp (1|32,R) cubic matrix model action expressed in terms of representations of the Lorentz groups SO(10,2) and SO(10,1).

We explore further the 11-dimensional case and compute an effective action for the BFSS-like degrees of freedom. We find the usual BFSS action with additional terms incorporating couplings to transverse 5-branes, as well as a mass-term and an infinite tower of higher-order interaction Maxime Bagnouda, Luca Carlevaroa, ,Adel Bilal, studied a similar conformal alterations, entwining and resemblances, with a pluralistic outlook Taking seriously the hypothesis that the full symmetry algebra of M-theory is $\mathfrak{osp}(1|32,\mathbb{R})$, we derive the supersymmetry transformations for all fields that appear in 11- and 12-dimensional realizations and give the associated SUSY algebras. We study the background-independent $\mathfrak{osp}(1|32,\mathbb{R})$ cubic matrix model action expressed in terms of representations of the Lorentz groups SO (10,2) and SO(10,1). We explore further the 11-dimensional case and compute an effective action for the BFSS-like degrees of freedom. We find the usual BFSS action with additional terms incorporating couplings to transverse 5-branes, as well as a mass-term and an infinite tower of higher-order interactions. Jeong-Hyuck Park concentrated on the homogeneity, specificity and inferential thought and diversity of Noncommutative superspace of arbitrary dimensions in a thematic way. Superfield theories on a noncommutative superspace can be formulated intwo folds, through the star product formalism and in terms of the super matrices.

Authors elab-orate the duality between them by constructing the isomorphism explicitly and relating the superspace integrations of the star product Lagrangian or the superpotential to the traces of the super matrices. We show there exists an interesting ¯ne tuned commutative limit where the duality can be still maintained. Namely on the commutative superspace too, there exists a Supermatrix model description for the superfield theory. This is interpreted as the result in the context of the wave particle duality. The dual particles for the superfields in even and odd spacetime dimensions are D-instantons and D0-branes respectively to be consistent with the T-duality. Continuum Schwinger-Dyson equations for nonperturbative two-dimensional quantum gravity coupled to various matter fields. The continuum Schwinger-Dyson equations for the one-matrix model are explicitly derived and turn out to be a formal Virasoro condition on the square root of the partition function, which is conjectured to be the τ function of the KdV hierarchy. Furthermore, we argue that general multi-matrix models are related to the W algebras and suitable reductions of KP hierarchy and its generalizations.( By MASAFUMI FUKUMA, HIKARU KAWAI, RYUICHI NAKAYAMA)

$T_{33}$ : CATEGORY TWO OF **SUPERMATRIX MODELS FOR M-THEORY**

In high-energy particle physics, we are interested in computing the probability for different outcomes in scattering experiments. These experiments can be broken down into three stages:1. Collide together a collection of incoming particles (usually two particles with high energies). 2. Allowing the incoming particles to interact. These interactions may change the types of particles present (e.g. if an electron and a positron annihilate they may produce two photons). 3. Measuring the resulting outgoing particles. The process by which the incoming particles are transformed (through their interaction) into the outgoing particles is called scattering. For particle physics, a physical theory of these processes must be able to compute the probability for different outgoing particles when we collide different incoming particles with different energies. The S-matrix in quantum field theory is used to do exactly this. It is assumed that the small-energy-density approximation is valid in these cases.

The S-matrix is closely related to the transition probability amplitude in quantum mechanics and to cross sections of various interactions; the elements (individual numerical entries) in the S-matrix are known as scattering amplitudes. Poles of the S-matrix in the complex-energy plane are identified with bound states, virtual states or resonances. Branch cuts of the S-matrix in the complex-energy plane are associated to the opening of a scattering channel. In the Hamiltonian approach to quantum field theory, the S-matrix may be calculated as a time-ordered exponential of the integrated Hamiltonian in the interaction picture; it may be also expressed using Feynman's path integrals. In both cases, the perturbative calculation of the S-matrix leads to Feynman diagrams. In scattering theory, the S-matrix is an operator mapping free particle in-states to free particle out-states (scattering channels) in the Heisenberg picture. This is very useful because often we cannot describe exactly the interaction (at least, not the most interesting ones). Matrix is a C++ package for high performance vector and matrix computations. It can be used only in problems when the size of the matrices is known at compile time, like in the tracking reconstruction of HEP experiments. It is based on a C++ technique, called expression templates, to achieve an high level optimization. The C++ templates can be used to implement vector and matrix expressions such that these expressions can be transformed at compile time to code which is equivalent to hand optimized code in a low-level language like FORTRAN or C

The SMatrix has been developed initially by T. Glebe of the Max-Planck-Institut, Heidelberg, as part of the HeraB analysis framework. A subset of the original package has been now incorporated in the ROOT distribution, with the aim to provide to the LHC experiments a stand-alone and high perform ant matrix package for reconstruction. The API of the current package differs from the original one, in order to be compliant to the ROOT coding conventions. SMatrix contains generic Matrix and Vector classes to describe matrix and vector of arbitrary dimensions and of arbitrary type. The classes are template on the scalar type and on the size of the matrix (number of rows and columns) or the vector. Therefore, the size has to be known at compile time. SMatrix supports symmetric matrices using a storage class contains only the N*(N+1)/2 independent element of a NxN symmetric matrix.It is not in the mandate of this package to provide a complete linear algebra functionality for these classes. What is provided are basic Matrix Template Functions and Vector Template Functions, such as the matrix-matrix, matrix-vector, vector-vector operations, plus some extra functionality for square matrices, like inversion, which is based on the optimized Cramer method for squared matrices of size up to 6x6, and determinant calculation?

$T_{34}$ : CATEGORY THREE OF **SUPERMATRIX MODELS FOR M-THEORY**

**Expansion Rate Of Quantum Field Describing Electrons And Other Fermions And Creation And Annihilation Of Operators according To Pauli's Exclusion Principle(Again We Talk Of The Systemic Characteristics To Which Pauli's Exclusion Principle Is Applied, And The Parametricization And Stratification Follows)**

## Module Numbered Seven

$G_{36}$ : CATEGORY ONE OF CREATION AND ANNIHILATION OPERATORS DUE TO PUALIS EXCLUSION PRINCIPLE (NOTE PAULIS EXCLUSION PRINCIPLE DENIES THE TWO STATES FOR THE ELECTRONS)

The differences between basic QM and Field Theory are these: in QM, the interactions between more than two particles are increasingly difficult to model, and the creation and destruction of particles cannot be modeled at all. In contrast, Field Theory can describe states containing arbitrary numbers of particles of different energies, masses, charges and types. Field Theory also provides an elegant framework for describing the interactions between particles, and the creation of new particles and destruction of old ones-- for example, the emission and absorption of photons by electrons, and vice versa. For a given interaction of particles, field theoretical calculations generally cannot be solved in a closed, analytical form-- that is, the predicted probabilities cannot be described by one simple equation; however, physicists have developed numerous approximation methods which produce estimates of ever-increasing precision (compared to experiment), with the precision depending upon how much mathematical work is put into the analysis. Field theoretical calculations, while extremely labor-intensive, have yielded predictions of unparalleled precision compared to experimental measurements. The most famous of these is the anomalous magnetic moment of the electron, a very difficult calculation that has (so far) yielded a prediction accurate to one part in a billion compared to experiment. Field theoretical calculations can be extremely labor-intensive, if math is employed by hand; or computer time-intensive, if software approximations are used. Computational attempts to model the field theoretical equations of the strong nuclear force within protons, neutrons and mesons are among the most intensive computational research projects ever attempted, but have produced good agreement with experiment. Field theoretical calculations become increasingly difficult as the number of (incoming or outgoing) particles increases; for very large numbers of particles (e.g. macroscopic liquids and solids), the methods of Solid State Physics (also based on QM) are employed instead. Quantum Field Theory has Special Relativity built in as an intrinsic part of the theory. However, the most naive application of Field Theory to General Relativity is known to be unworkable. This has led to attempts at alternative formulations to reconcile Field Theory and GR. In contrast to the other three forces of nature (electromagnetic, weak nuclear, strong nuclear), which have extensive and impressive experimental confirmation, no alternative formulations of quantum gravity have experimental confirmation. Field Bosons Mediate Action At a Distance.

How is it that particles affect each other-- for example, the repulsion of like charges, or attraction of unlike charges? Field theory can incorporate "action at a distance" models by the direct incorporation of an external potential field, e.g. V(x) where x is the distance from a fixed, unmoving charge, and V is the potential energy of the interaction. However, such models are of limited use because x becomes dependent on the motion of the other particle, if the other particle moves; and in addition this method ignores fluctuations of virtual particles expected from Heisenberg's Uncertainty Principle. Thus, most commonly field theory employs other methods. The forces of nature are mediated by particles called field bosons. Particles exert forces on each other, which appears to be action at a distance, by passing back and forth between them virtual field bosons, which exchange their momenta. As an analogy, consider two people on ice skates on an ice rink, playing football. The first throws a football, and the reaction force of the throw pushes him backward. The second catches the football, and the catch pushes her backward. The two ice skaters are now "repelled" and moving away from each other, with the football as mediator of the force. Thus, virtual bosons mediate attractive and repulsive forces.

Bosons in general are particles of integral spin (0, 1, 2, etc.) "Spin" here refers to quantum mechanical angular momentum of a particle spinning about its axis. In quantum mechanics, all elementary particles can only have angular momenta which are multiples or half-multiples of Planck's constant. In particular, the first three forces (but not gravity) are considered to be mediated by bosons of spin 1.

The electromagnetic force is mediated by the photon, a particle of light, which is massless and uncharged. The theory which describes the interactions of charged particles and photons is called Quantum

Electrodynamics (QED), attributed primarily to Richard Feynman and Julian Schwinger.

The weak nuclear force is mediated by particles called W and Z bosons. These are very massive particles, as massive as a heavy atomic nucleus. The W particles are charged (W+ and W-) and Z is uncharged.

The electromagnetic and weak forces have been mathematically unified into a single formalism, called Electroweak Theory. The unification means that photons, W and Z bosons are all considered to be different aspects of a more fundamental doublet of field bosons. The electroweak theory can describe electromagnetic and weak phenomena with fewer tuneable free parameters-- as the single most important goal of physics is to describe all forces with as few free parameters as possible. Electroweak Theory has been extremely successful, and predicted the existence and approximate mass of the Z boson before its observation in experiments.

The strong nuclear force is mediated by particles called gluons. Particles such as quarks are said to have "color charge", which gives them an ability to exchange gluons, in the same way that particles with electrical charge can exchange photons. The theory that describes quarks and gluons is called Quantum Chromodynamics, or QCD. A major complication in QCD is that the gluons themselves have "color charge", unlike, say, photons which have no electrical charge. This makes QCD calculations extremely difficult. On the other hand, it has the advantage of eliminating the "screening problem" identified by Landau. Also, the coupling between quarks and gluons (color charge) is, measured in absolute units, much larger than the electrical charges of charged particles like electrons.

$G_{37}$ : CATEGORY TWO OF CREATION AND ANNIHILATION OPERATORS DUE TO PUALIS EXCLUSION PRINCIPLE (NOTE PAULIS EXCLUSION PRINCIPLE DENIES THE TWO STATES FOR THE ELECTRONS)

In the Standard Model of physics, there are four forces of nature: electromagnetic, weak nuclear, strong nuclear, and gravity. The electromagnetic force models interactions between electrically charged particles, and historically resulted from a unification of the electrical and magnetic fields, which were once thought to be separate fields. The weak nuclear force is most well-known for mediating radioactive atomic decays, in which (for example) a proton in a nucleus will turn into a neutron (which remains in the nucleus), and a positron and neutrino, which are emitted. Non-nuclear particles such as electrons also participate in the weak force. Neutrinos only participate in the weak force, and have extremely low mass, making their observation very difficult. The weak nuclear force only exerts force when particles are extremely close together. The strong nuclear force holds together the protons and neutrons in an atomic nucleus, and the quarks within protons, neutrons, and mesons. Because protons all have the same charge, they repel each other strongly, and the strong nuclear force is necessary to overcome this and hold them together in a nucleus; likewise for the quarks inside protons, neutrons and mesons.

Unlike electromagnetism, which can extend over long distances, the strong nuclear force only exerts force when particles are extremely close together; but at close range, it is enormously stronger than electromagnetism. Field theory can model the first three forces with a high degree of precision and success, but fails for gravity, for reasons described below. Gravity is by far the weakest of all the forces of nature, when measured in absolute units. This may seem to us to be paradoxical. Electromagnetism (for example) appears to us to be weaker than gravity, because most matter we encounter is nearly equal in positive and negative charges, so that the opposing charges nearly cancel at long distances. However, with gravity, there is no observed "negative mass" to cancel out the effect of positive mass, so the gravitational forces of particles, while individually very tiny, are cumulatively enormous over long distances. (Antimatter has positive mass, but opposite properties to matter.) A difference between QED/Electroweak and QCD is that the coupling of electron and photons (electrical charge) is much less than 1 when measured in absolute units, while the color charges of quarks in QCD is greater than 1.

This means that predictions in QED can be approximated by a set of mathematical methods called "perturbative", while QCD calculations often requires methods called "non-perturbative." As a simple case, consider two electrons rushing toward each other. They should be repelled by like charges. In a QED perturbative calculation, this repulsion, while very complex, is modeled as a series of ever more complicated Feynman graphs each representing different exchanges of intermediate virtual particles. With each step in the calculation, more and more intermediate (internal) particles are exchanged, and the graphs grow more and more complicated. The zeroth. Order approximation is that the outgoing particles are the same as the incoming (no change). The first order approximation is that one virtual photon is passed from one electron to the other. Graphs for the second order approximation are: the electrons exchange two virtual photons; OR, one incoming electron emits a photon, which splits into an electron/position pair, the pair recombines to form a photon, which is finally absorbed by the other incoming electron. Each new graph is considered an additional "perturbation" of the zeroth. approximation (no change). By the addition of many such graphs, predictions of extraordinary precision can be computed; but this is very labor-intensive. In QCD, perturbative methods are of limited use because the color charge is so strong, that each new graph contributes more than the previous one, so the series may not terminate.

Consequently, the interactions of quarks are often modeled by other means, e.g. the use of very powerful computer simulations called "Lattice Gauge QCD" simulations. In these simulations, space and time are approximated as a lattice of points separated by fixed distances. The quarks and gluons do not have fixed positions on the lattice, as they are quantum mechanical wave functions; rather, the field of each quark and gluon must be continually computed all over the lattice, the value at each lattice point continually changing due to the strong interactions of the wave functions. These methods require enormous amounts of time on powerful supercomputers, but have produced several important recent successes.

$G_{36}$ : CATEGORY THREE OF ORTHOGONAL ENERGY STATE OF VACUUM (ENERGY EXCITATION OF THE VACUUM AND CONCOMITANT GENERATION OF ENERGY DIFFERENTIAL-TIME LAG OR INSTANTANEOUSNESS MIGHT EXISTS WHEREBY ACCENTUATION AND ATTRITIONS MODEL MAY ASSUME ZERO POSITIONS)

Following are the excerpts taken from the Blog of Professor Thomas Hertog at CERN. A further differentiation of the organization of action processes between electrons and fermions, when time dimension is taken in to account. All the experimental results are oriented towards the achievement of this indubitable goal which can come about with the intervention of other factors like fermions and electrons in the processes. Such an instrumental and goal discipline introduces a sense of verkrampte reactionariness and ensorcelled frenzy toward the achievement of the goal, in the real sense of term.A new paper by Cambridge physicist Stephen Hawking and Thomas Hertog of CERN (hertog@mail.cern.ch) suggests that it can. The leading explanation for the observed acceleration of the expansion of the universe is that a substance, dark energy, fills the vacuum and produces a uniform repulsive force between any two points in space -- a sort of anti-gravity. Quantum field theory allows for the existence of such a universal tendency. Unfortunately, its prediction for the value of the density of dark energy (a parameter referred to as the cosmological constant) is some 120 orders of magnitude larger than the observed value. In 2003, cosmologist Andrei Linde of Stanford University and his collaborators showed that string theory allows for the existence of dark energy, but without specifying the value of the cosmological constant. String theory, they found, produces a mathematical graph shaped like a mountainous landscape, where altitude represents the value of the cosmological constant.

After the big bang, the value would settle on a low point somewhere between the peaks and valleys of the landscape. But there could be on the order of 10500 possible low points -- with different corresponding values for the cosmological constant -- and no obvious reason for the universe to pick the one we observe in nature. Some experts hailed this multiplicity of values as a virtue of the theory. For example, Stanford University's Leonard Susskind in his book "The Cosmic Landscape: String Theory and the Illusion of Intelligent Design," argues that different values of the cosmological constant would be realized in different parallel worlds -- the pocket universes of Linde's "eternal inflation" theory.

We would just happen to live in one where the value is very small. But critics see the landscape as exemplifying the theory's inability to make useful predictions. The Hawking/Hertog paper is meant to address this concern. It looks at the universe as a quantum system in the framework of string theory. Quantum theory calculates the odds a system will evolve a certain way from given initial conditions, say, photons going through a double slit and hitting a certain spot on the other side. You repeat your experiment often enough and then you check that the odds you predicted were the correct ones. In Richard Feynman's formulation of quantum theory, the probability that a photon ends up at a particular spot is calculated by summing up over all possible trajectories for the photon. A photon goes through multiple paths at once and can even interfere with its other personas in the process. Hawking and Hertog argue that the universe itself must also follow different trajectories at once, evolving through many simultaneous, parallel histories, or "branches." (These parallel universes are not to be confused with those of eternal inflation, where multiple universes coexist in a classical rather than in a quantum sense.)

What we see in the present would be a particular, more or less probable, outcome of the "sum" over these histories. In particular, the sum should include all possible initial conditions, with all possible values of the cosmological constant. But applying quantum theory to the entire universe -- where the experimenters are part of the experiment -- is tricky. Here you have no control over the initial conditions, nor can your repeat the experiment again and again for statistical significance. Instead, the Hawking-Hertog approach starts with the present and uses what we know about our branch of the universe to trace its history backwards. Again, there will be multiple possible branches in our past, but most can be ignored in the Feynman summation because they are just too different from the universe we know, so the probability of going from one to the other is negligible. For example, Hertog says, knowledge that our universe is very close to being flat could allow one to concentrate on a very small portion of the string theory landscape whose values for the cosmological constant are compatible with that flatness. That could in turn lead to predictions that are experimentally testable. For example, one could calculate whether our universe is likely to produce the microwave background spectrum we actually observe.

$T_{36}$ : CATEGORY ONE OF EXPANSION RATES OF QUANTUM FIELDS DESCRIBING ELECTRONSAND OTHER FERMIONS

Quantum field theory (QFT) provides a theoretical framework for constructing quantum mechanical models of systems classically **represented** by an infinite number of degrees of freedom, that is, fields and (in a condensed matter context) many-body systems. It is the natural and quantitative language of particle physics and condensed matter physics. Most theories in modern particle physics, including the Standard Model of elementary particles and their interactions, **are formulated** as relativistic quantum field theories. Quantum field theories are **used in** many contexts, and are especially vital in elementary particle physics, where the particle count/number may change over the course of a reaction. They are also **used in the description** of critical phenomena and quantum phase transitions, such as in the BCS theory of superconductivity. In perturbative quantum field theory, the forces between particles **are mediated** by other particles.

The electromagnetic force between two electrons **is caused by** an exchange of photons. Intermediate vector bosons mediate the weak force and gluons **mediate** the strong force. There is currently no complete quantum theory of the remaining fundamental force, gravity, but many of the proposed theories postulate the existence of a graviton particle that mediates it. These force-carrying particles are virtual particles and, by definition, cannot be detected while carrying the force, **because such** detection will imply that the force is not being carried. In addition, the notion of "force mediating particle" comes from perturbation theory, and thus does not make sense in a context of bound states. In QFT, photons are not thought of as "little billiard balls" but are rather **viewed a**s field quanta – necessarily chunked ripples in a field, or "excitations", that "look like" particles. Fermions, like the electron, can also be described as ripples/excitations in a field, where each kind of fermion has its own field. In summary, the classical visualization of "everything is particles and fields", in quantum field theory, resolves into "everything is particles", which then resolves into "everything is fields". In the end, particles are regarded as excited states of a field (field quanta). The gravitational field and the electromagnetic field are the only two fundamental fields in Nature that have infinite range and a corresponding classical low-energy limit, which greatly diminishes and hides their "particle-like" excitations.

Albert Einstein, in 1905, attributed "particle-like" and discrete exchanges of momenta and energy, characteristic of "field quanta", to the electromagnetic field. Originally, his principal motivation was to explain the thermodynamics of radiation. Although it is often claimed that the photoelectric and Compton effects require a quantum description of the EM field, this is now understood to be untrue, and proper proof of the quantum nature of radiation is now taken up into modern quantum optics as in the antibunching effect. The word "photon" was coined in 1926 by physical chemist Gilbert Newton Lewis. In quantum field theory, a fermionic field is a quantum field whose quanta are fermions; that is, they obey Fermi–Dirac statistics. Fermionic fields obey canonical anticommutation relations rather than the canonical commutation relations of bosonic fields. The most prominent example of a fermionic field is the Dirac field, which describes fermions with spin-1/2: electrons, protons, quarks, etc. The Dirac field can be described as either a 4-component spinor or as a pair of 2-component Weyl spinors. Spin-1/2 Majorana fermions, such as the hypothetical neutralino, **can be described** as either a dependent 4-component Majorana spinor or a single 2-component Weyl spinor. It is not known whether the neutrino is a Majorana fermion or a Dirac fermion (see also Neutrinoless double-beta decay

$T_{37}$ : CATEGORY TWO OF EXPANSION RATES OF QUANTUM FIELDS DEXCRIBING ELECTRONSAND OTHER FERMIONS

How the flux of Yang-Mills gravitational exchange radiation (gravitons) being exchanged between all the masses in the universe physically **creates a**n observable gravitational acceleration field directed towards a cosmologically nearby or non-receding mass, labelled 'shield'. (The Hubble expansion rate and the distribution of masses **around us** are virtually isotropic, i.e., radially symmetric.) Internal differentiation, structural morphology, comparative dependability and normative aspect of resultant orientationality of expectations of experimental results and experience thereof is the prime prima Donna of the matter in question. The mass labelled 'shield' **creates an** asymmetry for the observer in the middle of the sphere, since it shields the graviton flux because it doesn't have an outward force relative to the observer (in the middle of the circle shown), and thus **doesn't produce a** forceful graviton flux in the direction of the observer according to Newton's 3rd law (action and reaction, an empirical fact, not a speculative assumption). Hence, any mass that is not at a vast cosmological distance (with significant redshift) physically **acts as a shield** for gravitons, and you get pressed towards that shield from the unshielded flux of gravitons on the other side. Gravitons act **by pushing**, they have spin-1.

Diagrammatically speaking, r is the distance to the mass that is shielding the graviton flux from receding masses located at the far greater distance R. As you can see from the simple but subtle geometry involved, the effective size of the area of sky **which is causing** gravity due to the asymmetry of mass at radius r is equal to the cross-sectional area of the mass for quantum gravity interactions (detailed calculations, included later in this post, show that this cross-section turns out to be the area of the event horizon of a black hole for the mass of the fundamental particle which is acting as the shield), multiplied by the factor (R/r)2, which is how the inverse square law, i.e., the 1/r2 dependence on gravitational force, occurs. Because this mechanism **is built on** solid facts of expansion from redshift data that can't be explained any other way than recession, and on experiment and observation **based** laws of nature such as Newton's, it is not just a geometric explanation of gravity but it uniquely **makes detailed predictions** including the strength of gravity, i.e., the value of G, and the cosmological expansion rate; it is a simple theory as it uses spin-1 gravitons which exert impulses that add up **to an** effective pressure or force when exchanged between masses.

It is quite a different theory to the mainstream model which ignores graviton interactions with other masses in the surrounding universe. The mainstream model in fact can't predict anything at all. It begins by ignoring all the masses in the universe except for two masses, such as two particles. It then represents gravity interactions between those two masses by a Lagrangian field equation which it **evaluates by** a Feynman path integral. It finds that if you ignore all the other masses in the universe, and just consider two masses, then spin-1 gauge boson exchange **will cause** repulsion, not attraction as we know occurs for gravity. It then 'corrects' the Lagrangian by changing the spin of the gauge boson to spin-2, which has 5 polarizations. This new 'corrected' Lagrangian with 5 tensor terms for the 5 polarizations of the spin-2 graviton being assumed gives an always-attractive force between two masses when put into the path integral and evaluated. However, it doesn't say how strong gravity is, or makes any predictions that can be checked. Thus, the mainstream first makes one error (ignoring all the graviton interactions between masses all over the universe) whose fatally flawed prediction (repulsion instead of attraction between two masses) it **'corrects' using** another error, a spin-2 graviton.

It is like a balancing error in Bank's books. Like there is one wrong entry on the debit side, there is also corresponding wrong entry on the credit side. Such balancing errors are very difficult to find for the books and Cosmic ledgers remain tallied, but still individual differences exist between the General Theory of

ISSN 2250-3153

Cosmic Ledger and the concomitant and corresponding individual portfolio balance. Like say that of electron totality in the universe or the fermion totality, or for that matter the dark matter existence as per experimental observation as and the theoretical predictions, projections and prognostications. So one reason why the actual spin-2 gravitons don't cause masses to repel is because the path integral isn't just a sum of interactions between two gravitational charges (composed of mass-energy) when dealing with gravity; it's instead a sum of interactions between all mass-energy in the universe.

The reason why mainstream people don't comprehend this is that the mathematics being used in the Lagrangian and path integral are already fairly complex, and they can't readily include the true dynamics so they ignore them and believe in a fiction instead. (There is a good analogy with the false mathematical epicycles of the Earth-centred universe. Whenever the theory was in difficulty, they simply added another epicycle to make the theory more complex, 'correcting' the error. Errors were actually celebrated and simply re-labelled being 'discoveries' that nature must contain more epicycles.)

Quantum field theory is the most successful physical theory ever, encompassing all known nuclear interactions and electromagnetism, and it has many more successful predictions and experimental tests than general relativity, so it is apparent that general relativity needs modification to accommodate quantum field theory, than the other way around. [General relativity necessitates a stress-energy tensor as the source of the gravitational field, and the gravitational field source in this tensor is represented by continuous differential equations, rather than discontinuous (lumpy) quantized matter. So the fact a smooth curvature (continuous, smooth acceleration curve on a Feynman diagram) results from general relativity is a product of the approximation used to statistically average the gravity field source, instead of properly representing the lumps. There other reasons why general relativity is just a flawed - classical - approximation as well. For instance, in addition to falsely assuming that mass is smoothly distributed in space instead of coming in lumps, general relativity also simply ignores any possibility of quantized field quanta, gravitons.] Quantum field theory has also been successfully applied to explain superfluid properties, because in condensed matter physics (low temperature phenomena generally) pairs of half integer spin fermions can associate to produce composite particles that have the properties of integer spin particles, bosons. In 1925, Max Born and Pascual Jordan recognised that a quantum transition, such as the fall of an electron from an excited to a ground state (accompanied by the emission of a photon), is a complicated problem because the number of particles changes (a photon is created or is absorbed).

Classical Maxwellian electrodynamics does describe radiation emission due to acceleration of charge, but does not explain why radiation is quantized. The quantum theory of Planck, and Bohr's atomic model, deal with specific problems (blackbody radiation spectra and line spectra, respectively), but are not general theories. It is now recognised that the correct explanation of quantum electrodynamics **lies in** Yang-Mills quantum field theory, in which exchange of radiation (as depicted by Feynman diagrams) is the underlying mechanism. In 1926, Werner Heisenberg, together with Born and Jordan, developed a quantum theory of electromagnetism by a process called canonical quantization, whereby quanta are treated as separate oscillators of given frequency. Their treatment neglected polarization and charge, and was inconsistent with relativity considerations. Jordan in 1927 employed a second quantization to include charges and thus quantum mechanics, while Dirac discovered a Hamiltonian for Schrödinger's time-dependent equation which is consistent with relativity. Schrödinger's time-dependent equation is essentially **saying the same thing** as this electromagnetic energy mechanism of Maxwell's 'displacement current': Hy= iħ.dy /dt = (½ih/p)dy /dt, where ħ = h/(2p). The energy flow is directly proportional to the rate of change of the wavefunction. This is identical to the classical Maxwell 'displacement current' term which states that the rate of flow of energy (via virtual 'displacement current') across vacuum in a capacitor or radio system **is directly proportional** to the rate of change of the electric field! In a charging capacitor, the displacement current falls as a function of time as the capacitor charges up. The solution for the fall of 'displacement current' flow across the vacuum (and through the circuit) as the capacitor charges up to a maximum capacity are: it = ioe- t / RC. This energy-based solution is similarly exponential to the solution to Schrödinger's equation: yt = yo exp[-2piH(t – to)/h].

$T_{38}$ : CATEGORY THREE OF EXPANSION RATES OF QUANTUM FIELDS DESCRIBING ELECTRONSAND OTHER FERMIONS

In particle physics, fundamental interactions (sometimes called interactive forces or fundamental forces) are the ways that elementary particles interact with one another. An interaction is fundamental when it **cannot be described** in terms of other interactions. The four known fundamental interactions are electromagnetism, strong interaction ("strong nuclear force"), weak interaction ("weak nuclear force"), and gravitation. All are non-contact forces. With the possible exception of gravitation, these interactions can usually **be described** in a set of calculational approximation methods known as perturbation theory, as being mediated by the exchange of gauge bosons between particles. However, there are situations where perturbation theory does not adequately **describe the** observed phenomena, such as bound states and solitons. n the conceptual model of fundamental interactions, matter consists of fermions, which carry properties called charges and spin ±1⁄2 (intrinsic angular momentum ±ħ⁄2, where ħ is the reduced Planck constant). They attract or repel each other by exchanging bosons.

The interaction of any pair of fermions in perturbation theory can then be modeled thus:

Two fermions go in → interaction by boson exchange → two changed fermions go out.

The exchange of bosons always implies the carriage of energy momentum between the fermions, thereby changing their speed and direction. The exchange may also transport a charge between the fermions, changing the charges of the fermions in the process (e.g., turn them from one type of fermion to another). Since bosons carry one unit of angular momentum, the fermion's spin direction will flip from +1⁄2 to −1⁄2 (or vice versa) during such an exchange (in units of the reduced Planck's constant).

Because an interaction **results in** fermions attracting and repelling each other, an older term for "interaction" is force.

**Heterotic M-Theory In Five Dimensions And Complexity Theory Interpretation Of High Energy Particle Physics**

## Module Numbered Eight

$G_{40}: CATEGORY\ ONE$ **HETEROTIC M-THEORY IN FIVE DIMENSIONS**

Andre Lukas, Burt A. Ovrut, K. S. Stelle, Daniel Waldram investigated the strongly coupled heterotic string theory and M theory in five dimensions, the five-dimensional effective action of strongly coupled heterotic string theory for the complete (1,1) sector of the theory by performing a reduction, on a Calabi-Yau three-fold, of M-theory on S^1/Z_2. A crucial ingredient for a consistent truncation is a non-zero mode of the antisymmetric tensor field strength which arises due to magnetic sources on the orbifold planes. The correct effective theory is a gauged version of five-dimensional N=1 supergravity coupled to Abelian vector multiplets, the universal hypermultiplet and four-dimensional boundary theories with gauge and gauge matter fields. The gauging is such that the dual of the four-form field strength in the universal multiplet is charged under a particular linear combination of the Abelian vector fields. In addition, the theory has potential terms for the moduli in the bulk as well as on the boundary.

Because of these potential terms, the supersymmetric ground state of the theory is a multi-charged BPS three-brane domain wall, which we construct in general. We show that the five-dimensional theory together with this solution provides the correct starting point for particle phenomenology as well as early universe cosmology. As an application, we compute the four-dimensional N=1 supergravity theory for the complete (1, 1) sector to leading nontrivial order by a reduction on the domain wall background. We find a correction to the matter field Kahler potential and threshold corrections to the gauge kinetic functions. Andre Lukas, Burt A. Ovrut, Daniel Waldram investigated the cosmological vacuum solutions of Horava-Witten theory and discuss their physical properties. by deriving the five-dimensional effective action of strongly coupled heterotic string theory by performing a reduction, on a Calabi-Yau three-fold, of M-theory on S1/Z2, and using the concomitant and corresponding results for the usage in cosmology The effective theory is shown to be a gauged version of five-dimensional N=1 supergravity coupled, for simplicity, to the universal hypermultiplet and four-dimensional boundary theories with gauge and universal gauge matter fields. The static vacuum of the theory is a pair of BPS three-brane domain walls.

We show that this five-dimensional theory, together with the domain wall vacuum solution, provides the correct starting point for early universe cosmology in Horava-Witten theory. Relevant cosmological solutions are those associated with the BPS domain wall vacuum. Such solutions must be inhomogeneous, depending on the orbifold coordinate as well as on time. We present two examples of this new type of cosmological solution, obtained by separation of variables. The first example represents the analog of a rolling radii solution with the radii specifying the geometry of the domain wall pair. This is generalized in the second example to include a nontrivial Ramond-Ramond scalar.

$G_{41}$: *CATEGORY TWO OF* **HETEROTIC M-THEORY IN FIVE DIMENSIONS**

In theoretical physics, M-theory **is an extension** of string theory in which 11 dimensions are identified. Because the dimensionality **exceeds that** of superstring theories in 10 dimensions, proponents believe that the 11-dimensional theory **unites all** five string theories (and supersedes them). Low-entropy dynamics are known to be supergravity**interacting** with 2- and 5-dimensional membranes. This idea is the unique supersymmetric theory in eleven dimensions, with its low-entropy matter content and interactions fully determined, and **can be obtained** as the strong coupling limit of type IIA string theory because a new dimension of space **emerges as** the coupling constant increases. Qualitative gradient of structural differentiation and ascribed particularistic solidarity abstraction various theories has lead to the formation of M-Theory. Drawing on the work of a number of string theorists (including Ashoke Sen, Chris Hull, Paul Townsend, Michael Duff and John Schwarz), Edward Witten of the Institute for Advanced Study suggested its existence at a conference at USC in 1995, and **used** M-theory **to explain a** number of previously observed dualities, initiating a flurry of new research in string theory called the second superstring revolution. In the early 1990s, it was shown that the various superstring theories **were related** by dualities **which allow** the description of an object in one super string theory to be related to the description of a different object in another super string theory.

These **relationships imply** that each of the super string theories is a different aspect of a single underlying theory, proposed by Witten, and named "M-theory". Originally the letter M in M-theory was taken from membrane, a construct designed to generalize the strings of string theory. However, as Witten was more skeptical about membranes than his colleagues, he opted for "M-theory" rather than "Membrane theory". Witten has since stated that the different interpretations of the M can be a matter of taste for the user, such as magic, mystery, and mother theory.M-theory (and string theory) has been criticized for lacking predictive power or being untestable. Further work continues to find mathematical constructs that join various surrounding theories. However, the tangible success of M-theory can be questioned, given its current incompleteness and limited predictive power. Supergravity theories were classified and stratified by Werner Nahm in the 1970s. In 10 dimensions, there are only **two** supergravity theories, which are denoted Type IIA and Type IIB. This similar denomination is not a coincidence; the Type IIA string theory has the Type IIA supergravity theory as its low-energy limit and the Type IIB string theory gives rise to Type IIB supergravity. The heterotic SO(32) and heterotic E8×E8 string theories also reduce to Type IIA and Type IIB supergravity in the low-energy limit.

This suggests that there may indeed be a relation between the heterotic/Type I theories and the Type II theories. In 1994, Edward Witten outlined the following relationship: The Type IIA supergravity (corresponding to the heterotic SO (32) and Type IIA string theories) can be obtained by dimensional reduction from the single unique eleven-dimensional supergravity theory. This means that if one studied supergravity on an eleven-dimensional spacetime that looks like the product of a ten-dimensional spacetime with another very small one-dimensional manifold, **one gets the** Type IIA supergravity theory. (And the Type IIB supergravity theory **can be obtained by** using T-duality.) However, eleven-dimensional supergravity is not consistent on its own — it does not make sense at extremely high energy, and likely requires some form of completion. It seems plausible, then, that there is some quantum theory — which Witten dubbed M-theory — in eleven-dimensions which gives rise at low energies to eleven-dimensional supergravity**, and is related** to ten-dimensional string theory by dimensional reduction. Dimensional reduction to a **circle yields the** Type IIA string theory, and dimensional **reduction t**o a line segment yields the heterotic SO (32) string theory.

$G_{42}$: *CATEGORY THREE OF***HETEROTIC M − THEORY IN FIVE DIMENSIONS**

Phenomenology of heterotic Mtheory plays an important part in the diverse practical critique of all experimentation.D. G. Cerdeño* and C. Muñoz† analyze some phenomenological implications of heterotic M theory with five-branes. Recent results for the effective four-dimensional action are used to perform a systematic analysis of the parameter space, finding the restrictions that result from requiring the volume of the Calabi-Yau manifold to remain positive. Then the different scales of the theory, namely the 11-dimensional Planck mass, the compactification scale, and the orbifold scale, are evaluated.

The expressions for the soft supersymmetry-breaking terms are computed and discussed in detail for the whole parameter space. With this information we study the theoretical predictions for the supersymmetric contribution to the muon anomalous magnetic moment, using the recent experimental result as a constraint on the parameter space. We finally analyze the neutralino as a dark matter candidate in this construction. In particular, the neutralino-nucleon cross section is computed and compared with the sensitivities explored by present dark matter detectors. Andre Lucas studied another interesting aspect of the internal and external pronominal connection between the non standard embedding and M Theory ,towards the end of which they constructed t vacua of M theory on S1/Z2 associated with Calabi-Yau threefolds. These vacua are appropriate for compactification to N=1 supersymmetry theories in both four and five dimensions. We allow for general E8×E8 gauge bundles and for the presence of five-branes. The five-branes span the four-dimensional uncompactified space and are wrapped on holomorphic curves in the Calabi-Yau manifold. Properties of these vacua, as well as of the resulting low-energy theories, are discussed.

We find that the low-energy gauge group is enlarged by gauge fields that originate on the five-brane world-volumes. In addition, the five-branes increase the types of new E8×E8 breaking patterns allowed by the non-standard embedding. Characteristic features of the low-energy theory, such as the threshold corrections to the gauge kinetic functions, are significantly modified due to the presence of the five-branes, as compared to the case of standard or non-standard embeddings without five-branes. Construction of cosmological solutions of four-dimensional effective heterotic M theory with a moving five-brane and evolving dilaton and T modulus. It is shown that the five-brane generates a transition between two asymptotic rolling-radii solutions. Moreover, the five-brane motion always drives the solutions towards strong coupling asymptotically. We present an explicit example of a negative-time branch solution which ends in a brane collision accompanied by a small-instanton transition. The five-dimensional origin of some of our solutions is also discussed by Edmund Copeland, James Gray l., etal.,

## $T_{40}$: $CATEGORY\ ONE$ COMPLEXITY THEORY INTERPRETATION OF HIGH ENERGY PARTICLE PHYSICS

M. S. El Naschie, S. Olsen, J. H. He, S. Nada, L. Marek-Crnjac, A. Helal expatiate and enucleate and delineate and disseminate information and particulars about the imperative compatibilities and structural variabilities of Need for Fractal Logic in High Energy Quantum in pure mathematics and particularly in transfinite set theory and introduced into the fundamentals of theoretical physics many novel concepts and devices such as fractal quasi manifolds with non-integer (Hausdorff) dimension for its geometry as well as infinite dimensional wild topology and non classical fuzzy logic.

In the present work transfinite fractal sets and fuzzy logic are combined to enable the introduction of a new theory termed fractal logic to the foundation of high energy particle physics. This leads naturally to a new look at quantum gravity. In particular we will show that to understand and develop quantum gravity we have to bring various fields together, particularly fractals and nonlinear dynamics as well as sphere packing, fuzzy set theory, number theory and quantum entanglement and irrationally q-deformed algebra. Particle physics is a branch of physics that studies the existence and interactions of particles that are the constituents of what is usually referred to as matter or radiation. In current understanding, particles are excitations of quantum fields and interact following their dynamics. Most of the interest in this area is in fundamental fields, each of which cannot be described as a bound state of other fields.

The current set of fundamental fields and their dynamics are summarized in a theory called the Standard Model; therefore particle physics is largely the study of the Standard Model's particle content and its possible extensions. Modern particle physics research is focused on subatomic particles, including atomic constituents such as electrons, protons, and neutrons (protons and neutrons are composite particles called baryons, made

ISSN 2250-3153

of quarks), produced by radioactive and scattering processes, such as photons, neutrinos, and muons, as well as a wide range of exotic particles. To be specific, the term particle is a misnomer from classical physics because the dynamics of particle physics are governed by quantum mechanics. As such, they exhibit wave-particle duality, displaying particle-like behavior under certain experimental conditions and wave-like behavior in others. In more technical terms, they are described by quantum state vectors in a Hilbert space, which is also treated in quantum field theory. Following the convention of particle physicists, elementary particles refer to objects such as electrons and photons as it is well known that these types of particles display wave-like properties as well. All particles and their interactions observed to date can be described almost entirely by a quantum field theory called theStandard Model. The Standard Model has 61 elementary particles. These elementary particles can combine to form composite particles, accounting for the hundreds of other species of particles discovered since the 1960s. The Standard Model has been found to agree with almost all the experimental tests conducted to date. However, most particle physicists believe that it is an incomplete description of nature, and that a more fundamental theory awaits discovery (See Theory of Everything). In recent years, measurements of neutrino mass have provided the first experimental deviations from the Standard Model.

The idea that all matter is composed of elementary particles dates to at least the 6th century BC. The philosophical doctrine of atomism and the nature of elementary particles were studied by ancient Greek philosophers such as Leucippus, Democritus, and Epicurus; ancient Indian philosophers such as Kanada, Dignāga, and Dharmakirti; Muslim scientists such as Ibn al-Haytham, Ibn Sina, and Mohammad al-Ghazali; and early modern European physicists such as Pierre Gassendi, Robert Boyle, and Isaac Newton. The particle theory of light was also proposed by Ibn al-Haytham, Ibn Sina, Gassendi, and Newton. These early ideas were founded in abstract, philosophical reasoning rather than experimentation and empirical observation.

## $T_{41}$:CATEGORY TWO OF COMPLEXITY THEORY INTERPRETATION OF HIGH ENERGY PARTICLE PHYSICS

When it comes to the study of the qualitative gradient of structural differentiation, comparative dependability and surface topology, predicational anteriority and ontological consonance and primordial exactitude of accolytish representation James Gowan takes the cake. Ervin Goldfain studied Non-Equilibrium Dynamics and Physics of the Terascale Sector Unitarity and locality are fundamental postulates of Quantum Field Theory (QFT). By construction, QFT is a replica of equilibrium thermodynamics, where evolution settles down to a steady state after all transients have vanished. Events unfolding in the TeV sector of particle physics are prone to slide outside equilibrium under the combined action of new fields and unsuppressed quantum corrections. In this region, the likely occurrence of critical behavior and the approach to scale invariance blur the distinction between "locality" and "non-locality".

We argue that a correct description of this far from equilibrium setting cannot be done outside nonlinear dynamics and complexity theory. Another topic that is investigated by illustrious and exemplary Gowan is The creation of matter during the "Big Bang" is apparently due to the asymmetric decay of electrically neutral leptoquarks and antileptoquarks, in which the antileptoquarks decay at a slightly faster rate than the leptoquarks. The leptoquarks in these decays (which are electrically neutral due to the fractionally charged quarks) are also colorless (in the limit of "asymptotic freedom"), due to the great compressive force exerted by the "X" IVB. A leptoquarks antineutrino is produced in this decay, balancing the baryon "number" charge of the eventual proton. This neutrino is a "dark matter" candidate.

The interaction is the initiating example of a general class of reactions between symmetric primary energy fields and asymmetric secondary or "alternative" information fields or charge carriers. Gowan also resorted to the imperative compatibilities and structural variabilities of interpretations of the activity of the Higgs boson: 1) the older, original interpretation of the Higgs as the scalar or gauge boson which determines the rest masses of the IVBs and elementary particles (which I can understand and endorse); 2) a newer (additional? alternative?) interpretation consisting of a "Higgs ether" which acts as the source of particle mass in the sense of inertial resistance to acceleration. In this latter interpretation, all massive particles interact with a universal Higgs field in proportion to their bound energy content, and it is this interaction or "Higgs ether drag" which causes the inertial resistance to acceleration we characterize as mass. It is this latter interpretation which I cannot understand or endorse, as it seems to force a distinction between rest mass and inertial mass, and has no power at all to explain Einstein's relativistic mass.

However, replacing the "Higgs ether drag" hypothesis (but retaining the Higgs scalar hypothesis) with a "gravitational field drag" hypothesis does allow us to understand the mechanism of relativistic variability in the metric and energetic parameters of mass, and crucially preserves the necessary equivalence between inertial and rest mass Noether's Theorem" states that in a continuous multicomponent field such as the electromagnetic field (or the metric field of spacetime), where one finds a symmetry one will find an associated conservation law, and vice versa. In matter, light's symmetries are conserved by charge and spin; in spacetime, by inertial and gravitational forces. Neutrinos carry "identity" charge (aka "number" or "flavor" charge), the symmetry debt of light's "anonymity". The charges of matter are the symmetry debts of light. The weak force is responsible for the creation of matter during the "Big Bang" apparently via the asymmetric decay of electrically neutral leptoquark-antileptoquark particle pairs, and for the subsequent creation, transformation, and destruction of single elementary particles - particles that do not exist in matter-antimatter pairs (seen as radioactivity, particle decay/transformation, and fission). Elementary particles created today must be interchangeable with those created during the "Big Bang" with respect to all conserved parameters - mass, spin, charge, etc. Creating absolutely invariant single elementary particles any time or place is the conservation challenge presented to and surmounted by the weak force, requiring the elaborate mechanism of the Higgs boson and the Intermediate Vector Bosons (IVBs). The great mass of the IVBs recreates the original energy density and unified force symmetry state in which the elementary particle classes (leptons and quarks) were first created, while the Higgs boson "gauges" (scales and selects) the IVBs and unified force symmetry state (there are several) appropriate to the transformation at hand. It is the quantization of the Higgs boson and the IVBs (plus virtual particles drawn from the global "vacuum sea") that ensures the invariance of the weak force transformation mechanism.

The weak force charge is "identity" charge (AKA "number" or "flavor" charge), and is carried implicitly by all massive leptons (including leptoquarks) and explicitly by neutrinos. We explore the hypothesis that there are 3 "families" or energy levels of the Higgs bosons and their associated Intermediate Vector Bosons (IVBs), analogously to the three families or energy levels of the quarks and leptons. With its origin in the "Multiverse", our Universe apparently devolved (rapidly) downward in an asymmetric "Higgs Cascade" to the electromagnetic ground state, and now evolves (slowly) upward again in a "rebound" driven by negentropic gravity and symmetry conservation (Noether's Theorem), toward the Multiverse or a state of pure electromagnetic radiation (light). CERN announced the discovery of a Higgs-like boson on 4 July, 2012 John A. Gowan concentrated upon the The "W" Intermediate Vector Boson and the Weak Force Mechanism. Elementary particles created today must be the same in every respect as those created eons ago during the "Big Bang". The conservation requirement of elementary particle invariance constrains the mechanism of weak force particle creation and transformation.

Weak force transformations recreate primordial symmetric energy states of the "Big Bang" force-unification eras (in the case of the "W", the electroweak .Bernard Riley studied Partners of the Su(3) Hadrons. The hadrons of the SU(3) JP= 0-, ½+ and 1- multiplets are shown to have partners of the same spin or of spin difference ½. Partnerships occur between hadrons with some quark content in common, there being no distinction between quarks and antiquarks. The partnerships are centred upon particle mass levels that descend in geometric progression from the Planck Mass. The mass differences characterizing partnerships are equal to the masses of levels. Isospin doublets behave as single particles, represented by the geometric mean of the hadron masses. The K-meson isospin doublets and the electron are arranged as partnerships, as are the π+ and π- isospin triplet states and the muon.

## $T_{42}$: *CATEGORY THREE OF* COMPLEXITY THEORY INTERPRETATION OF HIGH ENERGY PARTICLE PHYSICS

Following are the extracts, drawn from Sir Antony garret Lisi an Explanation of Fundamental Particle Physics ThatDoesn't Exist Yet. We have made necessary emphasis to suit our subject matter. I express profound homage Sir Lisi for publication of such myriad faceted article that has touched upon various aspectionalities nd attributions that have served the need of the researchers like me. Interrogation problem does not bear resemblance to the propositions it subsumes under it; it rather orders them based on its own engendered notions, nostrums, doctrines, dictums, axiomatic predications, postulation alcovishness, against the background of generalized significations and personalized manifestations. Interrogative problem is a problem reconstructed or projected based on empirical propositions. Research in fundamental particle physics has culminated in our current Standard Model of elementary particles.

Using ever larger machines, we have been able to identify and determine the properties of a whole zoo of elementary particles. These properties present many interesting patterns. All the matter we see around us is **composed of** electrons and up and down quarks, interacting differently with photons of electromagnetism, W and Z bosons of the weak force, gluons of the strong force, and gravity, according to their different values and kinds of charges. Additionally, an interaction between a W and an electron **produces an** electron neutrino, and these neutrinos are now known to permeate space—flying through us in great quantities, interacting only weakly. A neutrino passing through the earth probably wouldn't even notice it was there. Together, the electron, electron neutrino, and up and down quarks constitute what is called the first generation of fermions. Using high energy particle colliders, physicists have been able to see even more particles. It turns out the first generation fermions have second and third generation partners, with identical charges to the first but larger masses. And nobody knows why.

The second generation partner to the electron is called the muon, and the third generation partner is called the tau. Similarly, the down quark is partnered with the strange and bottom quarks and the up quark has partners called the charm and top, with the top discovered in 1995. Last and least, the electron neutrinos are partnered with muon and tau neutrinos. All of these fermions have different masses, arising from their interaction with a theorized background Higgs field. Once again, nobody knows why there are three generations, or why these particles have the masses they do. The Standard Model, our best current description of fundamental physics, lacks a good explanation.

The dominant research program in high energy theoretical physics, string theory, has effectively given up on finding an explanation for why the particle masses are what they are. The current non-explanation is that they arise by accident, from the infinite landscape of theoretical possibilities. This is a cop out. If a theory can't provide a satisfying explanation of an important pattern in nature, it's time to consider a different theory. Of course, it is possible that the pattern of particle masses arose by chance, or some complicated evolution, as did the orbital distances of our solar system's planets. But, as experimental data accumulates, patterns either fade or sharpen, and in the newest data on particle masses an intriguing pattern is sharpening. The answer may come from the shy neutrino.

The masses of the three generations of fermions are described by their interaction with the Higgs field. In more detail, this is described by "mixing matrices," involving a collection of angles and phases. There is no clear, a priori reason why these angles and phases should take particular values, but they are of great consequence. In fact, a small difference in these phases determines the prevalence of matter over antimatter in our universe. Now, in the mixing matrix for the quarks, the three angles and one phase are all quite small, with no discernible pattern. But for neutrinos this is not the case. Before the turn of the 21st century it was not even clear that neutrinos mixed. Too few electron neutrinos seemed to be coming from the sun, but scientists weren't sure why. In the past few years our knowledge has improved immensely. From the combined effort of many experimental teams we now know that, to a remarkable degree of precision, the three angles for neutrinos have sin squared equal to 1/2, 1/3, and 0. We do need to consider the possibility of coincidence, but as random numbers go, these do not seem very random. In fact, this mixing corresponds to a "tribimaximal" matrix, related to the geometric symmetry group of the tetrahedron.

What is tetrahedral symmetry doing in the masses of neutrinos?! Nobody knows. But you can bet there will be a good explanation. It is likely that this explanation will come from mathematicians and physicists working closely with Lie groups. The most important lesson from the great success of Einstein's theory of General Relativity is that our universe is fundamentally geometric, and this idea has extended to the geometric description of known forces and particles using group theory. It seems natural that a complete explanation of the Standard Model, including why there are three generations of fermions and why they have the masses they do, will come from the geometry of group theory. This explanation does not yet exist, but when it does it will be deep, elegant, and beautiful—and it will be my favorite.

## MODULENUMBEREDNINE

**Ads/cFt**

**Governing equation is :**

N= 4SYM at N=Infinity

Lambda= g ^2 subscript YM ,where N is related to String Tension

2(Phi)T=R^2/Factorial Alphs= Square root of Lambda

G subscript s is equivalent to lambda divided by 4(phi)N tends to zero

**NOTATION :**

$G_{44}$ :LHS Of the Highlighted equation(Category one)

$G_{45}$ : LHS of Highlighted equation(Category2)

$G_{46}$ :LHS of Highlighted equation (Category 3)

$T_{44}$ RHS of the highlighted equation (Category 1 for concomitant classification above)

$T_{45}$ : RHS of the highlighted equation (Category 2 for concomitant classification above)

$T_{46}$ : RHS of the highlighted equation(Category 3 for concomitant classification above)

$(a_{13})^{(1)}, (a_{14})^{(1)}, (a_{15})^{(1)}, (b_{13})^{(1)}, (b_{14})^{(1)}, (b_{15})^{(1)}$ $(a_{16})^{(2)}, (a_{17})^{(2)}, (a_{18})^{(2)}$ $(b_{16})^{(2)}, (b_{17})^{(2)}, (b_{18})^{(2)}$:
$(a_{20})^{(3)}, (a_{21})^{(3)}, (a_{22})^{(3)}$ , $(b_{20})^{(3)}, (b_{21})^{(3)}, (b_{22})^{(3)}$
$(a_{24})^{(4)}, (a_{25})^{(4)}, (a_{26})^{(4)}, (b_{24})^{(4)}, (b_{25})^{(4)}, (b_{26})^{(4)}, (b_{28})^{(5)}, (b_{29})^{(5)}, (b_{30})^{(5)}, (a_{28})^{(5)}, (a_{29})^{(5)}, (a_{30})^{(5)}$,
$(a_{32})^{(6)}, (a_{33})^{(6)}, (a_{34})^{(6)}, (b_{32})^{(6)}, (b_{33})^{(6)}, (b_{34})^{(6)}$
$(a_{36})^{(7)}, (a_{37})^{(7)}, (a_{38})^{(7)}, (b_{36})^{(7)}, (b_{37})^{(7)}, (b_{38})^{(7)}$
$(a_{40})^{(8)}, (a_{41})^{(8)}, (a_{42})^{(8)}, (b_{40})^{(8)}, (b_{41})^{(8)}, (b_{42})^{(8)}$
$(a_{44})^{(9)}, (a_{45})^{(9)}, (a_{46})^{(9)}, (b_{44})^{(9)}, (b_{45})^{(9)}, (b_{46})^{(9)}$

are Accentuation coefficients
$(a'_{13})^{(1)}, (a'_{14})^{(1)}, (a'_{15})^{(1)}, (b'_{13})^{(1)}, (b'_{14})^{(1)}, (b'_{15})^{(1)}, (a'_{16})^{(2)}, (a'_{17})^{(2)}, (a'_{18})^{(2)}$, $(b'_{16})^{(2)}, (b'_{17})^{(2)}, (b'_{18})^{(2)}$
, $(a'_{20})^{(3)}, (a'_{21})^{(3)}, (a'_{22})^{(3)}, (b'_{20})^{(3)}, (b'_{21})^{(3)}, (b'_{22})^{(3)}$
$(a'_{24})^{(4)}, (a'_{25})^{(4)}, (a'_{26})^{(4)}, (b'_{24})^{(4)}, (b'_{25})^{(4)}, (b'_{26})^{(4)}, (b'_{28})^{(5)}, (b'_{29})^{(5)}, (b'_{30})^{(5)}$ $(a'_{28})^{(5)}, (a'_{29})^{(5)}, (a'_{30})^{(5)}$ ,
$(a'_{32})^{(6)}, (a'_{33})^{(6)}, (a'_{34})^{(6)}, (b'_{32})^{(6)}, (b'_{33})^{(6)}, (b'_{34})^{(6)}$
$(a'_{36})^{(7)}, (a'_{37})^{(7)}, (a'_{38})^{(7)}, (b'_{36})^{(7)}, (b'_{37})^{(7)}, (b'_{38})^{(7)}$,
$(a'_{40})^{(8)}, (a'_{41})^{(8)}, (a'_{42})^{(8)}, (b'_{40})^{(8)}, (b'_{41})^{(8)}, (b'_{42})^{(8)}$,
$(a'_{44})^{(9)}, (a'_{45})^{(9)}, (a'_{46})^{(9)}, (b'_{44})^{(9)}, (b'_{45})^{(9)}, (b'_{46})^{(9)}$,

are Dissipation coefficients

**Module Numbered One**

**The differential system of this model is now (Module Numbered one)**

$\frac{dG_{13}}{dt} = (a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14},t)\right]G_{13}$ 1

$\frac{dG_{14}}{dt} = (a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14},t)\right]G_{14}$ 2

$\frac{dG_{15}}{dt} = (a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14},t)\right]G_{15}$ 3

$\frac{dT_{13}}{dt} = (b_{13})^{(1)}T_{14} - \left[(b'_{13})^{(1)} - (b''_{13})^{(1)}(G,t)\right]T_{13}$ 4

$\frac{dT_{14}}{dt} = (b_{14})^{(1)}T_{13} - \left[(b'_{14})^{(1)} - (b''_{14})^{(1)}(G,t)\right]T_{14}$ 5

$\frac{dT_{15}}{dt} = (b_{15})^{(1)}T_{14} - \left[(b'_{15})^{(1)} - (b''_{15})^{(1)}(G,t)\right]T_{15}$ 6

$+(a''_{13})^{(1)}(T_{14},t) =$ First augmentation factor

$-(b''_{13})^{(1)}(G,t) =$ First detritions factor

**Module Numbered Two**

**The differential system of this model is now ( Module numbered two)**

$\frac{dG_{16}}{dt} = (a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17},t)\right]G_{16}$ 7

$\frac{dG_{17}}{dt} = (a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17},t)\right]G_{17}$ 8

$\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[(a'_{18})^{(2)} + (a''_{18})^{(2)}(T_{17},t)\right]G_{18}$ 9

$\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[(b'_{16})^{(2)} - (b''_{16})^{(2)}\left((G_{19}),t\right)\right]T_{16}$ 10

$\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[(b'_{17})^{(2)} - (b''_{17})^{(2)}\left((G_{19}),t\right)\right]T_{17}$ 11

$\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[(b'_{18})^{(2)} - (b''_{18})^{(2)}\left((G_{19}),t\right)\right]T_{18}$ 12

$+(a''_{16})^{(2)}(T_{17},t) =$ First augmentation factor

$-(b''_{16})^{(2)}\left((G_{19}),t\right) =$ First detritions factor

**Module Numbered Three**

**The differential system of this model is now (Module numbered three)**

$\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[(a'_{20})^{(3)} + (a''_{20})^{(3)}(T_{21},t)\right]G_{20}$ 13

$\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[(a'_{21})^{(3)} + (a''_{21})^{(3)}(T_{21},t)\right]G_{21}$ 14

$\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[(a'_{22})^{(3)} + (a''_{22})^{(3)}(T_{21},t)\right]G_{22}$ 15

$\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[(b'_{20})^{(3)} - (b''_{20})^{(3)}(G_{23},t)\right]T_{20}$ 16

$\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[(b'_{21})^{(3)} - (b''_{21})^{(3)}(G_{23},t)\right]T_{21}$ 17

$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[(b'_{22})^{(3)} - (b''_{22})^{(3)}(G_{23}, t)\right]T_{22}$ 18

$+(a''_{20})^{(3)}(T_{21}, t) =$ First augmentation factor

$-(b''_{20})^{(3)}(G_{23}, t) =$ First detritions factor

**Module Numbered Four**

**The differential system of this model is now (Module numbered Four)**

$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[(a'_{24})^{(4)} + (a''_{24})^{(4)}(T_{25}, t)\right]G_{24}$ 19

$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \left[(a'_{25})^{(4)} + (a''_{25})^{(4)}(T_{25}, t)\right]G_{25}$ 20

$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \left[(a'_{26})^{(4)} + (a''_{26})^{(4)}(T_{25}, t)\right]G_{26}$ 21

$\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \left[(b'_{24})^{(4)} - (b''_{24})^{(4)}\left((G_{27}), t\right)\right]T_{24}$ 22

$\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \left[(b'_{25})^{(4)} - (b''_{25})^{(4)}\left((G_{27}), t\right)\right]T_{25}$ 23

$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \left[(b'_{26})^{(4)} - (b''_{26})^{(4)}\left((G_{27}), t\right)\right]T_{26}$ 24

$+(a''_{24})^{(4)}(T_{25}, t) =$ **First augmentation factor**

$-(b''_{24})^{(4)}\left((G_{27}), t\right) =$ **First detritions factor**

**Module Numbered Five**:

**The differential system of this model is now (Module number five)**

$\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \left[(a'_{28})^{(5)} + (a''_{28})^{(5)}(T_{29}, t)\right]G_{28}$ 25

$\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \left[(a'_{29})^{(5)} + (a''_{29})^{(5)}(T_{29}, t)\right]G_{29}$ 26

$\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \left[(a'_{30})^{(5)} + (a''_{30})^{(5)}(T_{29}, t)\right]G_{30}$ 27

$\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[(b'_{28})^{(5)} - (b''_{28})^{(5)}\left((G_{31}), t\right)\right]T_{28}$ 28

$\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[(b'_{29})^{(5)} - (b''_{29})^{(5)}\left((G_{31}), t\right)\right]T_{29}$ 29

$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[(b'_{30})^{(5)} - (b''_{30})^{(5)}\left((G_{31}), t\right)\right]T_{30}$ 30

$+(a''_{28})^{(5)}(T_{29}, t) =$ **First augmentation factor**

$-(b''_{28})^{(5)}\left((G_{31}), t\right) =$ **First detritions factor**

**Module Numbered Six**

**The differential system of this model is now (Module numbered Six)**

$\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[(a'_{32})^{(6)} + (a''_{32})^{(6)}(T_{33}, t)\right]G_{32}$ 31

$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[(a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33}, t)\right]G_{33}$ 32

$\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[(a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33}, t)\right]G_{34}$ 33

$\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[(b'_{32})^{(6)} - (b''_{32})^{(6)}\left((G_{35}), t\right)\right]T_{32}$ 34

ISSN 2250-3153

$\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[(b'_{33})^{(6)} - (b''_{33})^{(6)}\left((G_{35}),t\right)\right]T_{33}$ 35

$\frac{dT_{34}}{dt} = (b_{34})^{(6)}T_{33} - \left[(b'_{34})^{(6)} - (b''_{34})^{(6)}\left((G_{35}),t\right)\right]T_{34}$ 36

$+(a''_{32})^{(6)}(T_{33},t) =$ **First augmentation factor**

**Module Numbered Seven:**

**The differential system of this model is now (Seventh Module)**

$\frac{dG_{36}}{dt} = (a_{36})^{(7)}G_{37} - \left[(a'_{36})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right]G_{36}$ 37

$\frac{dG_{37}}{dt} = (a_{37})^{(7)}G_{36} - \left[(a'_{37})^{(7)} + (a''_{37})^{(7)}(T_{37},t)\right]G_{37}$ *38*

$\frac{dG_{38}}{dt} = (a_{38})^{(7)}G_{37} - \left[(a'_{38})^{(7)} + (a''_{38})^{(7)}(T_{37},t)\right]G_{38}$ *39*

$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \left[(b'_{36})^{(7)} - (b''_{36})^{(7)}\left((G_{39}),t\right)\right]T_{36}$ *40*

$\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \left[(b'_{37})^{(7)} - (b''_{37})^{(7)}\left((G_{39}),t\right)\right]T_{37}$ *41*

$\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \left[(b'_{38})^{(7)} - (b''_{38})^{(7)}\left((G_{39}),t\right)\right]T_{38}$ *42*

$+(a''_{36})^{(7)}(T_{37},t) =$ **First augmentation factor**

**Module Numbered Eight**

The differential system of this model is now

$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \left[(a'_{40})^{(8)} + (a''_{40})^{(8)}(T_{41},t)\right]G_{40}$ *43*

$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \left[(a'_{41})^{(8)} + (a''_{41})^{(8)}(T_{41},t)\right]G_{41}$ *44*

$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \left[(a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41},t)\right]G_{42}$ *45*

$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[(b'_{40})^{(8)} - (b''_{40})^{(8)}\left((G_{43}),t\right)\right]T_{40}$ *46*

$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[(b'_{41})^{(8)} - (b''_{41})^{(8)}\left((G_{43}),t\right)\right]T_{41}$ *47*

$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[(b'_{42})^{(8)} - (b''_{42})^{(8)}\left((G_{43}),t\right)\right]T_{42}$ *48*

**Module Numbered Nine**

The differential system of this model is now

$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[(a'_{44})^{(9)} + (a''_{44})^{(9)}(T_{45},t)\right]G_{44}$ 49

$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[(a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45},t)\right]G_{45}$ 50

$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[(a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45},t)\right]G_{46}$ 51

$\frac{dT_{44}}{dt} = (b_{44})^{(9)}T_{45} - \left[(b'_{44})^{(9)} - (b''_{44})^{(9)}\left((G_{47}),t\right)\right]T_{44}$ 52

$\frac{dT_{45}}{dt} = (b_{45})^{(9)}T_{44} - \left[(b'_{45})^{(9)} - (b''_{45})^{(9)}\left((G_{47}),t\right)\right]T_{45}$ 53

$$\frac{dT_{46}}{dt}=(b_{46})^{(9)}T_{45}-\left[(b'_{46})^{(9)}-(b''_{46})^{(9)}\left((G_{47}),t\right)\right]T_{46} \quad 54$$

$+(a''_{44})^{(9)}(T_{45},t)=$ **First augmentation factor**

$-(b''_{44})^{(9)}\left((G_{47}),t\right)=$ **First detrition factor**

$$\frac{dG_{13}}{dt}=(a_{13})^{(1)}G_{14}-\begin{bmatrix}(a'_{13})^{(1)}\boxed{+(a''_{13})^{(1)}(T_{14},t)}\boxed{+(a''_{16})^{(2,2,)}(T_{17},t)}\boxed{+(a''_{20})^{(3,3,)}(T_{21},t)}\\ \boxed{+(a''_{24})^{(4,4,4,4,)}(T_{25},t)}\boxed{+(a''_{28})^{(5,5,5,5,)}(T_{29},t)}\boxed{+(a''_{32})^{(6,6,6,6,)}(T_{33},t)}\\ \boxed{+(a''_{36})^{(7,7)}(T_{37},t)}\boxed{+(a''_{40})^{(8,8)}(T_{41},t)}\boxed{+(a''_{44})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}\end{bmatrix}G_{13} \quad 55$$

$$\frac{dG_{14}}{dt}=(a_{14})^{(1)}G_{13}-\begin{bmatrix}(a'_{14})^{(1)}\boxed{+(a''_{14})^{(1)}(T_{14},t)}\boxed{+(a''_{17})^{(2,2,)}(T_{17},t)}\boxed{+(a''_{21})^{(3,3,)}(T_{21},t)}\\ \boxed{+(a''_{25})^{(4,4,4,4,)}(T_{25},t)}\boxed{+(a''_{29})^{(5,5,5,5,)}(T_{29},t)}\boxed{+(a''_{33})^{(6,6,6,6,)}(T_{33},t)}\\ \boxed{+(a''_{37})^{(7,7)}(T_{37},t)}\boxed{+(a''_{41})^{(8,8)}(T_{41},t)}\boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}\end{bmatrix}G_{14} \quad 56$$

$$\frac{dG_{15}}{dt}=(a_{15})^{(1)}G_{14}-\begin{bmatrix}(a'_{15})^{(1)}\boxed{+(a''_{15})^{(1)}(T_{14},t)}\boxed{+(a''_{18})^{(2,2,)}(T_{17},t)}\boxed{+(a''_{22})^{(3,3,)}(T_{21},t)}\\ \boxed{+(a''_{26})^{(4,4,4,4,)}(T_{25},t)}\boxed{+(a''_{30})^{(5,5,5,5,)}(T_{29},t)}\boxed{+(a''_{34})^{(6,6,6,6,)}(T_{33},t)}\\ \boxed{+(a''_{38})^{(7,7)}(T_{37},t)}\boxed{+(a''_{42})^{(8,8)}(T_{41},t)}\boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}\end{bmatrix}G_{15} \quad 57$$

Where $\boxed{(a''_{13})^{(1)}(T_{14},t)}$ , $\boxed{(a''_{14})^{(1)}(T_{14},t)}$ , $\boxed{(a''_{15})^{(1)}(T_{14},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,)}(T_{17},t)}$ , $\boxed{+(a''_{17})^{(2,2,)}(T_{17},t)}$ , $\boxed{+(a''_{18})^{(2,2,)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,)}(T_{21},t)}$ , $\boxed{+(a''_{21})^{(3,3,)}(T_{21},t)}$ , $\boxed{+(a''_{22})^{(3,3,)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,)}(T_{25},t)}$ , $\boxed{+(a''_{25})^{(4,4,4,4,)}(T_{25},t)}$ , $\boxed{+(a''_{26})^{(4,4,4,4,)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,)}(T_{29},t)}$ , $\boxed{+(a''_{29})^{(5,5,5,5,)}(T_{29},t)}$ , $\boxed{+(a''_{30})^{(5,5,5,5,)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,)}(T_{33},t)}$ , $\boxed{+(a''_{33})^{(6,6,6,6,)}(T_{33},t)}$ , $\boxed{+(a''_{34})^{(6,6,6,6,)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7)}(T_{37},t)}\boxed{+(a''_{37})^{(7,7)}(T_{37},t)}\boxed{+(a''_{36})^{(7,7)}(T_{37},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8)}(T_{41},t)}\boxed{+(a''_{41})^{(8,8)}(T_{41},t)}\boxed{+(a''_{42})^{(8,8)}(T_{41},t)}$ are eight augmentation coefficient for 1,2,3

$\boxed{+(a''_{44})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ , $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ , $\boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{13}}{dt}=(b_{13})^{(1)}T_{14}-\left[\begin{array}{c}(b'_{13})^{(1)}\boxed{-(b''_{13})^{(1)}(G,t)}\ \boxed{-(b''_{16})^{(2,2,)}(G_{19},t)}\boxed{-(b''_{20})^{(3,3,)}(G_{23},t)}\\ \boxed{-(b''_{24})^{(4,4,4,4,)}(G_{27},t)}\boxed{-(b''_{28})^{(5,5,5,5,)}(G_{31},t)}\boxed{-(b''_{32})^{(6,6,6,6,)}(G_{35},t)}\\ \boxed{-(b''_{36})^{(7,7,)}(G_{39},t)}\boxed{-(b''_{40})^{(8,8)}(G_{43},t)}\boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9,9,9)}(G_{47},t)}\end{array}\right]T_{13} \quad 58$$

$$\frac{dT_{14}}{dt}=(b_{14})^{(1)}T_{13}-\left[\begin{array}{c}(b'_{14})^{(1)}\boxed{-(b''_{14})^{(1)}(G,t)}\ \boxed{-(b''_{17})^{(2,2,)}(G_{19},t)}\boxed{-(b''_{21})^{(3,3,)}(G_{23},t)}\\ \boxed{-(b''_{25})^{(4,4,4,4,)}(G_{27},t)}\boxed{-(b''_{29})^{(5,5,5,5,)}(G_{31},t)}\boxed{-(b''_{33})^{(6,6,6,6,)}(G_{35},t)}\\ \boxed{-(b''_{37})^{(7,7,)}(G_{39},t)}\boxed{-(b''_{41})^{(8,8)}(G_{43},t)}\boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9,9,9)}(G_{47},t)}\end{array}\right]T_{14} \quad 59$$

$$\frac{dT_{15}}{dt}=(b_{15})^{(1)}T_{14}-\left[\begin{array}{c}(b'_{15})^{(1)}\boxed{-(b''_{15})^{(1)}(G,t)}\ \boxed{-(b''_{18})^{(2,2,)}(G_{19},t)}\boxed{-(b''_{22})^{(3,3,)}(G_{23},t)}\\ \boxed{-(b''_{26})^{(4,4,4,4,)}(G_{27},t)}\boxed{-(b''_{30})^{(5,5,5,5,)}(G_{31},t)}\boxed{-(b''_{34})^{(6,6,6,6,)}(G_{35},t)}\\ \boxed{-(b''_{38})^{(7,7,)}(G_{39},t)}\boxed{-(b''_{42})^{(8,8)}(G_{43},t)}\boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9,9,9)}(G_{47},t)}\end{array}\right]T_{15} \quad 60$$

Where $\boxed{-(b''_{13})^{(1)}(G,t)}$ , $\boxed{-(b''_{14})^{(1)}(G,t)}$ , $\boxed{-(b''_{15})^{(1)}(G,t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,)}(G_{19},t)}$ , $\boxed{-(b''_{17})^{(2,2,)}(G_{19},t)}$ , $\boxed{-(b''_{18})^{(2,2,)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,)}(G_{23},t)}$ , $\boxed{-(b''_{21})^{(3,3,)}(G_{23},t)}$ , $\boxed{-(b''_{22})^{(3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,)}(G_{27},t)}$ , $\boxed{-(b''_{25})^{(4,4,4,4,)}(G_{27},t)}$ , $\boxed{-(b''_{26})^{(4,4,4,4,)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,)}(G_{31},t)}$ , $\boxed{-(b''_{29})^{(5,5,5,5,)}(G_{31},t)}$ , $\boxed{-(b''_{30})^{(5,5,5,5,)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,)}(G_{35},t)}$ , $\boxed{-(b''_{33})^{(6,6,6,6,)}(G_{35},t)}$ , $\boxed{-(b''_{34})^{(6,6,6,6,)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{37})^{(7,7,)}(G_{39},t)}$ , $\boxed{-(b''_{36})^{(7,7,)}(G_{39},t)}$ , $\boxed{-(b''_{38})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{40})^{(8,8)}(G_{43},t)}$ $\boxed{-(b''_{41})^{(8,8)}(G_{43},t)}$ $\boxed{-(b''_{42})^{(8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9,9,9)}(G_{47},t)}$ , $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9,9,9)}(G_{47},t)}$ , $\boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{16}}{dt}=(a_{16})^{(2)}G_{17}-\left[\begin{array}{c}(a'_{16})^{(2)}\boxed{+(a''_{16})^{(2)}(T_{17},t)}\boxed{+(a''_{13})^{(1,1,)}(T_{14},t)}\boxed{+(a''_{20})^{(3,3,3)}(T_{21},t)}\\ \boxed{+(a''_{24})^{(4,4,4,4,4)}(T_{25},t)}\boxed{+(a''_{28})^{(5,5,5,5,5)}(T_{29},t)}\boxed{+(a''_{32})^{(6,6,6,6,6)}(T_{33},t)}\\ \boxed{+(a''_{36})^{(7,7,7)}(T_{37},t)}\boxed{+(a''_{40})^{(8,8,8)}(T_{41},t)}\boxed{+(a''_{44})^{(9,9)}(T_{45},t)}\end{array}\right]G_{16} \quad 61$$

$$\frac{dG_{17}}{dt}=(a_{17})^{(2)}G_{16}-\left[\begin{array}{c}(a'_{17})^{(2)}\boxed{+(a''_{17})^{(2)}(T_{17},t)}\boxed{+(a''_{14})^{(1,1,)}(T_{14},t)}\boxed{+(a''_{21})^{(3,3,3)}(T_{21},t)}\\ \boxed{+(a''_{25})^{(4,4,4,4,4)}(T_{25},t)}\boxed{+(a''_{29})^{(5,5,5,5,5)}(T_{29},t)}\boxed{+(a''_{33})^{(6,6,6,6,6)}(T_{33},t)}\\ \boxed{+(a''_{37})^{(7,7,7)}(T_{37},t)}\boxed{+(a''_{41})^{(8,8,8)}(T_{41},t)}\boxed{+(a''_{45})^{(9,9)}(T_{45},t)}\end{array}\right]G_{17} \quad 62$$

$$\frac{dG_{18}}{dt} = (a_{18})^{(2)}G_{17} - \left[ \begin{matrix} (a'_{18})^{(2)} \boxed{+(a''_{18})^{(2)}(T_{17},t)} \boxed{+(a''_{15})^{(1,1,)}(T_{14},t)} \boxed{+(a''_{22})^{(3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4)}(T_{25},t)} \boxed{+(a''_{30})^{(5,5,5,5,5)}(T_{29},t)} \boxed{+(a''_{34})^{(6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{38})^{(7,7,7)}(T_{37},t)} \boxed{+(a''_{42})^{(8,8,8)}(T_{41},t)} \boxed{+(a''_{46})^{(9,9)}(T_{45},t)} \end{matrix} \right] G_{18}$$ *63*

Where $\boxed{+(a''_{16})^{(2)}(T_{17},t)}$ , $\boxed{+(a''_{17})^{(2)}(T_{17},t)}$ , $\boxed{+(a''_{18})^{(2)}(T_{17},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,)}(T_{14},t)}$ , $\boxed{+(a''_{14})^{(1,1,)}(T_{14},t)}$ , $\boxed{+(a''_{15})^{(1,1,)}(T_{14},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3)}(T_{21},t)}$ , $\boxed{+(a''_{21})^{(3,3,3)}(T_{21},t)}$ , $\boxed{+(a''_{22})^{(3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4)}(T_{25},t)}$ , $\boxed{+(a''_{25})^{(4,4,4,4,4)}(T_{25},t)}$ , $\boxed{+(a''_{26})^{(4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5)}(T_{29},t)}$ , $\boxed{+(a''_{29})^{(5,5,5,5,5)}(T_{29},t)}$ , $\boxed{+(a''_{30})^{(5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6)}(T_{33},t)}$ , $\boxed{+(a''_{33})^{(6,6,6,6,6)}(T_{33},t)}$ , $\boxed{+(a''_{34})^{(6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{36})^{(7,7,7)}(T_{37},t)}$ , $\boxed{+(a''_{37})^{(7,7,7)}(T_{37},t)}$ , $\boxed{+(a''_{38})^{(7,7,7)}(T_{37},t)}$ are seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{40})^{(8,8,8)}(T_{41},t)}$ , $\boxed{+(a''_{41})^{(8,8,8)}(T_{41},t)}$ , $\boxed{+(a''_{42})^{(8,8,8)}(T_{41},t)}$ are eight augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{44})^{(9,9)}(T_{45},t)}$ , $\boxed{+(a''_{45})^{(9,9)}(T_{45},t)}$ , $\boxed{+(a''_{46})^{(9,9)}(T_{45},t)}$ are ninth augmentation coefficient for category 1, 2 and 3

$$\frac{dT_{16}}{dt} = (b_{16})^{(2)}T_{17} - \left[ \begin{matrix} (b'_{16})^{(2)} \boxed{-(b''_{16})^{(2)}(G_{19},t)} \boxed{-(b''_{13})^{(1,1,)}(G,t)} \boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4)}(G_{27},t)} \boxed{-(b''_{28})^{(5,5,5,5,5)}(G_{31},t)} \boxed{-(b''_{32})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{36})^{(7,7,7)}(G_{39},t)} \boxed{-(b''_{40})^{(8,8,8)}(G_{43},t)} \boxed{-(b''_{44})^{(9,9)}(G_{47},t)} \end{matrix} \right] T_{16}$$ *64*

$$\frac{dT_{17}}{dt} = (b_{17})^{(2)}T_{16} - \left[ \begin{matrix} (b'_{17})^{(2)} \boxed{-(b''_{17})^{(2)}(G_{19},t)} \boxed{-(b''_{14})^{(1,1,)}(G,t)} \boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4)}(G_{27},t)} \boxed{-(b''_{29})^{(5,5,5,5,5)}(G_{31},t)} \boxed{-(b''_{33})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)} \boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)} \boxed{-(b''_{45})^{(9,9)}(G_{47},t)} \end{matrix} \right] T_{17}$$ *65*

$$\frac{dT_{18}}{dt} = (b_{18})^{(2)}T_{17} - \left[ \begin{matrix} (b'_{18})^{(2)} \boxed{-(b''_{18})^{(2)}(G_{19},t)} \boxed{-(b''_{15})^{(1,1,)}(G,t)} \boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4)}(G_{27},t)} \boxed{-(b''_{30})^{(5,5,5,5,5)}(G_{31},t)} \boxed{-(b''_{34})^{(6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)} \boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)} \boxed{-(b''_{46})^{(9,9)}(G_{47},t)} \end{matrix} \right] T_{18}$$ *66*

where $\boxed{-(b''_{16})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{17})^{(2)}(G_{19},t)}$ , $\boxed{-(b''_{18})^{(2)}(G_{19},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,)}(G,t)}$ , $\boxed{-(b''_{14})^{(1,1,)}(G,t)}$ , $\boxed{-(b''_{15})^{(1,1,)}(G,t)}$ are second detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{20})^{(3,3,3,)}(G_{23},t)}$ , $\boxed{-(b''_{21})^{(3,3,3,)}(G_{23},t)}$ , $\boxed{-(b''_{22})^{(3,3,3,)}(G_{23},t)}$ are third detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{36})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{40})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1,2 and 3

$\boxed{-(b''_{44})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2 and 3

$$\frac{dG_{20}}{dt} = (a_{20})^{(3)}G_{21} - \left[\begin{array}{c} (a'_{20})^{(3)} \boxed{+(a''_{20})^{(3)}(T_{21},t)} \boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)} \boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4)}(T_{25},t)} \boxed{+(a''_{28})^{(5,5,5,5,5,5)}(T_{29},t)} \boxed{+(a''_{32})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7)}(T_{37},t)} \boxed{+(a''_{40})^{(8,8,8,8)}(T_{41},t)} \boxed{+(a''_{44})^{(9,9,9)}(T_{45},t)} \end{array}\right] G_{20} \quad 67$$

$$\frac{dG_{21}}{dt} = (a_{21})^{(3)}G_{20} - \left[\begin{array}{c} (a'_{21})^{(3)} \boxed{+(a''_{21})^{(3)}(T_{21},t)} \boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)} \boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t)} \boxed{+(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t)} \boxed{+(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7)}(T_{37},t)} \boxed{+(a''_{41})^{(8,8,8,8)}(T_{41},t)} \boxed{+(a''_{45})^{(9,9,9)}(T_{45},t)} \end{array}\right] G_{21} \quad 68$$

$$\frac{dG_{22}}{dt} = (a_{22})^{(3)}G_{21} - \left[\begin{array}{c} (a'_{22})^{(3)} \boxed{+(a''_{22})^{(3)}(T_{21},t)} \boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)} \boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)} \boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)} \boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)} \boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)} \boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)} \end{array}\right] G_{22} \quad 69$$

$\boxed{+(a''_{20})^{(3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3)}(T_{21},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2)}(T_{17},t)}$ are second augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,)}(T_{14},t)}$ are third augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{36})^{(7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{37})^{(7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{40})^{(8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8)}(T_{41},t)}$ are eight augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{44})^{(9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1, 2 and 3

$$\frac{dT_{20}}{dt} = (b_{20})^{(3)}T_{21} - \left[ \begin{array}{c} (b'_{20})^{(3)} \boxed{-(b''_{20})^{(3)}(G_{23},t)} \boxed{-(b''_{16})^{(2,2,2)}(G_{19},t)} \boxed{-(b''_{13})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)} \boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)} \boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{36})^{(7,7,7,7)}(G_{39},t)} \boxed{-(b''_{40})^{(8,8,8,8)}(G_{43},t)} \boxed{-(b''_{44})^{(9,9,9)}(G_{47},t)} \end{array} \right] T_{20} \quad 70$$

$$\frac{dT_{21}}{dt} = (b_{21})^{(3)}T_{20} - \left[ \begin{array}{c} (b'_{21})^{(3)} \boxed{-(b''_{21})^{(3)}(G_{23},t)} \boxed{-(b''_{17})^{(2,2,2)}(G_{19},t)} \boxed{-(b''_{14})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)} \boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)} \boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7)}(G_{39},t)} \boxed{-(b''_{41})^{(8,8,8,8)}(G_{43},t)} \boxed{-(b''_{45})^{(9,9,9)}(G_{47},t)} \end{array} \right] T_{21} \quad 71$$

$$\frac{dT_{22}}{dt} = (b_{22})^{(3)}T_{21} - \left[ \begin{array}{c} (b'_{22})^{(3)} \boxed{-(b''_{22})^{(3)}(G_{23},t)} \boxed{-(b''_{18})^{(2,2,2)}(G_{19},t)} \boxed{-(b''_{15})^{(1,1,1,)}(G,t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)} \boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)} \boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7)}(G_{39},t)} \boxed{-(b''_{42})^{(8,8,8,8)}(G_{43},t)} \boxed{-(b''_{46})^{(9,9,9)}(G_{47},t)} \end{array} \right] T_{22} \quad 72$$

$\boxed{-(b''_{20})^{(3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3)}(G_{23},t)}$ , $\boxed{-(b''_{22})^{(3)}(G_{23},t)}$ *are first detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{16})^{(2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2)}(G_{19},t)}$ *are second detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{13})^{(1,1,1,)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,1,)}(G,t)}$ *are third detrition coefficients for category 1,2 and 3*

$\boxed{-(b''_{24})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4)}(G_{27},t)}$ *are fourth detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{28})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5)}(G_{31},t)}$ *are fifth detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{32})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6)}(G_{35},t)}$ *are sixth detrition coefficients for category 1, 2 and 3*

$\boxed{-(b''_{36})^{(7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7)}(G_{39},t)}$ $\boxed{-(b''_{38})^{(7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{40})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8)}(G_{43},t)}$ are eight detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{46})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{24}}{dt} = (a_{24})^{(4)}G_{25} - \left[ \begin{array}{c} (a'_{24})^{(4)} \boxed{+(a''_{24})^{(4)}(T_{25},t)} \boxed{+(a''_{28})^{(5,5,)}(T_{29},t)} \boxed{+(a''_{32})^{(6,6,)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1)}(T_{14},t)} \boxed{+(a''_{16})^{(2,2,2,2)}(T_{17},t)} \boxed{+(a''_{20})^{(3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7,7)}(T_{37},t)} \boxed{+(a''_{40})^{(8,8,8,8,8)}(T_{41},t)} \boxed{+(a''_{44})^{(9,9,9,9)}(T_{45},t)} \end{array} \right] G_{24} \quad 73$$

$$\frac{dG_{25}}{dt} = (a_{25})^{(4)}G_{24} - \begin{bmatrix} (a'_{25})^{(4)} \boxed{+(a''_{25})^{(4)}(T_{25},t)} \boxed{+(a''_{29})^{(5,5,)}(T_{29},t)} \boxed{+(a''_{33})^{(6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1)}(T_{14},t)} \boxed{+(a''_{17})^{(2,2,2,2)}(T_{17},t)} \boxed{+(a''_{21})^{(3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7,7)}(T_{37},t)} \boxed{+(a''_{41})^{(8,8,8,8,8)}(T_{41},t)} \boxed{+(a''_{45})^{(9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{25} \quad 74$$

$$\frac{dG_{26}}{dt} = (a_{26})^{(4)}G_{25} - \begin{bmatrix} (a'_{26})^{(4)} \boxed{+(a''_{26})^{(4)}(T_{25},t)} \boxed{+(a''_{30})^{(5,5,)}(T_{29},t)} \boxed{+(a''_{34})^{(6,6,)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1)}(T_{14},t)} \boxed{+(a''_{18})^{(2,2,2,2)}(T_{17},t)} \boxed{+(a''_{22})^{(3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7,7)}(T_{37},t)} \boxed{+(a''_{42})^{(8,8,8,8,8)}(T_{41},t)} \boxed{+(a''_{46})^{(9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{26} \quad 75$$

$\boxed{(a''_{24})^{(4)}(T_{25},t)}$ , $\boxed{(a''_{25})^{(4)}(T_{25},t)}$ , $\boxed{(a''_{26})^{(4)}(T_{25},t)}$ *are first augmentation coefficients category* 1, 2 3

$\boxed{+(a''_{28})^{(5,5,)}(T_{29},t)}$ , $\boxed{+(a''_{29})^{(5,5,)}(T_{29},t)}$ , $\boxed{+(a''_{30})^{(5,5,)}(T_{29},t)}$ *are second augmentation coefficient for category* 1, 2 *and* 3

$\boxed{+(a''_{32})^{(6,6,)}(T_{33},t)}$ , $\boxed{+(a''_{33})^{(6,6,)}(T_{33},t)}$ , $\boxed{+(a''_{34})^{(6,6,)}(T_{33},t)}$ *are third augmentation coefficient for category* 1, 2 *and* 3

$\boxed{+(a''_{13})^{(1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1)}(T_{14},t)}$ *are fourth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a''_{16})^{(2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2)}(T_{17},t)}$ *are fifth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a''_{20})^{(3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3)}(T_{21},t)}$ *are sixth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a''_{36})^{(7,7,7,7,7)}(T_{37},t)}$ , $\boxed{+(a''_{37})^{(7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7,7,7)}(T_{37},t)}$ *are seventh augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a''_{40})^{(8,8,8,8,8)}(T_{41},t)}$ , $\boxed{+(a''_{41})^{(8,8,8,8,8)}(T_{41},t)}$ , $\boxed{+(a''_{42})^{(8,8,8,8,8)}(T_{41},t)}$ *are eighth augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a''_{46})^{(9,9,9,9)}(T_{45},t)}$ , $\boxed{+(a''_{45})^{(9,9,9,9)}(T_{45},t)}$ , $\boxed{+(a''_{44})^{(9,9,9,9)}(T_{45},t)}$ are ninth detrition coefficients for category 1 2 3

$$\frac{dT_{24}}{dt} = (b_{24})^{(4)}T_{25} - \begin{bmatrix} (b'_{24})^{(4)} \boxed{-(b''_{24})^{(4)}(G_{27},t)} \boxed{-(b''_{28})^{(5,5,)}(G_{31},t)} \boxed{-(b''_{32})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1)}(G,t)} \boxed{-(b''_{16})^{(2,2,2,2)}(G_{19},t)} \boxed{-(b''_{20})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{36})^{(7,7,7,7,7)}(G_{39},t)} \boxed{-(b''_{40})^{(8,8,8,8,8)}(G_{43},t)} \boxed{-(b''_{44})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{24} \quad 76$$

$$\frac{dT_{25}}{dt} = (b_{25})^{(4)}T_{24} - \begin{bmatrix} (b'_{25})^{(4)} \boxed{-(b''_{25})^{(4)}(G_{27},t)} \boxed{-(b''_{29})^{(5,5,)}(G_{31},t)} \boxed{-(b''_{33})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1)}(G,t)} \boxed{-(b''_{17})^{(2,2,2,2)}(G_{19},t)} \boxed{-(b''_{21})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7,7)}(G_{39},t)} \boxed{-(b''_{41})^{(8,8,8,8,8)}(G_{43},t)} \boxed{-(b''_{45})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{25} \quad 77$$

$$\frac{dT_{26}}{dt} = (b_{26})^{(4)}T_{25} - \begin{bmatrix} (b'_{26})^{(4)} \boxed{-(b''_{26})^{(4)}(G_{27},t)} \boxed{-(b''_{30})^{(5,5,)}(G_{31},t)} \boxed{-(b''_{34})^{(6,6,)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1)}(G,t)} \boxed{-(b''_{18})^{(2,2,2,2)}(G_{19},t)} \boxed{-(b''_{22})^{(3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7)}(G_{39},t)} \boxed{-(b''_{42})^{(8,8,8,8,8)}(G_{43},t)} \boxed{-(b''_{46})^{(9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{26} \quad 78$$

*Where* $\boxed{-(b''_{24})^{(4)}(G_{27},t)}$ , $\boxed{-(b''_{25})^{(4)}(G_{27},t)}$ , $\boxed{-(b''_{26})^{(4)}(G_{27},t)}$ *are first detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{28})^{(5,5,)}(G_{31},t)}$ , $\boxed{-(b''_{29})^{(5,5,)}(G_{31},t)}$ , $\boxed{-(b''_{30})^{(5,5,)}(G_{31},t)}$ *are second detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{32})^{(6,6,)}(G_{35},t)}$ , $\boxed{-(b''_{33})^{(6,6,)}(G_{35},t)}$ , $\boxed{-(b''_{34})^{(6,6,)}(G_{35},t)}$ *are third detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{13})^{(1,1,1,1)}(G,t)}$ , $\boxed{-(b''_{14})^{(1,1,1,1)}(G,t)}$ , $\boxed{-(b''_{15})^{(1,1,1,1)}(G,t)}$
*are fourth detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{16})^{(2,2,2,2)}(G_{19},t)}$ , $\boxed{-(b''_{17})^{(2,2,2,2)}(G_{19},t)}$ , $\boxed{-(b''_{18})^{(2,2,2,2)}(G_{19},t)}$
*are fifth detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{20})^{(3,3,3,3)}(G_{23},t)}$ , $\boxed{-(b''_{21})^{(3,3,3,3)}(G_{23},t)}$ , $\boxed{-(b''_{22})^{(3,3,3,3)}(G_{23},t)}$
*are sixth detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{36})^{(7,7,7,7,7)}(G_{39},t)}$ , $\boxed{-(b''_{37})^{(7,7,7,7,7)}(G_{39},t)}$ , $\boxed{-(b''_{38})^{(7,7,7,7,7)}(G_{39},t)}$
*are seventh detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{40})^{(8,8,8,8,8)}(G_{43},t)}$ , $\boxed{-(b''_{41})^{(8,8,8,8,8)}(G_{43},t)}$ , $\boxed{-(b''_{42})^{(8,8,8,8,8)}(G_{43},t)}$
*are eighth detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{46})^{(9,9,9,9)}(G_{47},t)}$ , $\boxed{-(b''_{45})^{(9,9,9,9)}(G_{47},t)}$ , $\boxed{-(b''_{44})^{(9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1 2 3

$$\frac{dG_{28}}{dt} = (a_{28})^{(5)}G_{29} - \begin{bmatrix} (a'_{28})^{(5)} \boxed{+(a''_{28})^{(5)}(T_{29},t)} \boxed{+(a''_{24})^{(4,4,)}(T_{25},t)} \boxed{+(a''_{32})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1)}(T_{14},t)} \boxed{+(a''_{16})^{(2,2,2,2,2)}(T_{17},t)} \boxed{+(a''_{20})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7,7,7)}(T_{37},t)} \boxed{+(a''_{40})^{(8,8,8,8,8,8)}(T_{41},t)} \boxed{+(a''_{44})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{28} \quad 79$$

$$\frac{dG_{29}}{dt} = (a_{29})^{(5)}G_{28} - \begin{bmatrix} (a'_{29})^{(5)} \boxed{+(a''_{29})^{(5)}(T_{29},t)} \boxed{+(a''_{25})^{(4,4,)}(T_{25},t)} \boxed{+(a''_{33})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1)}(T_{14},t)} \boxed{+(a''_{17})^{(2,2,2,2,2)}(T_{17},t)} \boxed{+(a''_{21})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7,7,7)}(T_{37},t)} \boxed{+(a''_{41})^{(8,8,8,8,8,8)}(T_{41},t)} \boxed{+(a''_{45})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{29} \quad 80$$

$$\frac{dG_{30}}{dt} = (a_{30})^{(5)}G_{29} - \begin{bmatrix} (a'_{30})^{(5)} \boxed{+(a''_{30})^{(5)}(T_{29},t)} \boxed{+(a''_{26})^{(4,4,)}(T_{25},t)} \boxed{+(a''_{34})^{(6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1)}(T_{14},t)} \boxed{+(a''_{18})^{(2,2,2,2,2)}(T_{17},t)} \boxed{+(a''_{22})^{(3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7,7,7)}(T_{37},t)} \boxed{+(a''_{42})^{(8,8,8,8,8,8)}(T_{41},t)} \boxed{+(a''_{46})^{(9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{30} \quad 81$$

*Where* $\boxed{+(a''_{28})^{(5)}(T_{29},t)}$ , $\boxed{+(a''_{29})^{(5)}(T_{29},t)}$ , $\boxed{+(a''_{30})^{(5)}(T_{29},t)}$ *are first augmentation coefficients for category* 1, 2 *and* 3

*And* $\boxed{+(a''_{24})^{(4,4,)}(T_{25},t)}$ , $\boxed{+(a''_{25})^{(4,4,)}(T_{25},t)}$ , $\boxed{+(a''_{26})^{(4,4,)}(T_{25},t)}$ *are second augmentation coefficient for category* 1, 2 *and* 3

$\boxed{+(a''_{32})^{(6,6,6)}(T_{33},t)}$ , $\boxed{+(a''_{33})^{(6,6,6)}(T_{33},t)}$ , $\boxed{+(a''_{34})^{(6,6,6)}(T_{33},t)}$ *are third augmentation coefficient for category* 1, 2 *and* 3

$\boxed{+(a''_{13})^{(1,1,1,1,1)}(T_{14},t)}$ , $\boxed{+(a''_{14})^{(1,1,1,1,1)}(T_{14},t)}$ , $\boxed{+(a''_{15})^{(1,1,1,1,1)}(T_{14},t)}$ are fourth augmentation coefficients for category 1,2, and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2)}(T_{17},t)}$ are fifth augmentation coefficients for category 1,2,and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3)}(T_{21},t)}$ are sixth augmentation coefficients for category 1,2, 3

$\boxed{+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)}$ , $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)}$ are seventh augmentation coefficients for category 1,2, 3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8)}(T_{41},t)}$ are eighth augmentation coefficients for category 1,2, 3

$\boxed{+(a''_{46})^{(9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficients for category 1,2, 3

$$\frac{dT_{28}}{dt} = (b_{28})^{(5)}T_{29} - \left[\begin{matrix} (b'_{28})^{(5)} \boxed{-(b''_{28})^{(5)}(G_{31},t)} \boxed{-(b''_{24})^{(4,4,)}(G_{27},t)} \boxed{-(b''_{32})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1)}(G,t)} \boxed{-(b''_{16})^{(2,2,2,2,2)}(G_{19},t)} \boxed{-(b''_{20})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t)} \boxed{-(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t)} \boxed{-(b''_{44})^{(9,9,9,9,9)}(G_{47},t)} \end{matrix}\right] T_{28} \quad 82$$

$$\frac{dT_{29}}{dt} = (b_{29})^{(5)}T_{28} - \left[\begin{matrix} (b'_{29})^{(5)} \boxed{-(b''_{29})^{(5)}(G_{31},t)} \boxed{-(b''_{25})^{(4,4,)}(G_{27},t)} \boxed{-(b''_{33})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1)}(G,t)} \boxed{-(b''_{17})^{(2,2,2,2,2)}(G_{19},t)} \boxed{-(b''_{21})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t)} \boxed{-(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t)} \boxed{-(b''_{45})^{(9,9,9,9,9)}(G_{47},t)} \end{matrix}\right] T_{29} \quad 83$$

$$\frac{dT_{30}}{dt} = (b_{30})^{(5)}T_{29} - \left[\begin{matrix} (b'_{30})^{(5)} \boxed{-(b''_{30})^{(5)}(G_{31},t)} \boxed{-(b''_{26})^{(4,4,)}(G_{27},t)} \boxed{-(b''_{34})^{(6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,)}(G,t)} \boxed{-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)} \boxed{-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)} \boxed{-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)} \boxed{-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)} \end{matrix}\right] T_{30} \quad 84$$

*where* $\boxed{-(b''_{28})^{(5)}(G_{31},t)}$ , $\boxed{-(b''_{29})^{(5)}(G_{31},t)}$ , $\boxed{-(b''_{30})^{(5)}(G_{31},t)}$ *are first detrition coefficients for category* 1, 2 *and* 3

$\boxed{-(b''_{24})^{(4,4,)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,)}(G_{27},t)}$ *are second detrition coefficients for category* 1,2 *and* 3

$\boxed{-(b''_{32})^{(6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6)}(G_{35},t)}$ *are third detrition coefficients for category* 1,2 *and* 3

$\boxed{-(b''_{13})^{(1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1)}(G,t)}$ , $\boxed{-(b''_{15})^{(1,1,1,1,1,)}(G,t)}$ are fourth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t)}$ , $\boxed{-(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1,2, and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1,2, and 3

$$\frac{dG_{32}}{dt} = (a_{32})^{(6)}G_{33} - \left[ \begin{array}{c} (a'_{32})^{(6)} \boxed{+(a''_{32})^{(6)}(T_{33},t)} \boxed{+(a''_{28})^{(5,5,5)}(T_{29},t)} \boxed{+(a''_{24})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)} \boxed{+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)} \boxed{+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)} \boxed{+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)} \boxed{+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)} \end{array} \right] G_{32} \quad 85$$

$$\frac{dG_{33}}{dt} = (a_{33})^{(6)}G_{32} - \left[ \begin{array}{c} (a'_{33})^{(6)} \boxed{+(a''_{33})^{(6)}(T_{33},t)} \boxed{+(a''_{29})^{(5,5,5)}(T_{29},t)} \boxed{+(a''_{25})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)} \boxed{+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)} \boxed{+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)} \boxed{+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)} \boxed{+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)} \end{array} \right] G_{33} \quad 86$$

$$\frac{dG_{34}}{dt} = (a_{34})^{(6)}G_{33} - \left[ \begin{array}{c} (a'_{34})^{(6)} \boxed{+(a''_{34})^{(6)}(T_{33},t)} \boxed{+(a''_{30})^{(5,5,5)}(T_{29},t)} \boxed{+(a''_{26})^{(4,4,4,)}(T_{25},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)} \boxed{+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)} \boxed{+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)} \boxed{+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)} \boxed{+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)} \end{array} \right] G_{34} \quad 87$$

$\boxed{+(a''_{32})^{(6)}(T_{33},t)}, \boxed{+(a''_{33})^{(6)}(T_{33},t)}, \boxed{+(a''_{34})^{(6)}(T_{33},t)}$ *are first augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a''_{28})^{(5,5,5)}(T_{29},t)}, \boxed{+(a''_{29})^{(5,5,5)}(T_{29},t)}, \boxed{+(a''_{30})^{(5,5,5)}(T_{29},t)}$ *are second augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a''_{24})^{(4,4,4,)}(T_{25},t)}, \boxed{+(a''_{25})^{(4,4,4,)}(T_{25},t)}, \boxed{+(a''_{26})^{(4,4,4,)}(T_{25},t)}$ *are third augmentation coefficients for category* 1, 2 *and* 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1)}(T_{14},t)}, \boxed{+(a''_{14})^{(1,1,1,1,1,1)}(T_{14},t)}, \boxed{+(a''_{15})^{(1,1,1,1,1,1)}(T_{14},t)}$ - are fourth augmentation coefficients

$\boxed{+(a''_{16})^{(2,2,2,2,2,2)}(T_{17},t)}, \boxed{+(a''_{17})^{(2,2,2,2,2,2)}(T_{17},t)}, \boxed{+(a''_{18})^{(2,2,2,2,2,2)}(T_{17},t)}$ - fifth augmentation coefficients

$\boxed{+(a''_{20})^{(3,3,3,3,3,3)}(T_{21},t)}, \boxed{+(a''_{21})^{(3,3,3,3,3,3)}(T_{21},t)}, \boxed{+(a''_{22})^{(3,3,3,3,3,3)}(T_{21},t)}$ sixth augmentation coefficients

$\boxed{+(a''_{36})^{(7,7,7,7,7,7,7)}(T_{37},t)}, \boxed{+(a''_{37})^{(7,7,7,7,7,7,7)}(T_{37},t)}, \boxed{+(a''_{38})^{(7,7,7,7,7,7,7)}(T_{37},t)}$

seventh augmentation coefficients

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8)}(T_{41},t)}, \boxed{+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)}, \boxed{+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)}$

Eighth augmentation coefficients

$\boxed{+(a''_{44})^{(9,9,9,9,9,9)}(T_{45},t)}, \boxed{+(a''_{45})^{(9,9,9,9,9,9)}(T_{45},t)}, \boxed{+(a''_{46})^{(9,9,9,9,9,9)}(T_{45},t)}$ ninth augmentation coefficients

$$\frac{dT_{32}}{dt} = (b_{32})^{(6)}T_{33} - \left[ \begin{array}{c} (b'_{32})^{(6)} \boxed{-(b''_{32})^{(6)}(G_{35},t)} \boxed{-(b''_{28})^{(5,5,5)}(G_{31},t)} \boxed{-(b''_{24})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1)}(G,t)} \boxed{-(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t)} \boxed{-(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{36})^{(7,7,7,7,7,7,7)}(G_{39},t)} \boxed{-(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t)} \boxed{-(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t)} \end{array} \right] T_{32} \quad 88$$

$$\frac{dT_{33}}{dt} = (b_{33})^{(6)}T_{32} - \left[ \begin{array}{c} (b'_{33})^{(6)} \boxed{-(b''_{33})^{(6)}(G_{35},t)} \boxed{-(b''_{29})^{(5,5,5)}(G_{31},t)} \boxed{-(b''_{25})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1)}(G,t)} \boxed{-(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t)} \boxed{-(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{37})^{(7,7,7,7,7,7,7)}(G_{39},t)} \boxed{-(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t)} \boxed{-(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t)} \end{array} \right] T_{33} \quad 89$$

$$\frac{dT_{34}}{dt} = (b_{34})^{(6)} T_{33} - \left[ \begin{matrix} (b'_{34})^{(6)} \boxed{-(b''_{34})^{(6)}(G_{35},t)} \boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)} \boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1)}(G,t)} \boxed{-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)} \boxed{-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)} \boxed{-(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t)} \boxed{-(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t)} \end{matrix} \right] T_{34} \quad 90$$

$\boxed{-(b''_{32})^{(6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6)}(G_{35},t)}$ , $\boxed{-(b''_{34})^{(6)}(G_{35},t)}$ $are\ first\ detrition\ coefficients$ $for\ category$ 1, 2 $and$ 3

$\boxed{-(b''_{28})^{(5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5)}(G_{31},t)}$ $are\ second\ detrition\ coefficients$ $for\ category$ 1, 2 $and$ 3

$\boxed{-(b''_{24})^{(4,4,4,)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,)}(G_{27},t)}$ $are\ third\ detrition\ coefficients$ $for\ category$ 1,2 $and$ 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,1)}(G,t)}$ are fourth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2)}(G_{19},t)}$ are fifth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3)}(G_{23},t)}$ are sixth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{40})^{(8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{42})^{(8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2, and 3

$\boxed{-(b''_{46})^{(9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{44})^{(9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2, and 3

$$\frac{dG_{36}}{dt} = (a_{36})^{(7)} G_{37} - \left[ \begin{matrix} (a'_{36})^{(7)} \boxed{+(a''_{36})^{(7)}(T_{37},t)} \boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)} \boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)} \boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)} \boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,)}(T_{41},t)} \boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{matrix} \right] G_{13} \quad 91$$

$$\frac{dG_{37}}{dt} = (a_{37})^{(7)} G_{36} - \left[ \begin{matrix} (a'_{37})^{(7)} \boxed{+(a''_{37})^{(7)}(T_{37},t)} \boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)} \boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)} \boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)} \boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)} \boxed{+(a''_{41})^{(8,8,8,8,8,8,8)}(T_{41},t)} \boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{matrix} \right] G_{14} \quad 92$$

$$\frac{dG_{38}}{dt} = (a_{38})^{(7)} G_{37} - \left[ \begin{matrix} (a'_{38})^{(7)} \boxed{+(a''_{38})^{(7)}(T_{37},t)} \boxed{+(a''_{18})^{(2,2,2,2,2,2,2)}(T_{17},t)} \boxed{+(a''_{22})^{(3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4)}(T_{25},t)} \boxed{+(a''_{30})^{(5,5,5,5,5,5,5)}(T_{29},t)} \boxed{+(a''_{34})^{(6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1)}(T_{14},t)} \boxed{+(a''_{42})^{(8,8,8,8,8,8,8)}(T_{41},t)} \boxed{+(a''_{46})^{(9,9,9,9,9,9,9)}(T_{45},t)} \end{matrix} \right] G_{15} \quad 93$$

Where $\boxed{(a''_{36})^{(7)}(T_{37},t)}$, $\boxed{(a''_{37})^{(7)}(T_{37},t)}$, $\boxed{(a''_{38})^{(7)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2)}(T_{17},t)}$ , $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2)}(T_{17},t)}$ , $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3)}(T_{21},t)}$ , $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3)}(T_{21},t)}$ , $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4)}(T_{25},t)}$ , $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4)}(T_{25},t)}$ , $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5)}(T_{29},t)}$ , $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5)}(T_{29},t)}$ , $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6)}(T_{33},t)}$ , $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6)}(T_{33},t)}$ , $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1)}(T_{14},t)}$ , $\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1)}(T_{14},t)}$ , $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$ , $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8)}(T_{41},t)}$ , $\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,)}(T_{41},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{46})^{(9,9,9,9,9,9,9)}(T_{45},t)}$ , $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9)}(T_{45},t)}$ , $\boxed{+(a''_{44})^{(9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{36}}{dt} = (b_{36})^{(7)}T_{37} - \begin{bmatrix} (b'_{36})^{(7)}\boxed{-(b''_{36})^{(7)}(G_{39},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{13} \quad 94$$

$$\frac{dT_{37}}{dt} = (b_{37})^{(7)}T_{36} - \begin{bmatrix} (b'_{37})^{(7)}\boxed{-(b''_{37})^{(7)}(G_{39},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{14}$$

$$\frac{dT_{38}}{dt} = (b_{38})^{(7)}T_{37} - \begin{bmatrix} (b'_{38})^{(7)}\boxed{-(b''_{38})^{(7)}(G_{39},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8)}(G_{43},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9)}(G_{47},t)} \end{bmatrix} T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$ , $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$ , $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2)}(G_{19},t)}$ , $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2)}(G_{19},t)}$ , $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3)}(G_{23},t)}$ , $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3)}(G_{23},t)}$ , $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4)}(G_{27},t)}$ , $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4)}(G_{27},t)}$ , $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{28}'')^{(5,5,5,5,5,5,5)}(G_{31},t)}$ , $\boxed{-(b_{29}'')^{(5,5,5,5,5,5,5)}(G_{31},t)}$ , $\boxed{-(b_{30}'')^{(5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{32}'')^{(6,6,6,6,6,6,6)}(G_{35},t)}$ , $\boxed{-(b_{33}'')^{(6,6,6,6,6,6,6)}(G_{35},t)}$ , $\boxed{-(b_{34}'')^{(6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{15}'')^{(1,1,1,1,1,1,1,1)}(G,t)}$ , $\boxed{-(b_{14}'')^{(1,1,1,1,1,1,1,1)}(G,t)}$ , $\boxed{-(b_{13}'')^{(1,1,1,1,1,1,1,1)}(G,t)}$

are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{40}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ , $\boxed{-(b_{41}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ , $\boxed{-(b_{42}'')^{(8,8,8,8,8,8,8,8)}(G_{43},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b_{46}'')^{(9,9,9,9,9,9,9)}(G_{47},t)}$ , $\boxed{-(b_{45}'')^{(9,9,9,9,9,9,9)}(G_{47},t)}$ , $\boxed{-(b_{44}'')^{(9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{40}}{dt} = (a_{40})^{(8)}G_{41} - \begin{bmatrix} (a_{40}')^{(8)}\boxed{+(a_{40}'')^{(8)}(T_{41},t)}\boxed{+(a_{16}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}\boxed{+(a_{20}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{24}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}\boxed{+(a_{28}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}\boxed{+(a_{32}'')^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}'')^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}\boxed{+(a_{36}'')^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}\boxed{+(a_{44}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{13} \quad 95$$

$$\frac{dG_{41}}{dt} = (a_{41})^{(8)}G_{40} - \begin{bmatrix} (a_{41}')^{(8)}\boxed{+(a_{41}'')^{(8)}(T_{41},t)}\boxed{+(a_{17}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}\boxed{+(a_{21}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{25}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}\boxed{+(a_{29}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}\boxed{+(a_{33}'')^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{13}'')^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}\boxed{+(a_{37}'')^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}\boxed{+(a_{45}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{14}$$

$$\frac{dG_{42}}{dt} = (a_{42})^{(8)}G_{41} - \begin{bmatrix} (a_{42}')^{(8)}\boxed{+(a_{42}'')^{(8)}(T_{41},t)}\boxed{+(a_{18}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}\boxed{+(a_{22}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a_{26}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}\boxed{+(a_{30}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}\boxed{+(a_{34}'')^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a_{15}'')^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}\boxed{+(a_{38}'')^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}\boxed{+(a_{46}'')^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)} \end{bmatrix} G_{15}$$

Where $\boxed{+(a_{40}'')^{(8)}(T_{41},t)}$ , $\boxed{+(a_{41}'')^{(8)}(T_{41},t)}$ , $\boxed{+(a_{42}'')^{(8)}(T_{41},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a_{16}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ , $\boxed{+(a_{17}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ , $\boxed{+(a_{18}'')^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{20}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ , $\boxed{+(a_{21}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ , $\boxed{+(a_{22}'')^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{24}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ , $\boxed{+(a_{25}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ , $\boxed{+(a_{26}'')^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a_{28}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ , $\boxed{+(a_{29}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ , $\boxed{+(a_{30}'')^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are seventh augmentation coefficient for 1,2,3

$\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$, $\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{46})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$, $\boxed{+(a''_{44})^{(9,9,9,9,9,9,9,9,9)}(T_{45},t)}$ are ninth augmentation coefficient for 1,2,3

$$\frac{dT_{40}}{dt} = (b_{40})^{(8)}T_{41} - \left[\begin{matrix} (b'_{40})^{(8)}\boxed{-(b''_{40})^{(8)}(G_{43},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{matrix}\right] T_{13}$$

$$\frac{dT_{41}}{dt} = (b_{41})^{(8)}T_{40} - \left[\begin{matrix} (b'_{41})^{(8)}\boxed{-(b''_{41})^{(8)}(G_{43},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{matrix}\right] T_{14}$$

$$\frac{dT_{42}}{dt} = (b_{42})^{(8)}T_{41} - \left[\begin{matrix} (b'_{42})^{(8)}\boxed{-(b''_{42})^{(8)}(G_{43},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9,9)}(G_{47},t)} \end{matrix}\right] T_{15}$$

Where $\boxed{-(b''_{36})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7)}(G_{39},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,)}(G_{35},t)}$, $\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{38})^{(7,7,)}(G_{39},t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{44})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9,9,9,9,9,9,9,9)}(G_{47},t)}$ are ninth detrition coefficients for category 1, 2 and 3

$$\frac{dG_{44}}{dt} = (a_{44})^{(9)}G_{45} - \left[\begin{array}{ccc} (a'_{44})^{(9)} \boxed{+(a''_{44})^{(9)}(T_{45},t)} & \boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{13}$$ 96

$$\frac{dG_{45}}{dt} = (a_{45})^{(9)}G_{44} - \left[\begin{array}{ccc} (a'_{45})^{(9)} \boxed{+(a''_{45})^{(9)}(T_{45},t)} & \boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{14}$$

$$\frac{dG_{46}}{dt} = (a_{46})^{(9)}G_{45} - \left[\begin{array}{ccc} (a'_{46})^{(9)} \boxed{+(a''_{46})^{(9)}(T_{37},t)} & \boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)} & \boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)} \\ \boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)} & \boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)} & \boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)} \\ \boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)} & \boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)} & \boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)} \end{array}\right] G_{15}$$

Where $\boxed{+(a''_{44})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{45})^{(9)}(T_{45},t)}$, $\boxed{+(a''_{46})^{(9)}(T_{37},t)}$ are first augmentation coefficients for category 1, 2 and 3

$\boxed{+(a''_{16})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{17})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$, $\boxed{+(a''_{18})^{(2,2,2,2,2,2,2,2,2)}(T_{17},t)}$ are second augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{20})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{21})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$, $\boxed{+(a''_{22})^{(3,3,3,3,3,3,3,3,3)}(T_{21},t)}$ are third augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{24})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{25})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$, $\boxed{+(a''_{26})^{(4,4,4,4,4,4,4,4,4)}(T_{25},t)}$ are fourth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{28})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{29})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$, $\boxed{+(a''_{30})^{(5,5,5,5,5,5,5,5,5)}(T_{29},t)}$ are fifth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{32})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{33})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$, $\boxed{+(a''_{34})^{(6,6,6,6,6,6,6,6,6)}(T_{33},t)}$ are sixth augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{13})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{14})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$, $\boxed{+(a''_{15})^{(1,1,1,1,1,1,1,1,1)}(T_{14},t)}$ are Seventh augmentation coefficient for category 1, 2 and 3

$\boxed{+(a''_{38})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{37})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ $\boxed{+(a''_{36})^{(7,7,7,7,7,7,7,7,7)}(T_{37},t)}$ are eighth augmentation coefficient for 1,2,3

$\boxed{+(a''_{40})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{42})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$, $\boxed{+(a''_{41})^{(8,8,8,8,8,8,8,8,8)}(T_{41},t)}$ are ninth augmentation coefficient for 1,2,3

$\frac{dT_{44}}{dt} =$

$$(b_{44})^{(9)}T_{45} - \left[ \begin{matrix} (b'_{44})^{(9)} \boxed{-(b''_{44})^{(9)}(G_{47},t)} & \boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{matrix} \right] T_{13}$$

$\frac{dT_{45}}{dt} =$

$$(b_{45})^{(9)}T_{44} - \left[ \begin{matrix} (b'_{45})^{(9)} \boxed{-(b''_{45})^{(9)}(G_{47},t)} & \boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{matrix} \right] T_{14}$$

$\frac{dT_{46}}{dt} =$

$$(b_{46})^{(9)}T_{45} - \left[ \begin{matrix} (b'_{46})^{(9)} \boxed{-(b''_{46})^{(9)}(G_{47},t)} & \boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)} & \boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)} \\ \boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)} & \boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)} & \boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)} \\ \boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)} & \boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)} & \boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)} \end{matrix} \right] T_{15}$$

Where $\boxed{-(b''_{44})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{45})^{(9)}(G_{47},t)}$, $\boxed{-(b''_{46})^{(9)}(G_{47},t)}$ are first detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{16})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{17})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$, $\boxed{-(b''_{18})^{(2,2,2,2,2,2,2,2,2)}(G_{19},t)}$ are second detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{20})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{21})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$, $\boxed{-(b''_{22})^{(3,3,3,3,3,3,3,3,3)}(G_{23},t)}$ are third detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{24})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{25})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$, $\boxed{-(b''_{26})^{(4,4,4,4,4,4,4,4,4)}(G_{27},t)}$ are fourth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{28})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{29})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$, $\boxed{-(b''_{30})^{(5,5,5,5,5,5,5,5,5)}(G_{31},t)}$ are fifth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{32})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{33})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$, $\boxed{-(b''_{34})^{(6,6,6,6,6,6,6,6,6)}(G_{35},t)}$ are sixth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{15})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{14})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$, $\boxed{-(b''_{13})^{(1,1,1,1,1,1,1,1,1)}(G,t)}$ are seventh detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{37})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{36})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$, $\boxed{-(b''_{38})^{(7,7,7,7,7,7,7,7,7)}(G_{39},t)}$ are eighth detrition coefficients for category 1, 2 and 3

$\boxed{-(b''_{42})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{41})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$, $\boxed{-(b''_{40})^{(8,8,8,8,8,8,8,8,8)}(G_{43},t)}$ are ninth detrition coefficients for category 1, 2 and 3

Where we suppose

$$(a_i)^{(1)}, (a'_i)^{(1)}, (a''_i)^{(1)}, (b_i)^{(1)}, (b'_i)^{(1)}, (b''_i)^{(1)} > 0, \quad i,j = 13,14,15$$ *97*

The functions $(a_i'')^{(1)}, (b_i'')^{(1)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(1)}, \ (r_i)^{(1)}$:

$$(a_i'')^{(1)}(T_{14}, t) \leq (p_i)^{(1)} \leq (\hat{A}_{13})^{(1)}$$

$$(b_i'')^{(1)}(G, t) \leq \ (r_i)^{(1)} \leq (b_i')^{(1)} \leq (\hat{B}_{13})^{(1)}$$

$$lim_{T_2 \to \infty} (a_i'')^{(1)} (T_{14}, t) = (p_i)^{(1)}$$ 98

$$\lim_{G \to \infty} (b_i'')^{(1)} (G, t) = \ (r_i)^{(1)}$$

**Definition of** $(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}$ :

Where $\boxed{(\hat{A}_{13})^{(1)}, (\hat{B}_{13})^{(1)}, (p_i)^{(1)}, \ (r_i)^{(1)}}$ are positive constants and $\boxed{i = 13,14,15}$

They satisfy Lipschitz condition: 99

$$|(a_i'')^{(1)}(T_{14}', t) - (a_i'')^{(1)}(T_{14}, t)| \leq (\hat{k}_{13})^{(1)} |T_{14} - T_{14}'| e^{-(\hat{M}_{13})^{(1)} t}$$

$$|(b_i'')^{(1)}(G', t) - (b_i'')^{(1)}(G, t)| < (\hat{k}_{13})^{(1)} ||G - G'|| e^{-(\hat{M}_{13})^{(1)} t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(1)}(T_{14}', t)$ and $(a_i'')^{(1)}(T_{14}, t)$ . $(T_{14}', t)$ and $(T_{14}, t)$ are points belonging to the interval $\left[(\hat{k}_{13})^{(1)}, (\hat{M}_{13})^{(1)}\right]$ . It is to be noted that $(a_i'')^{(1)}(T_{14}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{13})^{(1)} = 1$ then the function $(a_i'')^{(1)}(T_{14}, t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$ : 100

$(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}$, are positive constants

$$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}} \ , \frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$$

**Definition of** $(\hat{P}_{13})^{(1)}, (\hat{Q}_{13})^{(1)}$ : 101

There exists two constants $(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ which together with $(\hat{M}_{13})^{(1)}, (\hat{k}_{13})^{(1)}, (\hat{A}_{13})^{(1)}$ and $(\hat{B}_{13})^{(1)}$ and the constants $(a_i)^{(1)}, (a_i')^{(1)}, (b_i)^{(1)}, (b_i')^{(1)}, (p_i)^{(1)}, \ (r_i)^{(1)}, i = 13,14,15,$
satisfy the inequalities

$$\frac{1}{(\hat{M}_{13})^{(1)}} [\, (a_i)^{(1)} + (a_i')^{(1)} + \ (\hat{A}_{13})^{(1)} + \ (\hat{P}_{13})^{(1)} \ (\hat{k}_{13})^{(1)}] < 1$$

$$\frac{1}{(\hat{M}_{13})^{(1)}} [\ (b_i)^{(1)} + (b_i')^{(1)} + \ (\hat{B}_{13})^{(1)} + \ (\hat{Q}_{13})^{(1)} \ \ (\hat{k}_{13})^{(1)}] < 1$$

Where we suppose

$$(a_i)^{(2)}, (a_i')^{(2)}, (a_i'')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (b_i'')^{(2)} > 0, \quad i, j = 16,17,18$$

The functions $(a_i'')^{(2)}, (b_i'')^{(2)}$ are positive continuous increasing and bounded.

**Definition of** $(\mathrm{p_i})^{(2)}, \ (\mathrm{r_i})^{(2)}$:

$(a_i'')^{(2)}(T_{17},t) \leq (p_i)^{(2)} \leq (\hat{A}_{16})^{(2)}$ 102

$(b_i'')^{(2)}(G_{19},t) \leq (r_i)^{(2)} \leq (b_i')^{(2)} \leq (\hat{B}_{16})^{(2)}$ 103

$\lim_{T_2\to\infty}(a_i'')^{(2)}(T_{17},t) = (p_i)^{(2)}$ 104

$\lim_{G\to\infty}(b_i'')^{(2)}\left((G_{19}),t\right) = (r_i)^{(2)}$ 105

<u>**Definition of**</u> $(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}$ : 106

Where $\boxed{(\hat{A}_{16})^{(2)}, (\hat{B}_{16})^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}}$ are positive constants and $\boxed{i = 16,17,18}$

They satisfy Lipschitz condition:

$|(a_i'')^{(2)}(T_{17}',t) - (a_i'')^{(2)}(T_{17},t)| \leq (\hat{k}_{16})^{(2)}|T_{17} - T_{17}'|e^{-(\hat{M}_{16})^{(2)}t}$ 107

$|(b_i'')^{(2)}((G_{19})',t) - (b_i'')^{(2)}\left((G_{19}),t\right)| < (\hat{k}_{16})^{(2)}||(G_{19}) - (G_{19})'||e^{-(\hat{M}_{16})^{(2)}t}$ 108

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(2)}(T_{17}',t)$ and$(a_i'')^{(2)}(T_{17},t)$ . $(T_{17}',t)$ and $(T_{17},t)$ are points belonging to the interval $\left[(\hat{k}_{16})^{(2)}, (\hat{M}_{16})^{(2)}\right]$ . It is to be noted that $(a_i'')^{(2)}(T_{17},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{16})^{(2)} = 1$ then the function $(a_i'')^{(2)}(T_{17},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous.

<u>**Definition of**</u> $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$ :

$(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}$, are positive constants 109

$$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}, \frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$$

<u>**Definition of**</u> $(\hat{P}_{13})^{(2)}, (\hat{Q}_{13})^{(2)}$ :

There exists two constants $(\hat{P}_{16})^{(2)}$ and $(\hat{Q}_{16})^{(2)}$ which together with $(\hat{M}_{16})^{(2)}, (\hat{k}_{16})^{(2)}, (\hat{A}_{16})^{(2)}$ *and* $(\hat{B}_{16})^{(2)}$ and the constants $(a_i)^{(2)}, (a_i')^{(2)}, (b_i)^{(2)}, (b_i')^{(2)}, (p_i)^{(2)}, (r_i)^{(2)}, i = 16,17,18,$

satisfy the inequalities

$\frac{1}{(\hat{M}_{16})^{(2)}}[(a_i)^{(2)} + (a_i')^{(2)} + (\hat{A}_{16})^{(2)} + (\hat{P}_{16})^{(2)}(\hat{k}_{16})^{(2)}] < 1$ 110

$\frac{1}{(\hat{M}_{16})^{(2)}}[(b_i)^{(2)} + (b_i')^{(2)} + (\hat{B}_{16})^{(2)} + (\hat{Q}_{16})^{(2)}(\hat{k}_{16})^{(2)}] < 1$ 111

Where we suppose

$(a_i)^{(3)}, (a_i')^{(3)}, (a_i'')^{(3)}, (b_i)^{(3)}, (b_i')^{(3)}, (b_i'')^{(3)} > 0, \quad i,j = 20,21,22$ 112

The functions $(a_i'')^{(3)}, (b_i'')^{(3)}$ are positive continuous increasing and bounded.

<u>**Definition of**</u> $(p_i)^{(3)}, (\mathrm{r_i})^{(3)}$:

$(a_i'')^{(3)}(T_{21},t) \leq (p_i)^{(3)} \leq (\hat{A}_{20})^{(3)}$

$(b_i'')^{(3)}(G_{23},t) \leq \ (r_i)^{(3)} \leq (b_i')^{(3)} \leq (\ \hat{B}_{20}\ )^{(3)}$

$lim_{T_2\to\infty}(a_i'')^{(3)}\ (T_{21},t) = (p_i)^{(3)}$ 113

$\lim_{G\to\infty}(b_i'')^{(3)}\ (G_{23},t) = \ \ (r_i)^{(3)}$

**Definition of** $(\ \hat{A}_{20}\ )^{(3)}, (\ \hat{B}_{20}\ )^{(3)}$ :

Where $\boxed{(\ \hat{A}_{20}\ )^{(3)}, (\ \hat{B}_{20}\ )^{(3)}, (p_i)^{(3)},\ \ (r_i)^{(3)}}$ are positive constants and $\boxed{i = 20,21,22}$

They satisfy Lipschitz condition: 114

$|(a_i'')^{(3)}(T_{21}',t) - (a_i'')^{(3)}(T_{21},t)| \leq (\ \hat{k}_{20}\ )^{(3)}|T_{21} - \ T_{21}'|e^{-(\ \hat{M}_{20}\ )^{(3)}t}$

$|(b_i'')^{(3)}(G_{23}{}',t) - (b_i'')^{(3)}(G_{23},t)| < (\ \hat{k}_{20}\ )^{(3)}||G_{23} - G_{23}{}'||e^{-(\ \hat{M}_{20}\ )^{(3)}t}$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(3)}(T_{21}',t)$ and$(a_i'')^{(3)}(T_{21},t)$ . $(T_{21}',t)$ And $(T_{21},t)$ are points belonging to the interval $\left[(\ \hat{k}_{20}\ )^{(3)}, (\ \hat{M}_{20}\ )^{(3)}\right]$. It is to be noted that $(a_i'')^{(3)}(T_{21},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\ \hat{M}_{20}\ )^{(3)} = 1$ then the function $(a_i'')^{(3)}(T_{21},t)$ , the first augmentation coefficient attributable to the system, would be absolutely continuous.

**Definition of** $(\ \hat{M}_{20}\ )^{(3)}, (\ \hat{k}_{20}\ )^{(3)}$ : 115

$(\ \hat{M}_{20}\ )^{(3)}, (\ \hat{k}_{20}\ )^{(3)}$, are positive constants

$$\frac{(a_i)^{(3)}}{(\ \hat{M}_{20}\ )^{(3)}}\ , \frac{(b_i)^{(3)}}{(\ \hat{M}_{20}\ )^{(3)}} < 1$$

There exists two constants There exists two constants $(\ \hat{P}_{20}\ )^{(3)}$ and $(\ \hat{Q}_{20}\ )^{(3)}$ which together with $(\ \hat{M}_{20}\ )^{(3)}, (\ \hat{k}_{20}\ )^{(3)}, (\hat{A}_{20})^{(3)} and\ (\ \hat{B}_{20}\ )^{(3)}$ and the constants $(a_i)^{(3)}, (a_i')^{(3)}, (b_i)^{(3)}, (b_i')^{(3)}, (p_i)^{(3)},\ \ (r_i)^{(3)}, i = 20,21,22,$ satisfy the inequalities 116

$$\frac{1}{(\ \hat{M}_{20}\ )^{(3)}}[\ (a_i)^{(3)} + (a_i')^{(3)} + \ \ (\ \hat{A}_{20}\ )^{(3)} + \ (\ \hat{P}_{20}\ )^{(3)}\ (\ \hat{k}_{20}\ )^{(3)}] < 1$$

$$\frac{1}{(\ \hat{M}_{20}\ )^{(3)}}[\ \ (b_i)^{(3)} + (b_i')^{(3)} + \ \ (\ \hat{B}_{20}\ )^{(3)} + \ (\ \hat{Q}_{20}\ )^{(3)}\ \ (\ \hat{k}_{20}\ )^{(3)}] < 1$$

Where we suppose

$(a_i)^{(4)}, (a_i')^{(4)}, (a_i'')^{(4)}, (b_i)^{(4)}, (b_i')^{(4)}, (b_i'')^{(4)} > 0, \quad i,j = 24,25,26$ 117

The functions $(a_i'')^{(4)}, (b_i'')^{(4)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(4)},\ \ (r_i)^{(4)}$:

$(a_i'')^{(4)}(T_{25},t) \leq (p_i)^{(4)} \leq (\ \hat{A}_{24}\ )^{(4)}$

$(b_i'')^{(4)}\big((G_{27}),t\big) \leq \ \ (r_i)^{(4)} \leq (b_i')^{(4)} \leq (\ \hat{B}_{24}\ )^{(4)}$

$lim_{T_2\to\infty}(a_i'')^{(4)}\ (T_{25},t) = (p_i)^{(4)}$ 118

$\lim_{G\to\infty}(b_i'')^{(4)}\ \big((G_{27}),t\big) = \ \ (r_i)^{(4)}$

<u>**Definition of**</u> $(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}$ :

Where $\boxed{(\hat{A}_{24})^{(4)}, (\hat{B}_{24})^{(4)}, (p_i)^{(4)}, \ (r_i)^{(4)}}$ are positive constants and $\boxed{i = 24,25,26}$

They satisfy Lipschitz condition: 119

$|(a_i'')^{(4)}(T_{25}', t) - (a_i'')^{(4)}(T_{25}, t)| \leq (\hat{k}_{24})^{(4)}|T_{25} - T_{25}'|e^{-(\hat{M}_{24})^{(4)}t}$

$|(b_i'')^{(4)}((G_{27})', t) - (b_i'')^{(4)}\big((G_{27}), t\big)| < (\hat{k}_{24})^{(4)}||(G_{27}) - (G_{27})'||e^{-(\hat{M}_{24})^{(4)}t}$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(4)}(T_{25}', t)$ and$(a_i'')^{(4)}(T_{25}, t)$ . $(T_{25}', t)$ and $(T_{25}, t)$ are points belonging to the interval $\big[(\hat{k}_{24})^{(4)}, (\hat{M}_{24})^{(4)}\big]$ . It is to be noted that $(a_i'')^{(4)}(T_{25}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{24})^{(4)} = 1$then the function $(a_i'')^{(4)}(T_{25}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

<u>**Definition of**</u> $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$ : 120

$(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}$, are positive constants

$$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}} \ , \frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$$

<u>**Definition of**</u> $(\hat{P}_{24})^{(4)}, (\hat{Q}_{24})^{(4)}$ : 121

There exists two constants $(\hat{P}_{24})^{(4)}$ and $(\hat{Q}_{24})^{(4)}$ which together with $(\hat{M}_{24})^{(4)}, (\hat{k}_{24})^{(4)}, (\hat{A}_{24})^{(4)} and \ (\hat{B}_{24})^{(4)}$ and the constants $(a_i)^{(4)}, (a_i')^{(4)}, (b_i)^{(4)}, (b_i')^{(4)}, (p_i)^{(4)}, \ (r_i)^{(4)}, i = 24,25,26$, satisfy the inequalities

$$\frac{1}{(\hat{M}_{24})^{(4)}}[\,(a_i)^{(4)} + (a_i')^{(4)} + \ (\hat{A}_{24})^{(4)} + \ (\hat{P}_{24})^{(4)}\,(\hat{k}_{24})^{(4)}] < 1$$

$$\frac{1}{(\hat{M}_{24})^{(4)}}[\ (b_i)^{(4)} + (b_i')^{(4)} + \ (\hat{B}_{24})^{(4)} + \ (\hat{Q}_{24})^{(4)}\ \ (\hat{k}_{24})^{(4)}] < 1$$

Where we suppose

$(a_i)^{(5)}, (a_i')^{(5)}, (a_i'')^{(5)}, (b_i)^{(5)}, (b_i')^{(5)}, (b_i'')^{(5)} > 0, \quad i, j = 28,29,30$ 122

The functions $(a_i'')^{(5)}, (b_i'')^{(5)}$ are positive continuous increasing and bounded.

<u>**Definition of**</u> $(p_i)^{(5)}, \ (r_i)^{(5)}$:

$$(a_i'')^{(5)}(T_{29}, t) \leq (p_i)^{(5)} \leq (\hat{A}_{28})^{(5)}$$

$$(b_i'')^{(5)}\big((G_{31}), t\big) \leq \ (r_i)^{(5)} \leq (b_i')^{(5)} \leq (\hat{B}_{28})^{(5)}$$

$lim_{T_2\to\infty}(a_i'')^{(5)}\,(T_{29}, t) = (p_i)^{(5)}$ 123

$$\lim_{G\to\infty}(b_i'')^{(5)}\,(G_{31}, t) = \ (r_i)^{(5)}$$

<u>**Definition of**</u> $(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}$ :

Where $\boxed{(\hat{A}_{28})^{(5)}, (\hat{B}_{28})^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}}$ are positive constants and $\boxed{i = 28,29,30}$

They satisfy Lipschitz condition: 124

$$|(a_i'')^{(5)}(T_{29}', t) - (a_i'')^{(5)}(T_{29}, t)| \leq (\hat{k}_{28})^{(5)}|T_{29} - T_{29}'|e^{-(\hat{M}_{28})^{(5)}t}$$

$$|(b_i'')^{(5)}((G_{31})', t) - (b_i'')^{(5)}((G_{31}), t)| < (\hat{k}_{28})^{(5)}||(G_{31}) - (G_{31})'||e^{-(\hat{M}_{28})^{(5)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(5)}(T_{29}', t)$ and $(a_i'')^{(5)}(T_{29}, t)$ . $(T_{29}', t)$ and $(T_{29}, t)$ are points belonging to the interval $[(\hat{k}_{28})^{(5)}, (\hat{M}_{28})^{(5)}]$ . It is to be noted that $(a_i'')^{(5)}(T_{29}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{28})^{(5)} = 1$ then the function $(a_i'')^{(5)}(T_{29}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

<u>**Definition of**</u> $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$ : 125

$(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}$, are positive constants

$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}, \frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$$

<u>**Definition of**</u> $(\hat{P}_{28})^{(5)}, (\hat{Q}_{28})^{(5)}$ : 126

There exists two constants $(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ which together with $(\hat{M}_{28})^{(5)}, (\hat{k}_{28})^{(5)}, (\hat{A}_{28})^{(5)} and (\hat{B}_{28})^{(5)}$ and the constants $(a_i)^{(5)}, (a_i')^{(5)}, (b_i)^{(5)}, (b_i')^{(5)}, (p_i)^{(5)}, (r_i)^{(5)}, i = 28,29,30,$ satisfy the inequalities

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(a_i)^{(5)} + (a_i')^{(5)} + (\hat{A}_{28})^{(5)} + (\hat{P}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$$

$$\frac{1}{(\hat{M}_{28})^{(5)}}[(b_i)^{(5)} + (b_i')^{(5)} + (\hat{B}_{28})^{(5)} + (\hat{Q}_{28})^{(5)}(\hat{k}_{28})^{(5)}] < 1$$

Where we suppose

$$(a_i)^{(6)}, (a_i')^{(6)}, (a_i'')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (b_i'')^{(6)} > 0, \quad i, j = 32,33,34$$ 127

The functions $(a_i'')^{(6)}, (b_i'')^{(6)}$ are positive continuous increasing and bounded.

<u>**Definition of**</u> $(p_i)^{(6)}, (r_i)^{(6)}$:

$$(a_i'')^{(6)}(T_{33}, t) \leq (p_i)^{(6)} \leq (\hat{A}_{32})^{(6)}$$

$$(b_i'')^{(6)}((G_{35}), t) \leq (r_i)^{(6)} \leq (b_i')^{(6)} \leq (\hat{B}_{32})^{(6)}$$

$$lim_{T_2 \to \infty}(a_i'')^{(6)}(T_{33}, t) = (p_i)^{(6)}$$ 128

$$\lim_{G \to \infty}(b_i'')^{(6)}((G_{35}), t) = (r_i)^{(6)}$$

**Definition of** $(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}$ :

Where $\boxed{(\hat{A}_{32})^{(6)}, (\hat{B}_{32})^{(6)}, (p_i)^{(6)},\ (r_i)^{(6)}}$ are positive constants and $\boxed{i = 32,33,34}$

They satisfy Lipschitz condition:

$|(a_i'')^{(6)}(T_{33}', t) - (a_i'')^{(6)}(T_{33}, t)| \leq (\hat{k}_{32})^{(6)}|T_{33} - T_{33}'|e^{-(\hat{M}_{32})^{(6)}t}$

$|(b_i'')^{(6)}((G_{35})', t) - (b_i'')^{(6)}((G_{35}), t)| < (\hat{k}_{32})^{(6)}||(G_{35}) - (G_{35})'||e^{-(\hat{M}_{32})^{(6)}t}$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(6)}(T_{33}', t)$ and$(a_i'')^{(6)}(T_{33}, t)$ . $(T_{33}', t)$ and $(T_{33}, t)$ are points belonging to the interval $[(\hat{k}_{32})^{(6)}, (\hat{M}_{32})^{(6)}]$ . It is to be noted that $(a_i'')^{(6)}(T_{33}, t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{32})^{(6)} = 1$ then the function $(a_i'')^{(6)}(T_{33}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$ : 129

$(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}$, are positive constants

$$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}} \ , \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$$

**Definition of** $(\hat{P}_{32})^{(6)}, (\hat{Q}_{32})^{(6)}$ : 130

There exists two constants $(\hat{P}_{32})^{(6)}$ and $(\hat{Q}_{32})^{(6)}$ which together with $(\hat{M}_{32})^{(6)}, (\hat{k}_{32})^{(6)}, (\hat{A}_{32})^{(6)} and\ (\hat{B}_{32})^{(6)}$ and the constants $(a_i)^{(6)}, (a_i')^{(6)}, (b_i)^{(6)}, (b_i')^{(6)}, (p_i)^{(6)},\ (r_i)^{(6)}, i = 32,33,34,$
satisfy the inequalities

$$\frac{1}{(\hat{M}_{32})^{(6)}}[\,(a_i)^{(6)} + (a_i')^{(6)} + (\hat{A}_{32})^{(6)} + (\hat{P}_{32})^{(6)}\,(\hat{k}_{32})^{(6)}] < 1$$

$$\frac{1}{(\hat{M}_{32})^{(6)}}[\,(b_i)^{(6)} + (b_i')^{(6)} + (\hat{B}_{32})^{(6)} + (\hat{Q}_{32})^{(6)}\ (\hat{k}_{32})^{(6)}] < 1$$

Where we suppose

$(a_i)^{(7)}, (a_i')^{(7)}, (a_i'')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (b_i'')^{(7)} > 0, \quad i, j = 36,37,38$ 131

The functions $(a_i'')^{(7)}, (b_i'')^{(7)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(7)},\ (r_i)^{(7)}$:

$(a_i'')^{(7)}(T_{37}, t) \leq (p_i)^{(7)} \leq (\hat{A}_{36})^{(7)}$

$(b_i'')^{(7)}(G_{39}, t) \leq\ (r_i)^{(7)} \leq (b_i')^{(7)} \leq (\hat{B}_{36})^{(7)}$

132

$\lim_{T_2\to\infty} (a_i'')^{(7)}(T_{37}, t) = (p_i)^{(7)}$

$\lim_{G\to\infty}(b_i'')^{(7)}((G_{39}), t) =\ (r_i)^{(7)}$

**Definition of** $( \hat{A}_{36} )^{(7)}, ( \hat{B}_{36} )^{(7)}$ :

Where $\boxed{( \hat{A}_{36} )^{(7)}, ( \hat{B}_{36} )^{(7)}, (p_i)^{(7)}, \ (r_i)^{(7)}}$ are positive constants and $\boxed{i = 36,37,38}$

They satisfy Lipschitz condition: 133

$$|(a_i'')^{(7)}(T_{37}', t) - (a_i'')^{(7)}(T_{37}, t)| \leq ( \hat{k}_{36} )^{(7)} |T_{37} - T_{37}'| e^{-( \hat{M}_{36} )^{(7)} t}$$

$$|(b_i'')^{(7)}((G_{39})', t) - (b_i'')^{(7)}\big((G_{39}), t\big)| < ( \hat{k}_{36} )^{(7)} ||(G_{39}) - (G_{39})'|| e^{-( \hat{M}_{36} )^{(7)} t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(7)}(T_{37}', t)$ and $(a_i'')^{(7)}(T_{37}, t)$ . $(T_{37}', t)$ and $(T_{37}, t)$ are points belonging to the interval $\big[( \hat{k}_{36} )^{(7)}, ( \hat{M}_{36} )^{(7)}\big]$ . It is to be noted that $(a_i'')^{(7)}(T_{37}, t)$ is uniformly continuous. In the eventuality of the fact, that if $( \hat{M}_{36} )^{(7)} = 1$ then the function $(a_i'')^{(7)}(T_{37}, t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $( \hat{M}_{36} )^{(7)}, ( \hat{k}_{36} )^{(7)}$ : 134

$( \hat{M}_{36} )^{(7)}, ( \hat{k}_{36} )^{(7)}$, are positive constants

$$\frac{(a_i)^{(7)}}{( \hat{M}_{36} )^{(7)}} \ , \frac{(b_i)^{(7)}}{( \hat{M}_{36} )^{(7)}} < 1$$

**Definition of** $( \hat{P}_{36} )^{(7)}, ( \hat{Q}_{36} )^{(7)}$ : 135

There exists two constants $( \hat{P}_{36} )^{(7)}$ and $( \hat{Q}_{36} )^{(7)}$ which together with $( \hat{M}_{36} )^{(7)}, ( \hat{k}_{36} )^{(7)}, (\hat{A}_{36})^{(7)} and \ ( \hat{B}_{36} )^{(7)}$ and the constants $(a_i)^{(7)}, (a_i')^{(7)}, (b_i)^{(7)}, (b_i')^{(7)}, (p_i)^{(7)}, \ (r_i)^{(7)}, i = 36,37,38$, satisfy the inequalities

$$\frac{1}{( \hat{M}_{36} )^{(7)}} [ \ (a_i)^{(7)} + (a_i')^{(7)} + \ ( \hat{A}_{36} )^{(7)} + \ ( \hat{P}_{36} )^{(7)} \ ( \hat{k}_{36} )^{(7)}] < 1$$

$$\frac{1}{( \hat{M}_{36} )^{(7)}} [ \ (b_i)^{(7)} + (b_i')^{(7)} + \ ( \hat{B}_{36} )^{(7)} + \ ( \hat{Q}_{36} )^{(7)} \ ( \hat{k}_{36} )^{(7)}] < 1$$

Where we suppose

$(a_i)^{(8)}, (a_i')^{(8)}, (a_i'')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (b_i'')^{(8)} > 0, \quad i, j = 40,41,42$ 136

The functions $(a_i'')^{(8)}, (b_i'')^{(8)}$ are positive continuous increasing and bounded

**Definition of** $(p_i)^{(8)}, \ (r_i)^{(8)}$: 137

$(a_i'')^{(8)}(T_{41}, t) \leq (p_i)^{(8)} \leq ( \hat{A}_{40} )^{(8)}$ 138

$(b_i'')^{(8)}((G_{43}), t) \leq \ (r_i)^{(8)} \leq (b_i')^{(8)} \leq ( \hat{B}_{40} )^{(8)}$ 139

$\lim_{T_2 \to \infty} (a_i'')^{(8)} (T_{41}, t) = (p_i)^{(8)}$ 140

$\lim_{G\to\infty}(b_i'')^{(8)}\left((G_{43}),t\right) = \ (r_i)^{(8)}$ 141

**Definition of** $(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}$ :

Where $\boxed{(\hat{A}_{40})^{(8)}, (\hat{B}_{40})^{(8)}, (p_i)^{(8)},\ (r_i)^{(8)}}$ are positive constants and $\boxed{i = 40,41,42}$

They satisfy Lipschitz condition:

$|(a_i'')^{(8)}(T_{41}',t) - (a_i'')^{(8)}(T_{41},t)| \leq (\hat{k}_{40})^{(8)}|T_{41} - T_{41}'|e^{-(\hat{M}_{40})^{(8)}t}$ 142

$|(b_i'')^{(8)}((G_{43})',t) - (b_i'')^{(8)}\left((G_{43}),t\right)| < (\hat{k}_{40})^{(8)}||(G_{43}) - (G_{43})'||e^{-(\hat{M}_{40})^{(8)}t}$ 143

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(8)}(T_{41}',t)$ and $(a_i'')^{(8)}(T_{41},t)$ . $(T_{41}',t)$ and $(T_{41},t)$ are points belonging to the interval $\left[(\hat{k}_{40})^{(8)}, (\hat{M}_{40})^{(8)}\right]$. It is to be noted that $(a_i'')^{(8)}(T_{41},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{40})^{(8)} = 1$ then the function $(a_i'')^{(8)}(T_{41},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$ :

$(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}$, are positive constants

$\frac{(a_i)^{(8)}}{(\hat{M}_{40})^{(8)}}\ , \frac{(b_i)^{(8)}}{(\hat{M}_{40})^{(8)}} < 1$ 144

**Definition of** $(\hat{P}_{40})^{(8)}, (\hat{Q}_{40})^{(8)}$ :

There exists two constants $(\hat{P}_{40})^{(8)}$ and $(\hat{Q}_{40})^{(8)}$ which together with $(\hat{M}_{40})^{(8)}, (\hat{k}_{40})^{(8)}, (\hat{A}_{40})^{(8)}$ $(\hat{B}_{40})^{(8)}$ and the constants $(a_i)^{(8)}, (a_i')^{(8)}, (b_i)^{(8)}, (b_i')^{(8)}, (p_i)^{(8)},\ (r_i)^{(8)}, i = 40,41,42,$
Satisfy the inequalities

$$\frac{1}{(\hat{M}_{40})^{(8)}}[\,(a_i)^{(8)} + (a_i')^{(8)} + \ (\hat{A}_{40})^{(8)} + \ (\hat{P}_{40})^{(8)}\ (\hat{k}_{40})^{(8)}] < 1$$ 145

$$\frac{1}{(\hat{M}_{40})^{(8)}}[\ (b_i)^{(8)} + (b_i')^{(8)} + \ (\hat{B}_{40})^{(8)} + \ (\hat{Q}_{40})^{(8)}\ \ (\hat{k}_{40})^{(8)}] < 1$$ 146

Where we suppose

$(a_i)^{(9)}, (a_i')^{(9)}, (a_i'')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (b_i'')^{(9)} > 0, \quad i,j = 44,45,46$ 146 A

The functions $(a_i'')^{(9)}, (b_i'')^{(9)}$ are positive continuous increasing and bounded.

**Definition of** $(p_i)^{(9)},\ (r_i)^{(9)}$:

$(a_i'')^{(9)}(T_{45},t) \leq (p_i)^{(9)} \leq (\hat{A}_{44})^{(9)}$

$(b_i'')^{(9)}(G_{47},t) \leq \ (r_i)^{(9)} \leq (b_i')^{(9)} \leq (\hat{B}_{44})^{(9)}$

$lim_{T_2\to\infty}(a_i'')^{(9)}\,(T_{45},t) = (p_i)^{(9)}$

$\lim_{G\to\infty}(b_i'')^{(9)}\,(G_{47},t) = \ (r_i)^{(9)}$

**Definition of** $(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}$ :

Where $\boxed{(\hat{A}_{44})^{(9)}, (\hat{B}_{44})^{(9)}, (p_i)^{(9)},\ (r_i)^{(9)}}$ are positive constants and $\boxed{i = 44,45,46}$

They satisfy Lipschitz condition:

$$|(a_i'')^{(9)}(T_{45}',t) - (a_i'')^{(9)}(T_{45},t)| \le (\hat{k}_{44})^{(9)}|T_{45} - T_{45}'|e^{-(\hat{M}_{44})^{(9)}t}$$

$$|(b_i'')^{(9)}((G_{47})',t) - (b_i'')^{(9)}\big((G_{47}),t\big)| < (\hat{k}_{44})^{(9)}||(G_{47}) - (G_{47})'||e^{-(\hat{M}_{44})^{(9)}t}$$

With the Lipschitz condition, we place a restriction on the behavior of functions $(a_i'')^{(9)}(T_{45}',t)$ and$(a_i'')^{(9)}(T_{45},t)$ . $(T_{45}',t)$ and $(T_{45},t)$ are points belonging to the interval $\big[(\hat{k}_{44})^{(9)}, (\hat{M}_{44})^{(9)}\big]$. It is to be noted that $(a_i'')^{(9)}(T_{45},t)$ is uniformly continuous. In the eventuality of the fact, that if $(\hat{M}_{44})^{(9)} = 1$ then the function $(a_i'')^{(9)}(T_{45},t)$ , the **first augmentation coefficient** attributable to the system, would be absolutely continuous.

**Definition of** $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$ :

$(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}$, are positive constants

$$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}\ ,\frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$$

**Definition of** $(\hat{P}_{44})^{(9)}, (\hat{Q}_{44})^{(9)}$ :

There exists two constants $(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ which together with $(\hat{M}_{44})^{(9)}, (\hat{k}_{44})^{(9)}, (\hat{A}_{44})^{(9)} and\ (\hat{B}_{44})^{(9)}$ and the constants $(a_i)^{(9)}, (a_i')^{(9)}, (b_i)^{(9)}, (b_i')^{(9)}, (p_i)^{(9)},\ (r_i)^{(9)}, i = 44,45,46,$

satisfy the inequalities

$$\frac{1}{(\hat{M}_{44})^{(9)}}[\,(a_i)^{(9)} + (a_i')^{(9)} + \ (\hat{A}_{44})^{(9)} + \ (\hat{P}_{44})^{(9)}\,(\hat{k}_{44})^{(9)}] < 1$$

$$\frac{1}{(\hat{M}_{44})^{(9)}}[\ (b_i)^{(9)} + (b_i')^{(9)} + \ (\hat{B}_{44})^{(9)} + \ (\hat{Q}_{44})^{(9)}\ \ (\hat{k}_{44})^{(9)}] < 1$$

**Theorem 1:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 147

Definition of $G_i(0)\,, T_i(0)$ :

$G_i(t) \le \big(\hat{P}_{13}\big)^{(1)}e^{(\hat{M}_{13})^{(1)}t}$ , $\boxed{G_i(0) = G_i^0 > 0}$

$T_i(t) \le \ (\hat{Q}_{13})^{(1)}e^{(\hat{M}_{13})^{(1)}t}$ , $\boxed{T_i(0) = T_i^0 > 0}$

**Theorem 2 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 148

**Definition of** $G_i(0)\,, T_i(0)$

$G_i(t) \leq \; (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $G_i(0) = G_i^0 > 0$

$T_i(t) \leq \; (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)} t}$ , $T_i(0) = T_i^0 > 0$

**Theorem 3 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 149

$G_i(t) \leq \; (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $G_i(0) = G_i^0 > 0$

$T_i(t) \leq \; (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$ , $T_i(0) = T_i^0 > 0$

**Theorem 4 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 150

<u>Definition of</u> $G_i(0)$ , $T_i(0)$ :

$G_i(t) \leq \left(\hat{P}_{24}\right)^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$

$T_i(t) \leq \; (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$

**Theorem 5 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 151

<u>Definition of</u> $G_i(0)$ , $T_i(0)$ :

$G_i(t) \leq \left(\hat{P}_{28}\right)^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$

$T_i(t) \leq \; (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$

**Theorem 6 :** if the conditions above are fulfilled, there exists a solution satisfying the conditions 152

<u>Definition of</u> $G_i(0)$ , $T_i(0)$ :

$G_i(t) \leq \left(\hat{P}_{32}\right)^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$

$T_i(t) \leq \; (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$

**Theorem 7:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 153

<u>Definition of</u> $G_i(0)$ , $T_i(0)$ :

$G_i(t) \leq \left(\hat{P}_{36}\right)^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$
$T_i(t) \leq \; (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$

**Theorem 8:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 153 A

<u>Definition of</u> $G_i(0)$ , $T_i(0)$ :

$G_i(t) \leq \left(\hat{P}_{40}\right)^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{G_i(0) = G_i^0 > 0}$
$T_i(t) \leq \; (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$ , $\boxed{T_i(0) = T_i^0 > 0}$

**Theorem 9:** if the conditions above are fulfilled, there exists a solution satisfying the conditions 153 B

**Definition of** $G_i(0)\,,T_i(0)$ :

$$G_i(t) \leq \left(\hat{P}_{44}\right)^{(9)} e^{(\hat{M}_{44})^{(9)}t} \quad , \quad \boxed{G_i(0) = G_i^0 > 0}$$

$$T_i(t) \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)}t} \quad , \quad \boxed{T_i(0) = T_i^0 > 0}$$

**Proof:** Consider operator $\mathcal{A}^{(1)}$ defined on the space of sextuples of continuous functions $G_i\,,\,T_i : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy 154

$$G_i(0) = G_i^0\,,\; T_i(0) = T_i^0\,,\; G_i^0 \leq (\hat{P}_{13})^{(1)}\,, T_i^0 \leq (\hat{Q}_{13})^{(1)},$$ 155

$$0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)}t}$$ 156

$$0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)}t}$$ 157

By 158

$$\bar{G}_{13}(t) = G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} G_{14}(s_{(13)}) - \left((a'_{13})^{(1)} + a''_{13})^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{13}(s_{(13)})\right] ds_{(13)}$$

$$\bar{G}_{14}(t) = G_{14}^0 + \int_0^t \left[(a_{14})^{(1)} G_{13}(s_{(13)}) - \left((a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{14}(s_{(13)})\right] ds_{(13)}$$

$$\bar{G}_{15}(t) = G_{15}^0 + \int_0^t \left[(a_{15})^{(1)} G_{14}(s_{(13)}) - \left((a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14}(s_{(13)}), s_{(13)})\right) G_{15}(s_{(13)})\right] ds_{(13)}$$

$$\bar{T}_{13}(t) = T_{13}^0 + \int_0^t \left[(b_{13})^{(1)} T_{14}(s_{(13)}) - \left((b'_{13})^{(1)} - (b''_{13})^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{13}(s_{(13)})\right] ds_{(13)}$$

$$\bar{T}_{14}(t) = T_{14}^0 + \int_0^t \left[(b_{14})^{(1)} T_{13}(s_{(13)}) - \left((b'_{14})^{(1)} - (b''_{14})^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{14}(s_{(13)})\right] ds_{(13)}$$

$$\bar{\mathrm{T}}_{15}(\mathrm{t}) = \mathrm{T}_{15}^0 + \int_0^t \left[(b_{15})^{(1)} T_{14}(s_{(13)}) - \left((b'_{15})^{(1)} - (b''_{15})^{(1)}(G(s_{(13)}), s_{(13)})\right) T_{15}(s_{(13)})\right] ds_{(13)}$$

Where $s_{(13)}$ is the integrand that is integrated over an interval $(0, t)$

**Proof:** 159

Consider operator $\mathcal{A}^{(2)}$ defined on the space of sextuples of continuous functions $G_i\,,\,T_i : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy

$$G_i(0) = G_i^0\,,\; T_i(0) = T_i^0\,,\; G_i^0 \leq (\hat{P}_{16})^{(2)}\,, T_i^0 \leq (\hat{Q}_{16})^{(2)},$$

$$0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)}t}$$

$$0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{16})^{(2)} e^{(\hat{M}_{16})^{(2)}t}$$

By 160

$$\bar{G}_{16}(t) = G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} G_{17}(s_{(16)}) - \left((a'_{16})^{(2)} + a''_{16})^{(2)}(T_{17}(s_{(16)}), s_{(16)})\right) G_{16}(s_{(16)})\right] ds_{(16)}$$

$$\bar{G}_{17}(t) = G_{17}^0 + \int_0^t \left[(a_{17})^{(2)} G_{16}\left(s_{(16)}\right) - \left((a'_{17})^{(2)} + (a''_{17})^{(2)}\left(T_{17}\left(s_{(16)}\right), s_{(17)}\right)\right) G_{17}\left(s_{(16)}\right)\right] ds_{(16)}$$

$$\bar{G}_{18}(t) = G_{18}^0 + \int_0^t \left[(a_{18})^{(2)} G_{17}\left(s_{(16)}\right) - \left((a'_{18})^{(2)} + (a''_{18})^{(2)}\left(T_{17}\left(s_{(16)}\right), s_{(16)}\right)\right) G_{18}\left(s_{(16)}\right)\right] ds_{(16)}$$

$$\bar{T}_{16}(t) = T_{16}^0 + \int_0^t \left[(b_{16})^{(2)} T_{17}\left(s_{(16)}\right) - \left((b'_{16})^{(2)} - (b''_{16})^{(2)}\left(G_{19}\left(s_{(16)}\right), s_{(16)}\right)\right) T_{16}\left(s_{(16)}\right)\right] ds_{(16)}$$

$$\bar{T}_{17}(t) = T_{17}^0 + \int_0^t \left[(b_{17})^{(2)} T_{16}\left(s_{(16)}\right) - \left((b'_{17})^{(2)} - (b''_{17})^{(2)}\left(G_{19}\left(s_{(16)}\right), s_{(16)}\right)\right) T_{17}\left(s_{(16)}\right)\right] ds_{(16)}$$

$$\bar{T}_{18}(t) = T_{18}^0 + \int_0^t \left[(b_{18})^{(2)} T_{17}\left(s_{(16)}\right) - \left((b'_{18})^{(2)} - (b''_{18})^{(2)}\left(G_{19}\left(s_{(16)}\right), s_{(16)}\right)\right) T_{18}\left(s_{(16)}\right)\right] ds_{(16)}$$

Where $s_{(16)}$ is the integrand that is integrated over an interval $(0, t)$

**Proof:**

Consider operator $\mathcal{A}^{(3)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy

$G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \leq (\hat{P}_{20})^{(3)}$, $T_i^0 \leq (\hat{Q}_{20})^{(3)}$,

$0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$

$0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} t}$

By 161

$$\bar{G}_{20}(t) = G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} G_{21}\left(s_{(20)}\right) - \left((a'_{20})^{(3)} + a''_{20})^{(3)}\left(T_{21}\left(s_{(20)}\right), s_{(20)}\right)\right) G_{20}\left(s_{(20)}\right)\right] ds_{(20)}$$

$$\bar{G}_{21}(t) = G_{21}^0 + \int_0^t \left[(a_{21})^{(3)} G_{20}\left(s_{(20)}\right) - \left((a'_{21})^{(3)} + (a''_{21})^{(3)}\left(T_{21}\left(s_{(20)}\right), s_{(20)}\right)\right) G_{21}\left(s_{(20)}\right)\right] ds_{(20)}$$

$$\bar{G}_{22}(t) = G_{22}^0 + \int_0^t \left[(a_{22})^{(3)} G_{21}\left(s_{(20)}\right) - \left((a'_{22})^{(3)} + (a''_{22})^{(3)}\left(T_{21}\left(s_{(20)}\right), s_{(20)}\right)\right) G_{22}\left(s_{(20)}\right)\right] ds_{(20)}$$

$$\bar{T}_{20}(t) = T_{20}^0 + \int_0^t \left[(b_{20})^{(3)} T_{21}\left(s_{(20)}\right) - \left((b'_{20})^{(3)} - (b''_{20})^{(3)}\left(G_{23}\left(s_{(20)}\right), s_{(20)}\right)\right) T_{20}\left(s_{(20)}\right)\right] ds_{(20)}$$

$$\bar{T}_{21}(t) = T_{21}^0 + \int_0^t \left[(b_{21})^{(3)} T_{20}\left(s_{(20)}\right) - \left((b'_{21})^{(3)} - (b''_{21})^{(3)}\left(G_{23}\left(s_{(20)}\right), s_{(20)}\right)\right) T_{21}\left(s_{(20)}\right)\right] ds_{(20)}$$

$$\bar{\mathrm{T}}_{22}(\mathrm{t}) = \mathrm{T}_{22}^0 + \int_0^t \left[(b_{22})^{(3)} T_{21}\left(s_{(20)}\right) - \left((b'_{22})^{(3)} - (b''_{22})^{(3)}\left(G_{23}\left(s_{(20)}\right), s_{(20)}\right)\right) T_{22}\left(s_{(20)}\right)\right] ds_{(20)}$$

Where $s_{(20)}$ is the integrand that is integrated over an interval $(0, t)$

**Proof:** Consider operator $\mathcal{A}^{(4)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy

$G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \leq (\hat{P}_{24})^{(4)}$, $T_i^0 \leq (\hat{Q}_{24})^{(4)}$,

$0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$

$0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{24})^{(4)} e^{(\hat{M}_{24})^{(4)} t}$

By 162

$$\bar{G}_{24}(t) = G_{24}^0 + \int_0^t \left[(a_{24})^{(4)} G_{25}(s_{(24)}) - \left((a'_{24})^{(4)} + a''_{24})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{24}(s_{(24)})\right] ds_{(24)}$$

$$\bar{G}_{25}(t) = G_{25}^0 + \int_0^t \left[(a_{25})^{(4)} G_{24}(s_{(24)}) - \left((a'_{25})^{(4)} + (a''_{25})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{25}(s_{(24)})\right] ds_{(24)}$$

$$\bar{G}_{26}(t) = G_{26}^0 + \int_0^t \left[(a_{26})^{(4)} G_{25}(s_{(24)}) - \left((a'_{26})^{(4)} + (a''_{26})^{(4)}(T_{25}(s_{(24)}), s_{(24)})\right) G_{26}(s_{(24)})\right] ds_{(24)}$$

$$\bar{T}_{24}(t) = T_{24}^0 + \int_0^t \left[(b_{24})^{(4)} T_{25}(s_{(24)}) - \left((b'_{24})^{(4)} - (b''_{24})^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{24}(s_{(24)})\right] ds_{(24)}$$

$$\bar{T}_{25}(t) = T_{25}^0 + \int_0^t \left[(b_{25})^{(4)} T_{24}(s_{(24)}) - \left((b'_{25})^{(4)} - (b''_{25})^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{25}(s_{(24)})\right] ds_{(24)}$$

$$\bar{T}_{26}(t) = T_{26}^0 + \int_0^t \left[(b_{26})^{(4)} T_{25}(s_{(24)}) - \left((b'_{26})^{(4)} - (b''_{26})^{(4)}(G_{27}(s_{(24)}), s_{(24)})\right) T_{26}(s_{(24)})\right] ds_{(24)}$$

Where $s_{(24)}$ is the integrand that is integrated over an interval $(0, t)$

**Proof:** Consider operator $\mathcal{A}^{(5)}$ defined on the space of sextuples of continuous functions $G_i, T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy

$$G_i(0) = G_i^0,\ T_i(0) = T_i^0,\ G_i^0 \le (\hat{P}_{28})^{(5)},\ T_i^0 \le (\hat{Q}_{28})^{(5)},$$

$$0 \le G_i(t) - G_i^0 \le (\hat{P}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$$

$$0 \le T_i(t) - T_i^0 \le (\hat{Q}_{28})^{(5)} e^{(\hat{M}_{28})^{(5)} t}$$

By 163

$$\bar{G}_{28}(t) = G_{28}^0 + \int_0^t \left[(a_{28})^{(5)} G_{29}(s_{(28)}) - \left((a'_{28})^{(5)} + a''_{28})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{28}(s_{(28)})\right] ds_{(28)}$$

$$\bar{G}_{29}(t) = G_{29}^0 + \int_0^t \left[(a_{29})^{(5)} G_{28}(s_{(28)}) - \left((a'_{29})^{(5)} + (a''_{29})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{29}(s_{(28)})\right] ds_{(28)}$$

$$\bar{G}_{30}(t) = G_{30}^0 + \int_0^t \left[(a_{30})^{(5)} G_{29}(s_{(28)}) - \left((a'_{30})^{(5)} + (a''_{30})^{(5)}(T_{29}(s_{(28)}), s_{(28)})\right) G_{30}(s_{(28)})\right] ds_{(28)}$$

$$\bar{T}_{28}(t) = T_{28}^0 + \int_0^t \left[(b_{28})^{(5)} T_{29}(s_{(28)}) - \left((b'_{28})^{(5)} - (b''_{28})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{28}(s_{(28)})\right] ds_{(28)}$$

$$\bar{T}_{29}(t) = T_{29}^0 + \int_0^t \left[(b_{29})^{(5)} T_{28}(s_{(28)}) - \left((b'_{29})^{(5)} - (b''_{29})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{29}(s_{(28)})\right] ds_{(28)}$$

$$\bar{T}_{30}(t) = T_{30}^0 + \int_0^t \left[(b_{30})^{(5)} T_{29}(s_{(28)}) - \left((b'_{30})^{(5)} - (b''_{30})^{(5)}(G_{31}(s_{(28)}), s_{(28)})\right) T_{30}(s_{(28)})\right] ds_{(28)}$$

Where $s_{(28)}$ is the integrand that is integrated over an interval $(0, t)$

**Proof:**

Consider operator $\mathcal{A}^{(6)}$ defined on the space of sextuples of continuous functions $G_i, T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy

$$G_i(0) = G_i^0,\ T_i(0) = T_i^0,\ G_i^0 \le (\hat{P}_{32})^{(6)},\ T_i^0 \le (\hat{Q}_{32})^{(6)},$$

$0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$

$0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{32})^{(6)} e^{(\hat{M}_{32})^{(6)} t}$

By 164

$$\bar{G}_{32}(t) = G_{32}^0 + \int_0^t \left[ (a_{32})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{32})^{(6)} + a''_{32})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{32}(s_{(32)}) \right] ds_{(32)}$$

$$\bar{G}_{33}(t) = G_{33}^0 + \int_0^t \left[ (a_{33})^{(6)} G_{32}(s_{(32)}) - \left( (a'_{33})^{(6)} + (a''_{33})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{33}(s_{(32)}) \right] ds_{(32)}$$

$$\bar{G}_{34}(t) = G_{34}^0 + \int_0^t \left[ (a_{34})^{(6)} G_{33}(s_{(32)}) - \left( (a'_{34})^{(6)} + (a''_{34})^{(6)} (T_{33}(s_{(32)}), s_{(32)}) \right) G_{34}(s_{(32)}) \right] ds_{(32)}$$

$$\bar{T}_{32}(t) = T_{32}^0 + \int_0^t \left[ (b_{32})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{32})^{(6)} - (b''_{32})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{32}(s_{(32)}) \right] ds_{(32)}$$

$$\bar{T}_{33}(t) = T_{33}^0 + \int_0^t \left[ (b_{33})^{(6)} T_{32}(s_{(32)}) - \left( (b'_{33})^{(6)} - (b''_{33})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{33}(s_{(32)}) \right] ds_{(32)}$$

$$\bar{\mathrm{T}}_{34}(\mathrm{t}) = \mathrm{T}_{34}^0 + \int_0^t \left[ (b_{34})^{(6)} T_{33}(s_{(32)}) - \left( (b'_{34})^{(6)} - (b''_{34})^{(6)} (G_{35}(s_{(32)}), s_{(32)}) \right) T_{34}(s_{(32)}) \right] ds_{(32)}$$

Where $s_{(32)}$ is the integrand that is integrated over an interval $(0, t)$

**Proof:**
Consider operator $\mathcal{A}^{(7)}$ defined on the space of sextuples of continuous functions $G_i$, $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy
$G_i(0) = G_i^0$, $T_i(0) = T_i^0$, $G_i^0 \leq (\hat{P}_{36})^{(7)}$, $T_i^0 \leq (\hat{Q}_{36})^{(7)}$,

$0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$

$0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} t}$

By 165

$$\bar{G}_{36}(t) = G_{36}^0 + \int_0^t \left[ (a_{36})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{36})^{(7)} + a''_{36})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{36}(s_{(36)}) \right] ds_{(36)}$$

$$\bar{G}_{37}(t) = G_{37}^0 + \int_0^t \left[ (a_{37})^{(7)} G_{36}(s_{(36)}) - \left( (a'_{37})^{(7)} + (a''_{37})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{37}(s_{(36)}) \right] ds_{(36)}$$

$$\bar{G}_{38}(t) = G_{38}^0 + \int_0^t \left[ (a_{38})^{(7)} G_{37}(s_{(36)}) - \left( (a'_{38})^{(7)} + (a''_{38})^{(7)} (T_{37}(s_{(36)}), s_{(36)}) \right) G_{38}(s_{(36)}) \right] ds_{(36)}$$

$$\bar{T}_{36}(t) = T_{36}^0 + \int_0^t \left[ (b_{36})^{(7)} T_{37}(s_{(36)}) - \left( (b'_{36})^{(7)} - (b''_{36})^{(7)} (G_{39}(s_{(36)}), s_{(36)}) \right) T_{36}(s_{(36)}) \right] ds_{(36)}$$

$$\bar{T}_{37}(t) = T_{37}^0 + \int_0^t \left[ (b_{37})^{(7)} T_{36}(s_{(36)}) - \left( (b'_{37})^{(7)} - (b''_{37})^{(7)} (G_{39}(s_{(36)}), s_{(36)}) \right) T_{37}(s_{(36)}) \right] ds_{(36)}$$

$$\bar{\mathrm{T}}_{38}(\mathrm{t}) = \mathrm{T}_{38}^0 + \int_0^t \left[ (b_{38})^{(7)} T_{37}(s_{(36)}) - \left( (b'_{38})^{(7)} - (b''_{38})^{(7)} (G_{39}(s_{(36)}), s_{(36)}) \right) T_{38}(s_{(36)}) \right] ds_{(36)}$$

Where $s_{(36)}$ is the integrand that is integrated over an interval $(0, t)$

**Proof:**

Consider operator $\mathcal{A}^{(8)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy

$G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{40})^{(8)}$ , $T_i^0 \leq (\hat{Q}_{40})^{(8)}$,

$0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$

$0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} t}$

By 166

$$\bar{G}_{40}(t) = G_{40}^0 + \int_0^t \left[(a_{40})^{(8)} G_{41}(s_{(40)}) - \left((a'_{40})^{(8)} + a''_{40})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{40}(s_{(40)})\right] ds_{(40)}$$

$$\bar{G}_{41}(t) = G_{41}^0 + \int_0^t \left[(a_{41})^{(8)} G_{40}(s_{(40)}) - \left((a'_{41})^{(8)} + (a''_{41})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{41}(s_{(40)})\right] ds_{(40)}$$

$$\bar{G}_{42}(t) = G_{42}^0 + \int_0^t \left[(a_{42})^{(8)} G_{41}(s_{(40)}) - \left((a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41}(s_{(40)}), s_{(40)})\right) G_{42}(s_{(40)})\right] ds_{(40)}$$

$$\bar{T}_{40}(t) = T_{40}^0 + \int_0^t \left[(b_{40})^{(8)} T_{41}(s_{(40)}) - \left((b'_{40})^{(8)} - (b''_{40})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{40}(s_{(40)})\right] ds_{(40)}$$

$$\bar{T}_{41}(t) = T_{41}^0 + \int_0^t \left[(b_{41})^{(8)} T_{40}(s_{(40)}) - \left((b'_{41})^{(8)} - (b''_{41})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{41}(s_{(40)})\right] ds_{(40)}$$

$$\bar{\mathrm{T}}_{42}(\mathrm{t}) = \mathrm{T}_{42}^0 + \int_0^t \left[(b_{42})^{(8)} T_{41}(s_{(40)}) - \left((b'_{42})^{(8)} - (b''_{42})^{(8)}(G_{43}(s_{(40)}), s_{(40)})\right) T_{42}(s_{(40)})\right] ds_{(40)}$$

Where $s_{(40)}$ is the integrand that is integrated over an interval $(0, t)$

**Proof:** 166A

Consider operator $\mathcal{A}^{(9)}$ defined on the space of sextuples of continuous functions $G_i$ , $T_i: \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy

$G_i(0) = G_i^0$ , $T_i(0) = T_i^0$ , $G_i^0 \leq (\hat{P}_{44})^{(9)}$ , $T_i^0 \leq (\hat{Q}_{44})^{(9)}$,

$0 \leq G_i(t) - G_i^0 \leq (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$

$0 \leq T_i(t) - T_i^0 \leq (\hat{Q}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} t}$

By

$$\bar{G}_{44}(t) = G_{44}^0 + \int_0^t \left[(a_{44})^{(9)} G_{45}(s_{(44)}) - \left((a'_{44})^{(9)} + a''_{44})^{(9)}(T_{45}(s_{(44)}), s_{(44)})\right) G_{44}(s_{(44)})\right] ds_{(44)}$$

$$\bar{G}_{45}(t) = G_{45}^0 + \int_0^t \left[(a_{45})^{(9)} G_{44}(s_{(44)}) - \left((a'_{45})^{(9)} + (a''_{45})^{(9)}\left(T_{45}(s_{(44)}), s_{(44)}\right)\right) G_{45}(s_{(44)})\right] ds_{(44)}$$

$$\bar{G}_{46}(t) = G_{46}^0 + \int_0^t \left[(a_{46})^{(9)} G_{45}(s_{(44)}) - \left((a'_{46})^{(9)} + (a''_{46})^{(9)}\left(T_{45}(s_{(44)}), s_{(44)}\right)\right) G_{46}(s_{(44)})\right] ds_{(44)}$$

$$\bar{T}_{44}(t) = T_{44}^0 + \int_0^t \left[(b_{44})^{(9)} T_{45}(s_{(44)}) - \left((b'_{44})^{(9)} - (b''_{44})^{(9)}\left(G_{47}(s_{(44)}), s_{(44)}\right)\right) T_{44}(s_{(44)})\right] ds_{(44)}$$

$$\bar{T}_{45}(t) = T_{45}^0 + \int_0^t \left[(b_{45})^{(9)} T_{44}(s_{(44)}) - \left((b'_{45})^{(9)} - (b''_{45})^{(9)}\left(G_{47}(s_{(44)}), s_{(44)}\right)\right) T_{45}(s_{(44)})\right] ds_{(44)}$$

$$\bar{\mathrm{T}}_{46}(\mathrm{t}) = \mathrm{T}_{46}^0 + \int_0^t \left[(b_{46})^{(9)} T_{45}(s_{(44)}) - \left((b'_{46})^{(9)} - (b''_{46})^{(9)}\left(G_{47}(s_{(44)}), s_{(44)}\right)\right) T_{46}(s_{(44)})\right] ds_{(44)}$$

Where $s_{(44)}$ is the integrand that is integrated over an interval $(0, t)$

The operator $\mathcal{A}^{(1)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that 167

$$G_{13}(t) \leq G_{13}^0 + \int_0^t \left[(a_{13})^{(1)} \left(G_{14}^0 + (\hat{P}_{13})^{(1)} e^{(\hat{M}_{13})^{(1)} s_{(13)}}\right)\right] ds_{(13)} =$$

$$\left(1 + (a_{13})^{(1)} t\right) G_{14}^0 + \frac{(a_{13})^{(1)} (\hat{P}_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left(e^{(\hat{M}_{13})^{(1)} t} - 1\right)$$

From which it follows that 168

$$(G_{13}(t) - G_{13}^0) e^{-(\hat{M}_{13})^{(1)} t} \leq \frac{(a_{13})^{(1)}}{(\hat{M}_{13})^{(1)}} \left[\left((\hat{P}_{13})^{(1)} + G_{14}^0\right) e^{\left(-\frac{(\hat{P}_{13})^{(1)} + G_{14}^0}{G_{14}^0}\right)} + (\hat{P}_{13})^{(1)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 1

Analogous inequalities hold also for $G_{14}$ , $G_{15}, T_{13}, T_{14}, T_{15}$

The operator $\mathcal{A}^{(2)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{16}(t) \leq G_{16}^0 + \int_0^t \left[(a_{16})^{(2)} \left(G_{17}^0 + (\hat{P}_{16})^{(6)} e^{(\hat{M}_{16})^{(2)} s_{(16)}}\right)\right] ds_{(16)} =$$ 169

$$\left(1 + (a_{16})^{(2)} t\right) G_{17}^0 + \frac{(a_{16})^{(2)} (\hat{P}_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left(e^{(\hat{M}_{16})^{(2)} t} - 1\right)$$

From which it follows that 170

$$(G_{16}(t) - G_{16}^0) e^{-(\hat{M}_{16})^{(2)} t} \leq \frac{(a_{16})^{(2)}}{(\hat{M}_{16})^{(2)}} \left[\left((\hat{P}_{16})^{(2)} + G_{17}^0\right) e^{\left(-\frac{(\hat{P}_{16})^{(2)} + G_{17}^0}{G_{17}^0}\right)} + (\hat{P}_{16})^{(2)}\right]$$

Analogous inequalities hold also for $G_{17}$ , $G_{18}, T_{16}, T_{17}, T_{18}$

The operator $\mathcal{A}^{(3)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that 171

$$G_{20}(t) \leq G_{20}^0 + \int_0^t \left[(a_{20})^{(3)} \left(G_{21}^0 + (\hat{P}_{20})^{(3)} e^{(\hat{M}_{20})^{(3)} s_{(20)}}\right)\right] ds_{(20)} =$$

$$\left(1 + (a_{20})^{(3)} t\right) G_{21}^0 + \frac{(a_{20})^{(3)} (\hat{P}_{20})^{(3)}}{(\hat{M}_{20})^{(3)}} \left(e^{(\hat{M}_{20})^{(3)} t} - 1\right)$$

From which it follows that 172

$$(G_{20}(t) - G_{20}^0)e^{-(\hat{M}_{20})^{(3)}t} \leq \frac{(a_{20})^{(3)}}{(\hat{M}_{20})^{(3)}}\left[\left((\hat{P}_{20})^{(3)} + G_{21}^0\right)e^{\left(-\frac{(\hat{P}_{20})^{(3)}+G_{21}^0}{G_{21}^0}\right)} + (\hat{P}_{20})^{(3)}\right]$$

Analogous inequalities hold also for $G_{21}$ , $G_{22}, T_{20}, T_{21}, T_{22}$

The operator $\mathcal{A}^{(4)}$ maps the space of functions satisfying into itself .Indeed it is obvious that 173

$$G_{24}(t) \leq G_{24}^0 + \int_0^t \left[(a_{24})^{(4)}\left(G_{25}^0 + (\hat{P}_{24})^{(4)}e^{(\hat{M}_{24})^{(4)}s_{(24)}}\right)\right] ds_{(24)} =$$

$$\left(1 + (a_{24})^{(4)}t\right)G_{25}^0 + \frac{(a_{24})^{(4)}(\hat{P}_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left(e^{(\hat{M}_{24})^{(4)}t} - 1\right)$$

From which it follows that 174

$$(G_{24}(t) - G_{24}^0)e^{-(\hat{M}_{24})^{(4)}t} \leq \frac{(a_{24})^{(4)}}{(\hat{M}_{24})^{(4)}}\left[\left((\hat{P}_{24})^{(4)} + G_{25}^0\right)e^{\left(-\frac{(\hat{P}_{24})^{(4)}+G_{25}^0}{G_{25}^0}\right)} + (\hat{P}_{24})^{(4)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 4

The operator $\mathcal{A}^{(5)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{28}(t) \leq G_{28}^0 + \int_0^t \left[(a_{28})^{(5)}\left(G_{29}^0 + (\hat{P}_{28})^{(5)}e^{(\hat{M}_{28})^{(5)}s_{(28)}}\right)\right] ds_{(28)} =$$

$$\left(1 + (a_{28})^{(5)}t\right)G_{29}^0 + \frac{(a_{28})^{(5)}(\hat{P}_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left(e^{(\hat{M}_{28})^{(5)}t} - 1\right)$$

From which it follows that 175

$$(G_{28}(t) - G_{28}^0)e^{-(\hat{M}_{28})^{(5)}t} \leq \frac{(a_{28})^{(5)}}{(\hat{M}_{28})^{(5)}}\left[\left((\hat{P}_{28})^{(5)} + G_{29}^0\right)e^{\left(-\frac{(\hat{P}_{28})^{(5)}+G_{29}^0}{G_{29}^0}\right)} + (\hat{P}_{28})^{(5)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 5

The operator $\mathcal{A}^{(6)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that 176

$$G_{32}(t) \leq G_{32}^0 + \int_0^t \left[(a_{32})^{(6)}\left(G_{33}^0 + (\hat{P}_{32})^{(6)}e^{(\hat{M}_{32})^{(6)}s_{(32)}}\right)\right] ds_{(32)} =$$

$$\left(1 + (a_{32})^{(6)}t\right)G_{33}^0 + \frac{(a_{32})^{(6)}(\hat{P}_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left(e^{(\hat{M}_{32})^{(6)}t} - 1\right)$$

From which it follows that 177

$$(G_{32}(t) - G_{32}^0)e^{-(\hat{M}_{32})^{(6)}t} \leq \frac{(a_{32})^{(6)}}{(\hat{M}_{32})^{(6)}}\left[\left((\hat{P}_{32})^{(6)} + G_{33}^0\right)e^{\left(-\frac{(\hat{P}_{32})^{(6)}+G_{33}^0}{G_{33}^0}\right)} + (\hat{P}_{32})^{(6)}\right]$$

$(G_i^0)$ is as defined in the statement of theorem 6

Analogous inequalities hold also for $G_{25}$ , $G_{26}, T_{24}, T_{25}, T_{26}$

The operator $\mathcal{A}^{(7)}$ maps the space of functions satisfying Equations into itself . Indeed it is obvious that 178

$$G_{36}(t) \leq G_{36}^0 + \int_0^t \left[ (a_{36})^{(7)} \left( G_{37}^0 + (\hat{P}_{36})^{(7)} e^{(\hat{M}_{36})^{(7)} s_{(36)}} \right) \right] ds_{(36)} =$$
$$\left(1 + (a_{36})^{(7)} t\right) G_{37}^0 + \frac{(a_{36})^{(7)} (\hat{P}_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left( e^{(\hat{M}_{36})^{(7)} t} - 1 \right)$$

From which it follows that

$$(G_{36}(t) - G_{36}^0) e^{-(\hat{M}_{36})^{(7)} t} \leq \frac{(a_{36})^{(7)}}{(\hat{M}_{36})^{(7)}} \left[ \left( (\hat{P}_{36})^{(7)} + G_{37}^0 \right) e^{\left( -\frac{(\hat{P}_{36})^{(7)} + G_{37}^0}{G_{37}^0} \right)} + (\hat{P}_{36})^{(7)} \right]$$

$(G_i^0)$ is as defined in the statement of theorem 7

179

The operator $\mathcal{A}^{(8)}$ maps the space of functions satisfying Equations into itself .Indeed it is obvious that

$$G_{40}(t) \leq G_{40}^0 + \int_0^t \left[ (a_{40})^{(8)} \left( G_{41}^0 + (\hat{P}_{40})^{(8)} e^{(\hat{M}_{40})^{(8)} s_{(40)}} \right) \right] ds_{(40)} =$$ 180
$$\left(1 + (a_{40})^{(8)} t\right) G_{41}^0 + \frac{(a_{40})^{(8)} (\hat{P}_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left( e^{(\hat{M}_{40})^{(8)} t} - 1 \right)$$

From which it follows that 181

$$(G_{40}(t) - G_{40}^0) e^{-(\hat{M}_{40})^{(8)} t} \leq \frac{(a_{40})^{(8)}}{(\hat{M}_{40})^{(8)}} \left[ \left( (\hat{P}_{40})^{(8)} + G_{41}^0 \right) e^{\left( -\frac{(\hat{P}_{40})^{(8)} + G_{41}^0}{G_{41}^0} \right)} + (\hat{P}_{40})^{(8)} \right]$$

$(G_i^0)$ is as defined in the statement of theorem 8

Analogous inequalities hold also for $G_{41}$ , $G_{42}, T_{40}, T_{41}, T_{42}$

The operator $\mathcal{A}^{(9)}$ maps the space of functions satisfying 34,35,36 into itself .Indeed it is obvious that

$$G_{44}(t) \leq G_{44}^0 + \int_0^t \left[ (a_{44})^{(9)} \left( G_{45}^0 + (\hat{P}_{44})^{(9)} e^{(\hat{M}_{44})^{(9)} s_{(44)}} \right) \right] ds_{(44)} =$$

$$\left(1 + (a_{44})^{(9)} t\right) G_{45}^0 + \frac{(a_{44})^{(9)} (\hat{P}_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left( e^{(\hat{M}_{44})^{(9)} t} - 1 \right)$$

From which it follows that

$$(G_{44}(t) - G_{44}^0) e^{-(\hat{M}_{44})^{(9)} t} \leq \frac{(a_{44})^{(9)}}{(\hat{M}_{44})^{(9)}} \left[ \left( (\hat{P}_{44})^{(9)} + G_{45}^0 \right) e^{\left( -\frac{(\hat{P}_{44})^{(9)} + G_{45}^0}{G_{45}^0} \right)} + (\hat{P}_{44})^{(9)} \right]$$

$(G_i^0)$ is as defined in the statement of theorem 9

Analogous inequalities hold also for $G_{45}$ , $G_{46}, T_{44}, T_{45}, T_{46}$

It is now sufficient to take $\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}$ , $\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}} < 1$ and to choose 182

$(\hat{P}_{13})^{(1)}$ and $(\hat{Q}_{13})^{(1)}$ large to have

$$\frac{(a_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[(\hat{P}_{13})^{(1)}+\left((\hat{P}_{13})^{(1)}+G_j^0\right)e^{-\left(\frac{(\hat{P}_{13})^{(1)}+G_j^0}{G_j^0}\right)}\right]\leq(\hat{P}_{13})^{(1)}$$ 183

$$\frac{(b_i)^{(1)}}{(\hat{M}_{13})^{(1)}}\left[\left((\hat{Q}_{13})^{(1)}+T_j^0\right)e^{-\left(\frac{(\hat{Q}_{13})^{(1)}+T_j^0}{T_j^0}\right)}+(\hat{Q}_{13})^{(1)}\right]\leq(\hat{Q}_{13})^{(1)}$$ 184

In order that the operator $\mathcal{A}^{(1)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(1)}$ is a contraction with respect to the metric 185

$$d\left(\left(G^{(1)},T^{(1)}\right),\left(G^{(2)},T^{(2)}\right)\right)=$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\hat{M}_{13})^{(1)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\hat{M}_{13})^{(1)}t}\}$$

Indeed if we denote

**Definition of** $\tilde{G},\tilde{T}$ : $(\tilde{G},\tilde{T})=\mathcal{A}^{(1)}(G,T)$

It results

$$\left|\tilde{G}_{13}^{(1)}-\tilde{G}_i^{(2)}\right|\leq\int_0^t(a_{13})^{(1)}\left|G_{14}^{(1)}-G_{14}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\,ds_{(13)}+$$

$$\int_0^t\{(a'_{13})^{(1)}\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}+$$

$$(a''_{13})^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)\left|G_{13}^{(1)}-G_{13}^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}+$$

$$G_{13}^{(2)}|(a''_{13})^{(1)}\left(T_{14}^{(1)},s_{(13)}\right)-(a''_{13})^{(1)}\left(T_{14}^{(2)},s_{(13)}\right)|\;e^{-(\widehat{M}_{13})^{(1)}s_{(13)}}e^{(\widehat{M}_{13})^{(1)}s_{(13)}}\}ds_{(13)}$$

Where $s_{(13)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$

From the hypotheses it follows

$$\left|G^{(1)}-G^{(2)}\right|e^{-(\widehat{M}_{13})^{(1)}t}\leq$$ 186

$$\frac{1}{(\widehat{M}_{13})^{(1)}}\left((a_{13})^{(1)}+\;(a'_{13})^{(1)}+(\widehat{A}_{13})^{(1)}+(\widehat{P}_{13})^{(1)}(\widehat{k}_{13})^{(1)}\right)d\left(\left(G^{(1)},T^{(1)};\;G^{(2)},T^{(2)}\right)\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

**Remark 1:** The fact that we supposed $(a''_{13})^{(1)}$ and $(b''_{13})^{(1)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ *and* $(\widehat{Q}_{13})^{(1)}e^{(\widehat{M}_{13})^{(1)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(1)}$ and $(b''_i)^{(1)}, i=13{,}14{,}15$ depend only on $\mathrm{T}_{14}$ and respectively on $G(and\ not\ on\ \ t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 2:** There does not exist any $t$ where $G_i\,(t)=0$ *and* $T_i\,(t)=0$

From 19 to 24 it results

$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(1)} - (a_i'')^{(1)}(T_{14}(s_{(13)}), s_{(13)})\} ds_{(13)}\right]} \geq 0$

$T_i\,(t) \geq T_i^0 e^{(-(b_i')^{(1)} t)} > 0$ for t > 0

**Definition of** $\left((\widehat{M}_{13})^{(1)}\right)_1, \left((\widehat{M}_{13})^{(1)}\right)_2$ *and* $\left((\widehat{M}_{13})^{(1)}\right)_3$ : 187

**Remark 3:** if $G_{13}$ is bounded, the same property have also $G_{14}$ *and* $G_{15}$ . indeed if

$G_{13} < (\widehat{M}_{13})^{(1)}$ it follows $\frac{dG_{14}}{dt} \leq \left((\widehat{M}_{13})^{(1)}\right)_1 - (a_{14}')^{(1)} G_{14}$ and by integrating

$G_{14} \leq \left((\widehat{M}_{13})^{(1)}\right)_2 = G_{14}^0 + 2(a_{14})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_1 / (a_{14}')^{(1)}$

In the same way , one can obtain

$G_{15} \leq \left((\widehat{M}_{13})^{(1)}\right)_3 = G_{15}^0 + 2(a_{15})^{(1)}\left((\widehat{M}_{13})^{(1)}\right)_2 / (a_{15}')^{(1)}$

If $G_{14}$ *or* $G_{15}$ is bounded, the same property follows for $G_{13}$ , $G_{15}$ and $G_{13}$ , $G_{14}$ respectively.

**Remark 4:** If $G_{13}$ *is* bounded, from below, the same property holds for $G_{14}$ *and* $G_{15}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{14}$ is bounded from below. 188

**Remark 5:** If $T_{13}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(1)}\,(G(t),t)) = (b_{14}')^{(1)}$ then $T_{14} \to \infty$. 189

**Definition of** $(m)^{(1)}$ and $\varepsilon_1$ :

Indeed let $t_1$ be so that for $t > t_1$

$(b_{14})^{(1)} - (b_i'')^{(1)}(G(t),t) < \varepsilon_1, T_{13}\,(t) > (m)^{(1)}$

Then $\frac{dT_{14}}{dt} \geq (a_{14})^{(1)}(m)^{(1)} - \varepsilon_1 T_{14}$ which leads to

$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{\varepsilon_1}\right)(1 - e^{-\varepsilon_1 t}) + T_{14}^0 e^{-\varepsilon_1 t}$ If we take t such that $e^{-\varepsilon_1 t} = \frac{1}{2}$ it results

$T_{14} \geq \left(\frac{(a_{14})^{(1)}(m)^{(1)}}{2}\right), \quad t = log\frac{2}{\varepsilon_1}$ By taking now $\varepsilon_1$ sufficiently small one sees that $T_{14}$ is unbounded. The same property holds for $T_{15}$ if $\lim_{t\to\infty}(b_{15}'')^{(1)}\,(G(t),t) = (b_{15}')^{(1)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations

It is now sufficient to take $\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}$ , $\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}} < 1$ and to choose 190

$(\hat{P}_{16})^{(2)}$ *and* $(\hat{Q}_{16})^{(2)}$ large to have

$$\frac{(a_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[(\hat{P}_{16})^{(2)} + ((\hat{P}_{16})^{(2)} + G_j^0)e^{-\left(\frac{(\hat{P}_{16})^{(2)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{16})^{(2)}$$ 191

$$\frac{(b_i)^{(2)}}{(\hat{M}_{16})^{(2)}}\left[((\hat{Q}_{16})^{(2)} + T_j^0)e^{-\left(\frac{(\hat{Q}_{16})^{(2)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{16})^{(2)}\right] \leq (\hat{Q}_{16})^{(2)}$$ 192

In order that the operator $\mathcal{A}^{(2)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself 193

The operator $\mathcal{A}^{(2)}$ is a contraction with respect to the metric 194

$$d\left(\left((G_{19})^{(1)},(T_{19})^{(1)}\right),\left((G_{19})^{(2)},(T_{19})^{(2)}\right)\right)=$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{16})^{(2)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{16})^{(2)}t}\}$$

Indeed if we denote 195

**Definition of** $\widetilde{G_{19}},\widetilde{T_{19}}$ : $\left(\widetilde{G_{19}},\widetilde{T_{19}}\right)=\mathcal{A}^{(2)}(G_{19},T_{19})$

It results 196

$$\left|\tilde{G}_{16}^{(1)}-\tilde{G}_i^{(2)}\right|\le\int_0^t(a_{16})^{(2)}\left|G_{17}^{(1)}-G_{17}^{(2)}\right|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}\,ds_{(16)}+$$

$$\int_0^t\{(a_{16}')^{(2)}\left|G_{16}^{(1)}-G_{16}^{(2)}\right|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}+$$

$$(a_{16}'')^{(2)}\left(T_{17}^{(1)},s_{(16)}\right)\left|G_{16}^{(1)}-G_{16}^{(2)}\right|e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}+$$

$$G_{16}^{(2)}|(a_{16}'')^{(2)}\left(T_{17}^{(1)},s_{(16)}\right)-(a_{16}'')^{(2)}\left(T_{17}^{(2)},s_{(16)}\right)|\;e^{-(\widehat{M}_{16})^{(2)}s_{(16)}}e^{(\widehat{M}_{16})^{(2)}s_{(16)}}\}ds_{(16)}$$

Where $s_{(16)}$ represents integrand that is integrated over the interval $[0,t]$ 197

From the hypotheses it follows

$$\left|(G_{19})^{(1)}-(G_{19})^{(2)}\right|\mathrm{e}^{-(\widehat{\mathrm{M}}_{16})^{(2)}\mathrm{t}}\le$$
$$\frac{1}{(\widehat{\mathrm{M}}_{16})^{(2)}}\left((a_{16})^{(2)}+\;(a_{16}')^{(2)}+(\widehat{\mathrm{A}}_{16})^{(2)}+(\widehat{\mathrm{P}}_{16})^{(2)}(\widehat{k}_{16})^{(2)}\right)\mathrm{d}\left(\left((G_{19})^{(1)},(T_{19})^{(1)};\;(G_{19})^{(2)},(T_{19})^{(2)}\right)\right)$$

And analogous inequalities for $\mathrm{G}_i$ and $\mathrm{T}_i$. Taking into account the hypothesis the result follows 198

**Remark 6:** The fact that we supposed $(a_{16}'')^{(2)}$ and $(b_{16}'')^{(2)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{\mathrm{P}}_{16})^{(2)}\mathrm{e}^{(\widehat{\mathrm{M}}_{16})^{(2)}\mathrm{t}}$ and $(\widehat{\mathrm{Q}}_{16})^{(2)}\mathrm{e}^{(\widehat{\mathrm{M}}_{16})^{(2)}\mathrm{t}}$ respectively of $\mathbb{R}_+$. 199

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(2)}$ and $(b_i'')^{(2)}, i=16{,}17{,}18$ depend only on $\mathrm{T}_{17}$ and respectively on $(G_{19})$(and not on t) and hypothesis can replaced by a usual Lipschitz condition.

**Remark 7:** There does not exist any t where $\mathrm{G}_i\,(\mathrm{t})=0$ and $\mathrm{T}_i\,(\mathrm{t})=0$ 200

it results

$$\mathrm{G}_i\,(\mathrm{t})\ge\mathrm{G}_i^0\mathrm{e}^{\left[-\int_0^{\mathrm{t}}\{(a_i')^{(2)}-(a_i'')^{(2)}(\mathrm{T}_{17}(s_{(16)}),s_{(16)})\}\mathrm{d}s_{(16)}\right]}\ge 0$$

$$\mathrm{T}_i\,(\mathrm{t})\ge\mathrm{T}_i^0\mathrm{e}^{\left(-(b_i')^{(2)}\mathrm{t}\right)}>0\quad\text{for } \mathrm{t}>0$$

**Definition of** $\left((\widehat{\mathrm{M}}_{16})^{(2)}\right)_1,\left((\widehat{\mathrm{M}}_{16})^{(2)}\right)_2$ and $\left((\widehat{\mathrm{M}}_{16})^{(2)}\right)_3$ : 201

**Remark 8:** if $G_{16}$ is bounded, the same property have also $G_{17}$ and $G_{18}$ . indeed if

$G_{16} < (\widehat{M}_{16})^{(2)}$ it follows $\frac{dG_{17}}{dt} \leq \left((\widehat{M}_{16})^{(2)}\right)_1 - (a'_{17})^{(2)} G_{17}$ and by integrating

$G_{17} \leq \left((\widehat{M}_{16})^{(2)}\right)_2 = G_{17}^0 + 2(a_{17})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_1 / (a'_{17})^{(2)}$

In the same way , one can obtain

$G_{18} \leq \left((\widehat{M}_{16})^{(2)}\right)_3 = G_{18}^0 + 2(a_{18})^{(2)}\left((\widehat{M}_{16})^{(2)}\right)_2 / (a'_{18})^{(2)}$

If $G_{17}$ or $G_{18}$ is bounded, the same property follows for $G_{16}$ , $G_{18}$ and $G_{16}$ , $G_{17}$ respectively.

**Remark 9:** If $G_{16}$ is bounded, from below, the same property holds for $G_{17}$ and $G_{18}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{17}$ is bounded from below. 202

**Remark 10:** If $T_{16}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(2)}((G_{19})(t),t)) = (b'_{17})^{(2)}$ then $T_{17} \to \infty$. 203

**Definition of** $(m)^{(2)}$ and $\varepsilon_2$ :

Indeed let $t_2$ be so that for $t > t_2$

$(b_{17})^{(2)} - (b_i'')^{(2)}((G_{19})(t),t) < \varepsilon_2, T_{16}(t) > (m)^{(2)}$

Then $\frac{dT_{17}}{dt} \geq (a_{17})^{(2)}(m)^{(2)} - \varepsilon_2 T_{17}$ which leads to 204

$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{\varepsilon_2}\right)(1 - e^{-\varepsilon_2 t}) + T_{17}^0 e^{-\varepsilon_2 t}$ If we take t such that $e^{-\varepsilon_2 t} = \frac{1}{2}$ it results

$T_{17} \geq \left(\frac{(a_{17})^{(2)}(m)^{(2)}}{2}\right), \quad t = \log\frac{2}{\varepsilon_2}$ By taking now $\varepsilon_2$ sufficiently small one sees that $T_{17}$ is unbounded. The same property holds for $T_{18}$ if $\lim_{t\to\infty}(b''_{18})^{(2)}\left((G_{19})(t),t\right) = (b'_{18})^{(2)}$ 205

We now state a more precise theorem about the behaviors at infinity of the solutions of equations

206

It is now sufficient to take $\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}$ , $\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}} < 1$ and to choose 207

$(\widehat{P}_{20})^{(3)}$ and $(\widehat{Q}_{20})^{(3)}$ large to have

$$\frac{(a_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[(\hat{P}_{20})^{(3)} + \left((\hat{P}_{20})^{(3)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{20})^{(3)} + G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{20})^{(3)}$$ 208

$$\frac{(b_i)^{(3)}}{(\hat{M}_{20})^{(3)}}\left[\left((\hat{Q}_{20})^{(3)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{20})^{(3)} + T_j^0}{T_j^0}\right)} + (\hat{Q}_{20})^{(3)}\right] \leq (\hat{Q}_{20})^{(3)}$$ 209

In order that the operator $\mathcal{A}^{(3)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying Equations into itself 210

The operator $\mathcal{A}^{(3)}$ is a contraction with respect to the metric 211

$d\left(\left((G_{23})^{(1)},(T_{23})^{(1)}\right),\left((G_{23})^{(2)},(T_{23})^{(2)}\right)\right)=$

$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\widehat{M}_{20})^{(3)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\widehat{M}_{20})^{(3)}t}\}$

Indeed if we denote 212

**Definition of** $\widetilde{G_{23}},\widetilde{T_{23}}$ :$\left(\widetilde{(G_{23})},\widetilde{(T_{23})}\right)=\mathcal{A}^{(3)}\left((G_{23}),(T_{23})\right)$

It results 213

$\left|\tilde{G}_{20}^{(1)}-\tilde{G}_i^{(2)}\right|\leq\int_0^t(a_{20})^{(3)}\left|G_{21}^{(1)}-G_{21}^{(2)}\right|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\,ds_{(20)}+$

$\int_0^t\{(a'_{20})^{(3)}\left|G_{20}^{(1)}-G_{20}^{(2)}\right|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}+$

$(a''_{20})^{(3)}\left(T_{21}^{(1)},s_{(20)}\right)\left|G_{20}^{(1)}-G_{20}^{(2)}\right|e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}+$

$G_{20}^{(2)}|(a''_{20})^{(3)}\left(T_{21}^{(1)},s_{(20)}\right)-(a''_{20})^{(3)}\left(T_{21}^{(2)},s_{(20)}\right)|\ \ e^{-(\widehat{M}_{20})^{(3)}s_{(20)}}e^{(\widehat{M}_{20})^{(3)}s_{(20)}}\}ds_{(20)}$

Where $s_{(20)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$

From the hypotheses it follows

$\left|G_{23}{}^{(1)}-G_{23}{}^{(2)}\right|e^{-(\widehat{M}_{20})^{(3)}t}\leq$ 214
$\frac{1}{(\widehat{M}_{20})^{(3)}}\left((a_{20})^{(3)}+\ (a'_{20})^{(3)}+(\widehat{A}_{20})^{(3)}+(\widehat{P}_{20})^{(3)}(\widehat{k}_{20})^{(3)}\right)d\left(\left((G_{23})^{(1)},(T_{23})^{(1)};\ (G_{23})^{(2)},(T_{23})^{(2)}\right)\right)$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

**Remark 11:** The fact that we supposed $(a''_{20})^{(3)}$ and $(b''_{20})^{(3)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ *and* $(\widehat{Q}_{20})^{(3)}e^{(\widehat{M}_{20})^{(3)}t}$ respectively of $\mathbb{R}_+$. 215

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(3)}$ and $(b''_i)^{(3)},i=20,21,22$ depend only on $\mathrm{T}_{21}$ and respectively on $(G_{23})(\text{and not on }t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 12:** There does not exist any $t$ where $G_i\,(t)=0$ *and* $T_i\,(t)=0$ 216

it results

$G_i\,(t)\geq G_i^0e^{\left[-\int_0^t\{(a'_i)^{(3)}-(a''_i)^{(3)}(T_{21}(s_{(20)}),s_{(20)})\}ds_{(20)}\right]}\geq 0$

$T_i\,(t)\geq T_i^0e^{(-(b'_i)^{(3)}t)}>0\quad\text{for t}>0$

**Definition of** $\left((\widehat{M}_{20})^{(3)}\right)_1,\left((\widehat{M}_{20})^{(3)}\right)_2$ *and* $\left((\widehat{M}_{20})^{(3)}\right)_3$ : 217

**Remark 13:** if $G_{20}$ is bounded, the same property have also $G_{21}$ *and* $G_{22}$ . indeed if

$G_{20}<(\widehat{M}_{20})^{(3)}$ it follows $\frac{dG_{21}}{dt}\leq\left((\widehat{M}_{20})^{(3)}\right)_1-(a'_{21})^{(3)}G_{21}$ and by integrating

$G_{21} \leq \left( (\widehat{M}_{20})^{(3)} \right)_2 = G_{21}^0 + 2(a_{21})^{(3)} \left( (\widehat{M}_{20})^{(3)} \right)_1 / (a'_{21})^{(3)}$

In the same way , one can obtain

$G_{22} \leq \left( (\widehat{M}_{20})^{(3)} \right)_3 = G_{22}^0 + 2(a_{22})^{(3)} \left( (\widehat{M}_{20})^{(3)} \right)_2 / (a'_{22})^{(3)}$

If $G_{21}$ *or* $G_{22}$ is bounded, the same property follows for $G_{20}$ , $G_{22}$ and $G_{20}$ , $G_{21}$ respectively.

**Remark 14:** If $G_{20}$ *is* bounded, from below, the same property holds for $G_{21}$ *and* $G_{22}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{21}$ is bounded from below. 218

**Remark 15:** If $T_{20}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(3)} \left( (G_{23})(t), t \right) = (b'_{21})^{(3)}$ then $T_{21} \to \infty$. 219

**Definition of** $(m)^{(3)}$ and $\varepsilon_3$ :

Indeed let $t_3$ be so that for $t > t_3$

$(b_{21})^{(3)} - (b_i'')^{(3)} \left( (G_{23})(t), t \right) < \varepsilon_3, T_{20}\,(t) > (m)^{(3)}$

Then $\frac{dT_{21}}{dt} \geq (a_{21})^{(3)} (m)^{(3)} - \varepsilon_3 T_{21}$ which leads to 220

$T_{21} \geq \left( \frac{(a_{21})^{(3)} (m)^{(3)}}{\varepsilon_3} \right) (1 - e^{-\varepsilon_3 t}) + T_{21}^0 e^{-\varepsilon_3 t}$ If we take t such that $e^{-\varepsilon_3 t} = \frac{1}{2}$ it results

$T_{21} \geq \left( \frac{(a_{21})^{(3)} (m)^{(3)}}{2} \right), \quad t = log \frac{2}{\varepsilon_3}$ By taking now $\varepsilon_3$ sufficiently small one sees that $T_{21}$ is unbounded. The same property holds for $T_{22}$ if $\lim_{t\to\infty}(b''_{22})^{(3)} \left( (G_{23})(t), t \right) = (b'_{22})^{(3)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations

It is now sufficient to take $\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}}$ , $\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} < 1$ and to choose 221

$(\widehat{P}_{24})^{(4)}$ and $(\widehat{Q}_{24})^{(4)}$ large to have

$$\frac{(a_i)^{(4)}}{(\hat{M}_{24})^{(4)}} \left[ (\widehat{P}_{24})^{(4)} + ((\widehat{P}_{24})^{(4)} + G_j^0) e^{-\left( \frac{(\widehat{P}_{24})^{(4)} + G_j^0}{G_j^0} \right)} \right] \leq (\widehat{P}_{24})^{(4)}$$ 222

$$\frac{(b_i)^{(4)}}{(\hat{M}_{24})^{(4)}} \left[ ((\widehat{Q}_{24})^{(4)} + T_j^0) e^{-\left( \frac{(\widehat{Q}_{24})^{(4)} + T_j^0}{T_j^0} \right)} + (\widehat{Q}_{24})^{(4)} \right] \leq (\widehat{Q}_{24})^{(4)}$$ 223

In order that the operator $\mathcal{A}^{(4)}$ transforms the space of sextuples of functions $G_i , T_i$ satisfying Equations into itself 224

The operator $\mathcal{A}^{(4)}$ is a contraction with respect to the metric 225

$d \left( \left( (G_{27})^{(1)}, (T_{27})^{(1)} \right), \left( (G_{27})^{(2)}, (T_{27})^{(2)} \right) \right) =$

$\sup_i \{ \max_{t\in\mathbb{R}_+} \left| G_i^{(1)}(t) - G_i^{(2)}(t) \right| e^{-(\hat{M}_{24})^{(4)} t}, \max_{t\in\mathbb{R}_+} \left| T_i^{(1)}(t) - T_i^{(2)}(t) \right| e^{-(\hat{M}_{24})^{(4)} t} \}$

Indeed if we denote

**Definition of** $\widetilde{(G_{27})}, \widetilde{(T_{27})}$ : $\left( \widetilde{(G_{27})}, \widetilde{(T_{27})} \right) = \mathcal{A}^{(4)}((G_{27}), (T_{27}))$

It results

$$\left|\tilde{G}_{24}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{24})^{(4)} \left|G_{25}^{(1)} - G_{25}^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{(\widehat{M}_{24})^{(4)} s_{(24)}} \, ds_{(24)} +$$

$$\int_0^t \{(a_{24}')^{(4)} \left|G_{24}^{(1)} - G_{24}^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} +$$

$$(a_{24}'')^{(4)} \left(T_{25}^{(1)}, s_{(24)}\right) \left|G_{24}^{(1)} - G_{24}^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{(\widehat{M}_{24})^{(4)} s_{(24)}} +$$

$$G_{24}^{(2)} |(a_{24}'')^{(4)} \left(T_{25}^{(1)}, s_{(24)}\right) - (a_{24}'')^{(4)} \left(T_{25}^{(2)}, s_{(24)}\right)| \; e^{-(\widehat{M}_{24})^{(4)} s_{(24)}} e^{(\widehat{M}_{24})^{(4)} s_{(24)}} \} ds_{(24)}$$

Where $s_{(24)}$ represents integrand that is integrated over the interval $[0, \mathrm{t}]$

From the hypotheses on Equations it follows

$$\left|(G_{27})^{(1)} - (G_{27})^{(2)}\right| e^{-(\widehat{M}_{24})^{(4)} t} \leq$$ 226

$$\frac{1}{(\widehat{M}_{24})^{(4)}} \left((a_{24})^{(4)} + (a_{24}')^{(4)} + (\widehat{A}_{24})^{(4)} + (\widehat{P}_{24})^{(4)} (\widehat{k}_{24})^{(4)}\right) d \left(\left((G_{27})^{(1)}, (T_{27})^{(1)}; \; (G_{27})^{(2)}, (T_{27})^{(2)}\right)\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

**Remark 16:** The fact that we supposed $(a_{24}'')^{(4)}$ and $(b_{24}'')^{(4)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{24})^{(4)} e^{(\widehat{M}_{24})^{(4)} t}$ *and* $(\widehat{Q}_{24})^{(4)} e^{(\widehat{M}_{24})^{(4)} t}$ respectively of $\mathbb{R}_+$. 227

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a_i'')^{(4)}$ and $(b_i'')^{(4)}, i = 24,25,26$ depend only on $\mathrm{T}_{25}$ and respectively on $(G_{27})(\textit{and not on } t)$ and hypothesis can replaced by a usual Lipschitz condition.

**Remark 17:** There does not exist any $t$ where $G_i\,(t) = 0$ *and* $T_i\,(t) = 0$ 228

it results

$$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a_i')^{(4)} - (a_i'')^{(4)} (T_{25}(s_{(24)}), s_{(24)})\} ds_{(24)}\right]} \geq 0$$

$$T_i\,(t) \geq T_i^0 e^{(-(b_i')^{(4)} t)} > 0 \quad \text{for } \mathrm{t} > 0$$

**Definition of** $\left((\widehat{M}_{24})^{(4)}\right)_1, \left((\widehat{M}_{24})^{(4)}\right)_2$ *and* $\left((\widehat{M}_{24})^{(4)}\right)_3$ : 229

**Remark 18:** if $G_{24}$ is bounded, the same property have also $G_{25}$ *and* $G_{26}$ . indeed if

$G_{24} < (\widehat{M}_{24})^{(4)}$ it follows $\frac{dG_{25}}{dt} \leq \left((\widehat{M}_{24})^{(4)}\right)_1 - (a_{25}')^{(4)} G_{25}$ and by integrating

$$G_{25} \leq \left((\widehat{M}_{24})^{(4)}\right)_2 = G_{25}^0 + 2(a_{25})^{(4)} \left((\widehat{M}_{24})^{(4)}\right)_1 / (a_{25}')^{(4)}$$

In the same way , one can obtain

$G_{26} \leq \left((\widehat{M}_{24})^{(4)}\right)_3 = G_{26}^0 + 2(a_{26})^{(4)}\left((\widehat{M}_{24})^{(4)}\right)_2/(a'_{26})^{(4)}$

If $G_{25}$ *or* $G_{26}$ is bounded, the same property follows for $G_{24}$ , $G_{26}$ and $G_{24}$ , $G_{25}$ respectively.

**Remark 19:** If $G_{24}$ *is* bounded, from below, the same property holds for $G_{25}$ *and* $G_{26}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{25}$ is bounded from below. 230

**Remark 20:** If $T_{24}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(4)}((G_{27})(t),t)) = (b'_{25})^{(4)}$ then $T_{25} \to \infty$. 231

**Definition of** $(m)^{(4)}$ and $\varepsilon_4$ :

Indeed let $t_4$ be so that for $t > t_4$

$(b_{25})^{(4)} - (b_i'')^{(4)}((G_{27})(t),t) < \varepsilon_4, T_{24}(t) > (m)^{(4)}$

Then $\frac{dT_{25}}{dt} \geq (a_{25})^{(4)}(m)^{(4)} - \varepsilon_4 T_{25}$ which leads to 232

$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{\varepsilon_4}\right)(1 - e^{-\varepsilon_4 t}) + T_{25}^0 e^{-\varepsilon_4 t}$ If we take t such that $e^{-\varepsilon_4 t} = \frac{1}{2}$ it results

$T_{25} \geq \left(\frac{(a_{25})^{(4)}(m)^{(4)}}{2}\right), \quad t = \log\frac{2}{\varepsilon_4}$ By taking now $\varepsilon_4$ sufficiently small one sees that $T_{25}$ is unbounded. The same property holds for $T_{26}$ if $\lim_{t\to\infty}(b''_{26})^{(4)}\left((G_{27})(t),t\right) = (b'_{26})^{(4)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 42

Analogous inequalities hold also for $G_{29}$ , $G_{30}, T_{28}, T_{29}, T_{30}$

It is now sufficient to take $\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}$ , $\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}} < 1$ and to choose 233

$(\hat{P}_{28})^{(5)}$ and $(\hat{Q}_{28})^{(5)}$ large to have

$$\frac{(a_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\left[(\hat{P}_{28})^{(5)} + \left((\hat{P}_{28})^{(5)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{28})^{(5)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{28})^{(5)} \quad 234$$

$$\frac{(b_i)^{(5)}}{(\hat{M}_{28})^{(5)}}\left[\left((\hat{Q}_{28})^{(5)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{28})^{(5)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{28})^{(5)}\right] \leq (\hat{Q}_{28})^{(5)} \quad 235$$

In order that the operator $\mathcal{A}^{(5)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(5)}$ is a contraction with respect to the metric 236

$d\left(\left((G_{31})^{(1)},(T_{31})^{(1)}\right),\left((G_{31})^{(2)},(T_{31})^{(2)}\right)\right) =$

$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\hat{M}_{28})^{(5)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\hat{M}_{28})^{(5)}t}\}$

Indeed if we denote

**Definition of** $\widetilde{(G_{31})},\widetilde{(T_{31})}$ : $\left(\widetilde{(G_{31})},\widetilde{(T_{31})}\right)=\mathcal{A}^{(5)}\big((G_{31}),(T_{31})\big)$

It results

$\left|\tilde{G}_{28}^{(1)}-\tilde{G}_i^{(2)}\right|\leq\int_0^t(a_{28})^{(5)}\left|G_{29}^{(1)}-G_{29}^{(2)}\right|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\,ds_{(28)}+$

$\int_0^t\{(a'_{28})^{(5)}\left|G_{28}^{(1)}-G_{28}^{(2)}\right|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}+$

$(a''_{28})^{(5)}\big(T_{29}^{(1)},s_{(28)}\big)\left|G_{28}^{(1)}-G_{28}^{(2)}\right|e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}+$

$G_{28}^{(2)}|(a''_{28})^{(5)}\big(T_{29}^{(1)},s_{(28)}\big)-(a''_{28})^{(5)}\big(T_{29}^{(2)},s_{(28)}\big)|\;e^{-(\widehat{M}_{28})^{(5)}s_{(28)}}e^{(\widehat{M}_{28})^{(5)}s_{(28)}}\}ds_{(28)}$

Where $s_{(28)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$

From the hypotheses on it follows

$\left|(G_{31})^{(1)}-(G_{31})^{(2)}\right|e^{-(\widehat{M}_{28})^{(5)}t}\leq$ 237
$\frac{1}{(\widehat{M}_{28})^{(5)}}\big((a_{28})^{(5)}+(a'_{28})^{(5)}+(\widehat{A}_{28})^{(5)}+(\widehat{P}_{28})^{(5)}(\widehat{k}_{28})^{(5)}\big)d\left(\left((G_{31})^{(1)},(T_{31})^{(1)};\;(G_{31})^{(2)},(T_{31})^{(2)}\right)\right)$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

**Remark 21:** The fact that we supposed $(a''_{28})^{(5)}$ and $(b''_{28})^{(5)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ *and* $(\widehat{Q}_{28})^{(5)}e^{(\widehat{M}_{28})^{(5)}t}$ respectively of $\mathbb{R}_+$. 238

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(5)}$ and $(b''_i)^{(5)}, i=28,29,30$ depend only on $\mathrm{T}_{29}$ and respectively on $(G_{31})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition.

**Remark 22:** There does not exist any $t$ where $G_i\,(t)=0$ *and* $T_i\,(t)=0$ 239

it results

$G_i\,(t)\geq G_i^0e^{\left[-\int_0^t\{(a'_i)^{(5)}-(a''_i)^{(5)}(T_{29}(s_{(28)}),s_{(28)})\}ds_{(28)}\right]}\geq 0$

$T_i\,(t)\geq T_i^0e^{(-(b'_i)^{(5)}t)}>0$ for $\mathrm{t}>0$

**Definition of** $\big((\widehat{M}_{28})^{(5)}\big)_1,\big((\widehat{M}_{28})^{(5)}\big)_2$ *and* $\big((\widehat{M}_{28})^{(5)}\big)_3$ : 240

**Remark 23:** if $G_{28}$ is bounded, the same property have also $G_{29}$ *and* $G_{30}$ . indeed if

$G_{28}<(\widehat{M}_{28})^{(5)}$ it follows $\frac{dG_{29}}{dt}\leq\big((\widehat{M}_{28})^{(5)}\big)_1-(a'_{29})^{(5)}G_{29}$ and by integrating

$G_{29} \leq \left((\widehat{M}_{28})^{(5)}\right)_2 = G_{29}^0 + 2(a_{29})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_1/(a'_{29})^{(5)}$

In the same way , one can obtain

$G_{30} \leq \left((\widehat{M}_{28})^{(5)}\right)_3 = G_{30}^0 + 2(a_{30})^{(5)}\left((\widehat{M}_{28})^{(5)}\right)_2/(a'_{30})^{(5)}$

If $G_{29}$ *or* $G_{30}$ is bounded, the same property follows for $G_{28}$ , $G_{30}$ and $G_{28}$ , $G_{29}$ respectively.

**Remark 24:** If $G_{28}$ *is* bounded, from below, the same property holds for $G_{29}$ *and* $G_{30}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{29}$ is bounded from below. 241

**Remark 25:** If $T_{28}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(5)}((G_{31})(t),t)) = (b'_{29})^{(5)}$ then $T_{29} \to \infty$. 242

**Definition of** $(m)^{(5)}$ and $\varepsilon_5$ :

Indeed let $t_5$ be so that for $t > t_5$

$(b_{29})^{(5)} - (b_i'')^{(5)}((G_{31})(t),t) < \varepsilon_5, T_{28}(t) > (m)^{(5)}$

Then $\frac{dT_{29}}{dt} \geq (a_{29})^{(5)}(m)^{(5)} - \varepsilon_5 T_{29}$ which leads to 243

$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{\varepsilon_5}\right)(1 - e^{-\varepsilon_5 t}) + T_{29}^0 e^{-\varepsilon_5 t}$ If we take t such that $e^{-\varepsilon_5 t} = \frac{1}{2}$ it results

$T_{29} \geq \left(\frac{(a_{29})^{(5)}(m)^{(5)}}{2}\right), \quad t = log\frac{2}{\varepsilon_5}$ By taking now $\varepsilon_5$ sufficiently small one sees that $T_{29}$ is unbounded.
The same property holds for $T_{30}$ if $\lim_{t\to\infty}(b''_{30})^{(5)}\left((G_{31})(t),t\right) = (b'_{30})^{(5)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations

Analogous inequalities hold also for $G_{33}, G_{34}, T_{32}, T_{33}, T_{34}$

It is now sufficient to take $\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}, \frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}} < 1$ and to choose 244

$(\widehat{P}_{32})^{(6)}$ and $(\widehat{Q}_{32})^{(6)}$ large to have

$$\frac{(a_i)^{(6)}}{(\hat{M}_{32})^{(6)}}\left[(\widehat{P}_{32})^{(6)} + \left((\widehat{P}_{32})^{(6)} + G_j^0\right)e^{-\left(\frac{(\widehat{P}_{32})^{(6)}+G_j^0}{G_j^0}\right)}\right] \leq (\widehat{P}_{32})^{(6)} \quad 245$$

$$\frac{(b_i)^{(6)}}{(\hat{M}_{32})^{(6)}}\left[\left((\widehat{Q}_{32})^{(6)} + T_j^0\right)e^{-\left(\frac{(\widehat{Q}_{32})^{(6)}+T_j^0}{T_j^0}\right)} + (\widehat{Q}_{32})^{(6)}\right] \leq (\widehat{Q}_{32})^{(6)} \quad 246$$

In order that the operator $\mathcal{A}^{(6)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself

The operator $\mathcal{A}^{(6)}$ is a contraction with respect to the metric 247

$d\left(\left((G_{35})^{(1)},(T_{35})^{(1)}\right),\left((G_{35})^{(2)},(T_{35})^{(2)}\right)\right) =$

$sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t)-G_i^{(2)}(t)\right|e^{-(\hat{M}_{32})^{(6)}t},\max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t)-T_i^{(2)}(t)\right|e^{-(\hat{M}_{32})^{(6)}t}\}$

Indeed if we denote

<u>Definition of</u> $\widetilde{(G_{35})},\widetilde{(T_{35})}$ : $\left(\widetilde{(G_{35})},\widetilde{(T_{35})}\right)=\mathcal{A}^{(6)}\left((G_{35}),(T_{35})\right)$

It results

$\left|\tilde{G}_{32}^{(1)}-\tilde{G}_i^{(2)}\right|\le\int_0^t(a_{32})^{(6)}\left|G_{33}^{(1)}-G_{33}^{(2)}\right|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\,ds_{(32)}+$

$\int_0^t\{(a'_{32})^{(6)}\left|G_{32}^{(1)}-G_{32}^{(2)}\right|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}+$

$(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)\left|G_{32}^{(1)}-G_{32}^{(2)}\right|e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}+$

$G_{32}^{(2)}|(a''_{32})^{(6)}\left(T_{33}^{(1)},s_{(32)}\right)-(a''_{32})^{(6)}\left(T_{33}^{(2)},s_{(32)}\right)|\ e^{-(\widehat{M}_{32})^{(6)}s_{(32)}}e^{(\widehat{M}_{32})^{(6)}s_{(32)}}\}ds_{(32)}$

Where $s_{(32)}$ represents integrand that is integrated over the interval $[0,\mathrm{t}]$

From the hypotheses it follows

$\left|(G_{35})^{(1)}-(G_{35})^{(2)}\right|e^{-(\widehat{M}_{32})^{(6)}t}\le$ 248
$\frac{1}{(\widehat{M}_{32})^{(6)}}\left((a_{32})^{(6)}+(a'_{32})^{(6)}+(\widehat{A}_{32})^{(6)}+(\widehat{P}_{32})^{(6)}(\widehat{k}_{32})^{(6)}\right)d\left(\left((G_{35})^{(1)},(T_{35})^{(1)};\ (G_{35})^{(2)},(T_{35})^{(2)}\right)\right)$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

<u>Remark 26:</u> The fact that we supposed $(a''_{32})^{(6)}$ and $(b''_{32})^{(6)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ *and* $(\widehat{Q}_{32})^{(6)}e^{(\widehat{M}_{32})^{(6)}t}$ respectively of $\mathbb{R}_+$. 249

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(6)}$ and $(b''_i)^{(6)}, i=32,33,34$ depend only on $\mathrm{T}_{33}$ and respectively on $(G_{35})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition.

<u>Remark 27:</u> There does not exist any $t$ where $G_i\,(t)=0$ *and* $T_i\,(t)=0$ 250

it results

$G_i\,(t)\ge G_i^0e^{\left[-\int_0^t\{(a'_i)^{(6)}-(a''_i)^{(6)}(T_{33}(s_{(32)}),s_{(32)})\}ds_{(32)}\right]}\ge 0$

$T_i\,(t)\ge T_i^0e^{(-(b'_i)^{(6)}t)}>0\quad\text{for t}>0$

<u>Definition of</u> $\left((\widehat{M}_{32})^{(6)}\right)_1,\left((\widehat{M}_{32})^{(6)}\right)_2$ *and* $\left((\widehat{M}_{32})^{(6)}\right)_3$ : 251

<u>Remark 28:</u> if $G_{32}$ is bounded, the same property have also $G_{33}$ *and* $G_{34}$ . indeed if

$G_{32}<(\widehat{M}_{32})^{(6)}$ it follows $\frac{dG_{33}}{dt}\le\left((\widehat{M}_{32})^{(6)}\right)_1-(a'_{33})^{(6)}G_{33}$ and by integrating

$G_{33} \leq \left((\widehat{M}_{32})^{(6)}\right)_2 = G_{33}^0 + 2(a_{33})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_1/(a'_{33})^{(6)}$

In the same way , one can obtain

$G_{34} \leq \left((\widehat{M}_{32})^{(6)}\right)_3 = G_{34}^0 + 2(a_{34})^{(6)}\left((\widehat{M}_{32})^{(6)}\right)_2/(a'_{34})^{(6)}$

If $G_{33}$ *or* $G_{34}$ is bounded, the same property follows for $G_{32}$ , $G_{34}$ and $G_{32}$ , $G_{33}$ respectively.

**Remark 29:** If $G_{32}$ *is* bounded, from below, the same property holds for $G_{33}$ *and* $G_{34}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{33}$ is bounded from below. 252

**Remark 30:** If $T_{32}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(6)}((G_{35})(t),t)) = (b'_{33})^{(6)}$ then $T_{33} \to \infty$. 253

**Definition of** $(m)^{(6)}$ and $\varepsilon_6$ :

Indeed let $t_6$ be so that for $t > t_6$

$(b_{33})^{(6)} - (b_i'')^{(6)}\left((G_{35})(t),t\right) < \varepsilon_6, T_{32}\,(t) > (m)^{(6)}$

Then $\frac{dT_{33}}{dt} \geq (a_{33})^{(6)}(m)^{(6)} - \varepsilon_6 T_{33}$ which leads to 254

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{\varepsilon_6}\right)(1 - e^{-\varepsilon_6 t}) + T_{33}^0 e^{-\varepsilon_6 t}$ If we take t such that $e^{-\varepsilon_6 t} = \frac{1}{2}$ it results

$T_{33} \geq \left(\frac{(a_{33})^{(6)}(m)^{(6)}}{2}\right), \quad t = log\frac{2}{\varepsilon_6}$ By taking now $\varepsilon_6$ sufficiently small one sees that $T_{33}$ is unbounded. The same property holds for $T_{34}$ if $\lim_{t\to\infty}(b_{34}'')^{(6)}\left((G_{35})(t),t(t),t\right) = (b'_{34})^{(6)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations

Analogous inequalities hold also for $G_{37}$ , $G_{38}, T_{36}, T_{37}, T_{38}$ 255

It is now sufficient to take $\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}$ , $\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}} < 1$ and to choose
$(\widehat{P}_{36})^{(7)}$ and $(\widehat{Q}_{36})^{(7)}$ large to have

256

$$\frac{(a_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[(\widehat{P}_{36})^{(7)} + \left((\hat{P}_{36})^{(7)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{36})^{(7)}+G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{36})^{(7)}$$

257

$$\frac{(b_i)^{(7)}}{(\hat{M}_{36})^{(7)}}\left[\left((\hat{Q}_{36})^{(7)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{36})^{(7)}+T_j^0}{T_j^0}\right)} + (\hat{Q}_{36})^{(7)}\right] \leq (\hat{Q}_{36})^{(7)}$$

In order that the operator $\mathcal{A}^{(7)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself
The operator $\mathcal{A}^{(7)}$ is a contraction with respect to the metric 258

$d\left(\left((G_{39})^{(1)},(T_{39})^{(1)}\right),\left((G_{39})^{(2)},(T_{39})^{(2)}\right)\right) =$

$\sup_i \{\max_{t\in\mathbb{R}_+} \left|G_i^{(1)}(t) - G_i^{(2)}(t)\right| e^{-(\hat{M}_{36})^{(7)}t}, \max_{t\in\mathbb{R}_+} \left|T_i^{(1)}(t) - T_i^{(2)}(t)\right| e^{-(\hat{M}_{36})^{(7)}t}\}$

Indeed if we denote

**Definition of** $\widetilde{(G_{39})}, \widetilde{(T_{39})} : \left( \widetilde{(G_{39})}, \widetilde{(T_{39})} \right) = \mathcal{A}^{(7)}((G_{39}), (T_{39}))$
It results

$$\left|\tilde{G}_{36}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{36})^{(7)} \left|G_{37}^{(1)} - G_{37}^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{(\widehat{M}_{36})^{(7)} s_{(36)}}\, ds_{(36)} +$$
$$\int_0^t \{(a'_{36})^{(7)} \left|G_{36}^{(1)} - G_{36}^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} +$$
$$(a''_{36})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right)\left|G_{36}^{(1)} - G_{36}^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{(\widehat{M}_{36})^{(7)} s_{(36)}} +$$
$$G_{36}^{(2)} |(a''_{36})^{(7)}\left(T_{37}^{(1)}, s_{(36)}\right) - (a''_{36})^{(7)}\left(T_{37}^{(2)}, s_{(36)}\right)|\ \ e^{-(\widehat{M}_{36})^{(7)} s_{(36)}} e^{(\widehat{M}_{36})^{(7)} s_{(36)}}\} ds_{(36)}$$

Where $s_{(36)}$ represents integrand that is integrated over the interval $[0, \mathrm{t}]$

From the hypotheses on it follows

$$\left|(G_{39})^{(1)} - (G_{39})^{(2)}\right| e^{-(\widehat{M}_{36})^{(7)} t} \leq$$ 259
$$\frac{1}{(\widehat{M}_{36})^{(7)}} \left((a_{36})^{(7)} + (a'_{36})^{(7)} + (\widehat{A}_{36})^{(7)} + (\widehat{P}_{36})^{(7)} (\widehat{k}_{36})^{(7)}\right) d\left(\left((G_{39})^{(1)}, (T_{39})^{(1)};\ (G_{39})^{(2)}, (T_{39})^{(2)}\right)\right)$$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

**Remark 31:** The fact that we supposed $(a''_{36})^{(7)}$ and $(b''_{36})^{(7)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{36})^{(7)} e^{(\widehat{M}_{36})^{(7)} t}$ *and* $(\widehat{Q}_{36})^{(7)} e^{(\widehat{M}_{36})^{(7)} t}$ respectively of $\mathbb{R}_+$. 260

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(7)}$ and $(b''_i)^{(7)}, i = 36,37,38$ depend only on $\mathrm{T}_{37}$ and respectively on $(G_{39})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition.

**Remark 32:** There does not exist any $t$ where $G_i(t) = 0$ *and* $T_i(t) = 0$ 261

it results

$$G_i(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(7)} - (a''_i)^{(7)}(T_{37}(s_{(36)}), s_{(36)})\} ds_{(36)}\right]} \geq 0$$

$$T_i(t) \geq T_i^0 e^{(-(b'_i)^{(7)} t)} > 0 \quad \text{for } t > 0$$

**Definition of** $\left((\widehat{M}_{36})^{(7)}\right)_1, \left((\widehat{M}_{36})^{(7)}\right)_2$ *and* $\left((\widehat{M}_{36})^{(7)}\right)_3$ : 262

**Remark 33:** if $G_{36}$ is bounded, the same property have also $G_{37}$ *and* $G_{38}$ . indeed if

$G_{36} < (\widehat{M}_{36})^{(7)}$ it follows $\frac{dG_{37}}{dt} \leq \left((\widehat{M}_{36})^{(7)}\right)_1 - (a'_{37})^{(7)} G_{37}$ and by integrating

$$G_{37} \leq \left((\widehat{M}_{36})^{(7)}\right)_2 = G_{37}^0 + 2(a_{37})^{(7)}\left((\widehat{M}_{36})^{(7)}\right)_1 / (a'_{37})^{(7)}$$

In the same way , one can obtain

ISSN 2250-3153

$G_{38} \leq \left( (\widehat{M}_{36})^{(7)} \right)_3 = G_{38}^0 + 2(a_{38})^{(7)} \left( (\widehat{M}_{36})^{(7)} \right)_2 / (a'_{38})^{(7)}$

If $G_{37}$ *or* $G_{38}$ is bounded, the same property follows for $G_{36}$ , $G_{38}$ and $G_{36}$ , $G_{37}$ respectively.

**Remark 34:** If $G_{36}$ *is* bounded, from below, the same property holds for $G_{37}$ *and* $G_{38}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{37}$ is bounded from below. 263

**Remark 35:** If $T_{36}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(7)}((G_{39})(t),t)) = (b'_{37})^{(7)}$ then $T_{37} \to \infty$. 264

**Definition of** $(m)^{(7)}$ and $\varepsilon_7$ :

Indeed let $t_7$ be so that for $t > t_7$

$(b_{37})^{(7)} - (b_i'')^{(7)}((G_{39})(t),t) < \varepsilon_7, T_{36}(t) > (m)^{(7)}$

Then $\frac{dT_{37}}{dt} \geq (a_{37})^{(7)}(m)^{(7)} - \varepsilon_7 T_{37}$ which leads to 265

$T_{37} \geq \left( \frac{(a_{37})^{(7)}(m)^{(7)}}{\varepsilon_7} \right)(1 - e^{-\varepsilon_7 t}) + T_{37}^0 e^{-\varepsilon_7 t}$ If we take t such that $e^{-\varepsilon_7 t} = \frac{1}{2}$ it results

$T_{37} \geq \left( \frac{(a_{37})^{(7)}(m)^{(7)}}{2} \right), \quad t = log \frac{2}{\varepsilon_7}$ By taking now $\varepsilon_7$ sufficiently small one sees that $T_{37}$ is unbounded.
The same property holds for $T_{38}$ if $\lim_{t\to\infty}(b''_{38})^{(7)}\left((G_{39})(t),t\right) = (b'_{38})^{(7)}$
We now state a more precise theorem about the behaviors at infinity of the solutions of equations

It is now sufficient to take $\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}, \frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}} < 1$ and to choose 266
$(\widehat{P}_{40})^{(8)}$ and $(\widehat{Q}_{40})^{(8)}$ large to have

$$\frac{(a_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[ (\widehat{P}_{40})^{(8)} + \left( (\widehat{P}_{40})^{(8)} + G_j^0 \right) e^{-\left( \frac{(\widehat{P}_{40})^{(8)} + G_j^0}{G_j^0} \right)} \right] \leq (\widehat{P}_{40})^{(8)}$$ 267

268
$$\frac{(b_i)^{(8)}}{(\widehat{M}_{40})^{(8)}}\left[ \left( (\widehat{Q}_{40})^{(8)} + T_j^0 \right) e^{-\left( \frac{(\widehat{Q}_{40})^{(8)} + T_j^0}{T_j^0} \right)} + (\widehat{Q}_{40})^{(8)} \right] \leq (\widehat{Q}_{40})^{(8)}$$

In order that the operator $\mathcal{A}^{(8)}$ transforms the space of sextuples of functions $G_i, T_i$ satisfying Equations into itself
The operator $\mathcal{A}^{(8)}$ is a contraction with respect to the metric

$$d\left( \left( (G_{43})^{(1)}, (T_{43})^{(1)} \right), \left( (G_{43})^{(2)}, (T_{43})^{(2)} \right) \right) =$$ 269
$$\sup_i \{ \max_{t\in\mathbb{R}_+} \left| G_i^{(1)}(t) - G_i^{(2)}(t) \right| e^{-(\widehat{M}_{40})^{(8)} t}, \max_{t\in\mathbb{R}_+} \left| T_i^{(1)}(t) - T_i^{(2)}(t) \right| e^{-(\widehat{M}_{40})^{(8)} t} \}$$

Indeed if we denote 270

**Definition of** $\widetilde{(G_{43})}, \widetilde{(T_{43})}$ : $\left( \widetilde{(G_{43})}, \widetilde{(T_{43})} \right) = \mathcal{A}^{(8)}((G_{43}), (T_{43}))$

It results 271

$\left|\tilde{G}_{40}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{40})^{(8)} \left|G_{41}^{(1)} - G_{41}^{(2)}\right| e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{(\widehat{M}_{40})^{(8)} s_{(40)}}\, ds_{(40)} +$
$\int_0^t \{(a'_{40})^{(8)} \left|G_{40}^{(1)} - G_{40}^{(2)}\right| e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} +$
$(a''_{40})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right)\left|G_{40}^{(1)} - G_{40}^{(2)}\right| e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{(\widehat{M}_{40})^{(8)} s_{(40)}} +$
$G_{40}^{(2)} |(a''_{40})^{(8)}\left(T_{41}^{(1)}, s_{(40)}\right) - (a''_{40})^{(8)}\left(T_{41}^{(2)}, s_{(40)}\right)|\ e^{-(\widehat{M}_{40})^{(8)} s_{(40)}} e^{(\widehat{M}_{40})^{(8)} s_{(40)}}\} ds_{(40)}$

Where $s_{(40)}$ represents integrand that is integrated over the interval $[0, \mathrm{t}]$ 272

From the hypotheses it follows

$\left|(G_{43})^{(1)} - (G_{43})^{(2)}\right| e^{-(\widehat{M}_{40})^{(8)} t} \leq$ 273
$\frac{1}{(\widehat{M}_{40})^{(8)}}\left((a_{40})^{(8)} + (a'_{40})^{(8)} + (\widehat{A}_{40})^{(8)} + (\widehat{P}_{40})^{(8)} (\widehat{k}_{40})^{(8)}\right) d\left(\left((G_{43})^{(1)}, (T_{43})^{(1)};\ (G_{43})^{(2)}, (T_{43})^{(2)}\right)\right)$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis the result follows

<u>**Remark 36:**</u> The fact that we supposed $(a''_{40})^{(8)}$ and $(b''_{40})^{(8)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)} t}$ *and* $(\widehat{Q}_{40})^{(8)} e^{(\widehat{M}_{40})^{(8)} t}$ respectively of $\mathbb{R}_+$. 274

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(8)}$ and $(b''_i)^{(8)}, i = 40,41,42$ depend only on $\mathrm{T}_{41}$ and respectively on $(G_{43})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition.

<u>**Remark 37**</u> There does not exist any $t$ where $G_i\,(t) = 0$ *and* $T_i\,(t) = 0$ 275

it results

$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t \{(a'_i)^{(8)} - (a''_i)^{(8)}(T_{41}(s_{(40)}), s_{(40)})\} ds_{(40)}\right]} \geq 0$

$T_i\,(t) \geq T_i^0 e^{\left(-(b'_i)^{(8)} t\right)} > 0$ for $\mathrm{t} > 0$

<u>**Definition of**</u> $\left((\widehat{M}_{40})^{(8)}\right)_1, \left((\widehat{M}_{40})^{(8)}\right)_2$ *and* $\left((\widehat{M}_{40})^{(8)}\right)_3$ : 276

<u>**Remark 38:**</u> if $G_{40}$ is bounded, the same property have also $G_{41}$ *and* $G_{42}$ . indeed if

$G_{40} < (\widehat{M}_{40})^{(8)}$ it follows $\frac{dG_{41}}{dt} \leq \left((\widehat{M}_{40})^{(8)}\right)_1 - (a'_{41})^{(8)} G_{41}$ and by integrating

$G_{41} \leq \left((\widehat{M}_{40})^{(8)}\right)_2 = G_{41}^0 + 2(a_{41})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_1 / (a'_{41})^{(8)}$

In the same way , one can obtain

$G_{42} \leq \left((\widehat{M}_{40})^{(8)}\right)_3 = G_{42}^0 + 2(a_{42})^{(8)}\left((\widehat{M}_{40})^{(8)}\right)_2 / (a'_{42})^{(8)}$

If $G_{41}$ *or* $G_{42}$ is bounded, the same property follows for $G_{40}$ , $G_{42}$ and $G_{40}$ , $G_{41}$ respectively.

<u>**Remark 39:**</u> If $G_{40}$ *is* bounded, from below, the same property holds for $G_{41}$ *and* $G_{42}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{41}$ is bounded from below. 277

<u>**Remark 40:**</u> If $\mathrm{T}_{40}$ is bounded from below and $\lim_{t\to\infty}((b''_i)^{(8)}\,((G_{43})(t), t)) = (b'_{41})^{(8)}$ then $T_{41} \to \infty$. 278

**Definition of** $(m)^{(8)}$ and $\varepsilon_8$ :

Indeed let $t_8$ be so that for $t > t_8$

$$(b_{41})^{(8)} - (b_i'')^{(8)}\big((G_{43})(t), t\big) < \varepsilon_8, T_{40}(t) > (m)^{(8)}$$

Then $\frac{dT_{41}}{dt} \geq (a_{41})^{(8)}(m)^{(8)} - \varepsilon_8 T_{41}$ which leads to 279

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{\varepsilon_8}\right)(1 - e^{-\varepsilon_8 t}) + T_{41}^0 e^{-\varepsilon_8 t}$ If we take t such that $e^{-\varepsilon_8 t} = \frac{1}{2}$ it results

$T_{41} \geq \left(\frac{(a_{41})^{(8)}(m)^{(8)}}{2}\right), \quad t = log\frac{2}{\varepsilon_8}$ By taking now $\varepsilon_8$ sufficiently small one sees that $T_{41}$ is unbounded. The same property holds for $T_{42}$ if $\lim_{t\to\infty}(b_{42}'')^{(8)}\left((G_{43})(t), t(t), t\right) = (b_{42}')^{(8)}$

It is now sufficient to take $\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}$ , $\frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}} < 1$ and to choose 279 A

$(\hat{P}_{44})^{(9)}$ and $(\hat{Q}_{44})^{(9)}$ large to have

$$\frac{(a_i)^{(9)}}{(\hat{M}_{44})^{(9)}}\left[(\hat{P}_{44})^{(9)} + \left((\hat{P}_{44})^{(9)} + G_j^0\right)e^{-\left(\frac{(\hat{P}_{44})^{(9)} + G_j^0}{G_j^0}\right)}\right] \leq (\hat{P}_{44})^{(9)}$$

$$\frac{(b_i)^{(9)}}{(\hat{M}_{44})^{(9)}}\left[\left((\hat{Q}_{44})^{(9)} + T_j^0\right)e^{-\left(\frac{(\hat{Q}_{44})^{(9)} + T_j^0}{T_j^0}\right)} + (\hat{Q}_{44})^{(9)}\right] \leq (\hat{Q}_{44})^{(9)}$$

In order that the operator $\mathcal{A}^{(9)}$ transforms the space of sextuples of functions $G_i$ , $T_i$ satisfying 39,35,36 into itself

The operator $\mathcal{A}^{(9)}$ is a contraction with respect to the metric

$$d\left(\left((G_{47})^{(1)}, (T_{47})^{(1)}\right), \left((G_{47})^{(2)}, (T_{47})^{(2)}\right)\right) =$$

$$\sup_i\{\max_{t\in\mathbb{R}_+}\left|G_i^{(1)}(t) - G_i^{(2)}(t)\right|e^{-(\hat{M}_{44})^{(9)}t}, \max_{t\in\mathbb{R}_+}\left|T_i^{(1)}(t) - T_i^{(2)}(t)\right|e^{-(\hat{M}_{44})^{(9)}t}\}$$

Indeed if we denote

**Definition of** $\widetilde{(G_{47})}, \widetilde{(T_{47})} : \left(\widetilde{(G_{47})}, \widetilde{(T_{47})}\right) = \mathcal{A}^{(9)}\left((G_{47}), (T_{47})\right)$

It results

$$\left|\tilde{G}_{44}^{(1)} - \tilde{G}_i^{(2)}\right| \leq \int_0^t (a_{44})^{(9)}\left|G_{45}^{(1)} - G_{45}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\,ds_{(44)} +$$

$$\int_0^t\{(a_{44}')^{(9)}\left|G_{44}^{(1)} - G_{44}^{(2)}\right|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{-(\widehat{M}_{44})^{(9)}s_{(44)}} +$$

$(a''_{44})^{(9)}\big(T_{45}^{(1)}, s_{(44)}\big)\big|G_{44}^{(1)} - G_{44}^{(2)}\big|e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}} +$

$G_{44}^{(2)}|(a''_{44})^{(9)}\big(T_{45}^{(1)}, s_{(44)}\big) - (a''_{44})^{(9)}\big(T_{45}^{(2)}, s_{(44)}\big)|\ e^{-(\widehat{M}_{44})^{(9)}s_{(44)}}e^{(\widehat{M}_{44})^{(9)}s_{(44)}}\}ds_{(44)}$

Where $s_{(44)}$ represents integrand that is integrated over the interval $[0, t]$

From the hypotheses on 45,46,47,28 and 29 it follows

$\big|(G_{47})^{(1)} - G^{(2)}\big|e^{-(\widehat{M}_{44})^{(9)}t} \leq$
$\frac{1}{(\widehat{M}_{44})^{(9)}}\big((a_{44})^{(9)} + (a'_{44})^{(9)} + (\widehat{A}_{44})^{(9)} + (\widehat{P}_{44})^{(9)}(\widehat{k}_{44})^{(9)}\big)d\left(\left((G_{47})^{(1)}, (T_{47})^{(1)};\ (G_{47})^{(2)}, (T_{47})^{(2)}\right)\right)$

And analogous inequalities for $G_i$ *and* $T_i$. Taking into account the hypothesis (39,35,36) the result follows

**Remark 41:** The fact that we supposed $(a''_{44})^{(9)}$ and $(b''_{44})^{(9)}$ depending also on t can be considered as not conformal with the reality, however we have put this hypothesis ,in order that we can postulate condition necessary to prove the uniqueness of the solution bounded by $(\widehat{P}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ *and* $(\widehat{Q}_{44})^{(9)}e^{(\widehat{M}_{44})^{(9)}t}$ respectively of $\mathbb{R}_+$.

If instead of proving the existence of the solution on $\mathbb{R}_+$, we have to prove it only on a compact then it suffices to consider that $(a''_i)^{(9)}$ and $(b''_i)^{(9)}, i = 44,45,46$ depend only on $T_{45}$ and respectively on $(G_{47})$(*and not on* $t$) and hypothesis can replaced by a usual Lipschitz condition.

**Remark 42:** There does not exist any $t$ where $G_i\,(t) = 0$ *and* $T_i\,(t) = 0$

From 99 to 44 it results

$G_i\,(t) \geq G_i^0 e^{\left[-\int_0^t\{(a'_i)^{(9)} - (a''_i)^{(9)}(T_{45}(s_{(44)}), s_{(44)})\}ds_{(44)}\right]} \geq 0$

$T_i\,(t) \geq T_i^0 e^{(-(b'_i)^{(9)}t)} > 0$ for $t > 0$

**Definition of** $\big((\widehat{M}_{44})^{(9)}\big)_1, \big((\widehat{M}_{44})^{(9)}\big)_2$ *and* $\big((\widehat{M}_{44})^{(9)}\big)_3$ :

**Remark 43:** if $G_{44}$ is bounded, the same property have also $G_{45}$ *and* $G_{46}$ . indeed if

$G_{44} < (\widehat{M}_{44})^{(9)}$ it follows $\frac{dG_{45}}{dt} \leq \big((\widehat{M}_{44})^{(9)}\big)_1 - (a'_{45})^{(9)}G_{45}$ and by integrating

$G_{45} \leq \big((\widehat{M}_{44})^{(9)}\big)_2 = G_{45}^0 + 2(a_{45})^{(9)}\big((\widehat{M}_{44})^{(9)}\big)_1/(a'_{45})^{(9)}$

In the same way , one can obtain

$G_{46} \leq \big((\widehat{M}_{44})^{(9)}\big)_3 = G_{46}^0 + 2(a_{46})^{(9)}\big((\widehat{M}_{44})^{(9)}\big)_2/(a'_{46})^{(9)}$

If $G_{45}$ *or* $G_{46}$ is bounded, the same property follows for $G_{44}$ , $G_{46}$ and $G_{44}$ , $G_{45}$ respectively.

**Remark 44:** If $G_{44}$ *is* bounded, from below, the same property holds for $G_{45}$ *and* $G_{46}$ . The proof is analogous with the preceding one. An analogous property is true if $G_{45}$ is bounded from below.

**Remark 45:** If $T_{44}$ is bounded from below and $\lim_{t\to\infty}((b_i'')^{(9)}\left((G_{47})(t),t\right)) = (b_{45}')^{(9)}$ then $T_{45} \to \infty$.

**Definition of** $(m)^{(9)}$ and $\varepsilon_9$ :

Indeed let $t_9$ be so that for $t > t_9$

$(b_{45})^{(9)} - (b_i'')^{(9)}\left((G_{47})(t),t\right) < \varepsilon_9, T_{44}(t) > (m)^{(9)}$

Then $\frac{dT_{45}}{dt} \geq (a_{45})^{(9)}(m)^{(9)} - \varepsilon_9 T_{45}$ which leads to

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{\varepsilon_9}\right)(1-e^{-\varepsilon_9 t}) + T_{45}^0 e^{-\varepsilon_9 t}$ If we take t such that $e^{-\varepsilon_9 t} = \frac{1}{2}$ it results

$T_{45} \geq \left(\frac{(a_{45})^{(9)}(m)^{(9)}}{2}\right), \quad t = \log\frac{2}{\varepsilon_9}$ By taking now $\varepsilon_9$ sufficiently small one sees that $T_{45}$ is unbounded. The same property holds for $T_{46}$ if $\lim_{t\to\infty}(b_{46}'')^{(9)}\left((G_{47})(t),t\right) = (b_{46}')^{(9)}$

We now state a more precise theorem about the behaviors at infinity of the solutions of equations 37 to 92

**Behavior of the solutions of equation** 280

**Theorem** If we denote and define

**Definition of** $(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ :

$(\sigma_1)^{(1)}, (\sigma_2)^{(1)}, (\tau_1)^{(1)}, (\tau_2)^{(1)}$ four constants satisfying

$-(\sigma_2)^{(1)} \leq -(a_{13}')^{(1)} + (a_{14}')^{(1)} - (a_{13}'')^{(1)}(T_{14},t) + (a_{14}'')^{(1)}(T_{14},t) \leq -(\sigma_1)^{(1)}$

$-(\tau_2)^{(1)} \leq -(b_{13}')^{(1)} + (b_{14}')^{(1)} - (b_{13}'')^{(1)}(G,t) - (b_{14}'')^{(1)}(G,t) \leq -(\tau_1)^{(1)}$

**Definition of** $(\nu_1)^{(1)}, (\nu_2)^{(1)}, (u_1)^{(1)}, (u_2)^{(1)}, \nu^{(1)}, u^{(1)}$ : 281

By $(\nu_1)^{(1)} > 0, (\nu_2)^{(1)} < 0$ and respectively $(u_1)^{(1)} > 0, (u_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_1)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_1)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(\bar{\nu}_1)^{(1)},, (\bar{\nu}_2)^{(1)}, (\bar{u}_1)^{(1)}, (\bar{u}_2)^{(1)}$ : 282

By $(\bar{\nu}_1)^{(1)} > 0, (\bar{\nu}_2)^{(1)} < 0$ and respectively $(\bar{u}_1)^{(1)} > 0, (\bar{u}_2)^{(1)} < 0$ the roots of the equations $(a_{14})^{(1)}\left(\nu^{(1)}\right)^2 + (\sigma_2)^{(1)}\nu^{(1)} - (a_{13})^{(1)} = 0$ and $(b_{14})^{(1)}\left(u^{(1)}\right)^2 + (\tau_2)^{(1)}u^{(1)} - (b_{13})^{(1)} = 0$

**Definition of** $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}, (\nu_0)^{(1)}$ :- 283

If we define $(m_1)^{(1)}, (m_2)^{(1)}, (\mu_1)^{(1)}, (\mu_2)^{(1)}$ by

$(m_2)^{(1)} = (\nu_0)^{(1)}, (m_1)^{(1)} = (\nu_1)^{(1)}, \ if\ (\nu_0)^{(1)} < (\nu_1)^{(1)}$

$(m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\bar{\nu}_1)^{(1)}, if\ (\nu_1)^{(1)} < (\nu_0)^{(1)} < (\bar{\nu}_1)^{(1)},$

and $\boxed{(\nu_0)^{(1)} = \frac{G_{13}^0}{G_{14}^0}}$

$( m_2)^{(1)} = (\nu_1)^{(1)}, (m_1)^{(1)} = (\nu_0)^{(1)},\ if\ (\bar{\nu}_1)^{(1)} < (\nu_0)^{(1)}$

and analogously 284

$(\mu_2)^{(1)} = (u_0)^{(1)}, (\mu_1)^{(1)} = (u_1)^{(1)},\ if\ (u_0)^{(1)} < (u_1)^{(1)}$

$(\mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (\bar{u}_1)^{(1)}\ , if\ (u_1)^{(1)} < (u_0)^{(1)} < (\bar{u}_1)^{(1)},$

and $\boxed{(u_0)^{(1)} = \frac{T_{13}^0}{T_{14}^0}}$

$( \mu_2)^{(1)} = (u_1)^{(1)}, (\mu_1)^{(1)} = (u_0)^{(1)}, if\ (\bar{u}_1)^{(1)} < (u_0)^{(1)}$ where $(u_1)^{(1)}, (\bar{u}_1)^{(1)}$

are defined

Then the solution of global equations satisfies the inequalities 285

$G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \leq G_{13}(t) \leq G_{13}^0 e^{(S_1)^{(1)} t}$

where $(p_i)^{(1)}$ is defined by equation

$\frac{1}{(m_1)^{(1)}} G_{13}^0 e^{((S_1)^{(1)} - (p_{13})^{(1)})t} \leq G_{14}(t) \leq \frac{1}{(m_2)^{(1)}} G_{13}^0 e^{(S_1)^{(1)} t}$

$\left( \frac{(a_{15})^{(1)} G_{13}^0}{(m_1)^{(1)}((S_1)^{(1)} - (p_{13})^{(1)} - (S_2)^{(1)})} \left[ e^{((S_1)^{(1)} - (p_{13})^{(1)})t} - e^{-(S_2)^{(1)} t} \right] + G_{15}^0 e^{-(S_2)^{(1)} t} \leq G_{15}(t) \leq \right.$ 286
$\left. \frac{(a_{15})^{(1)} G_{13}^0}{(m_2)^{(1)}((S_1)^{(1)} - (a'_{15})^{(1)})} [e^{(S_1)^{(1)} t} - e^{-(a'_{15})^{(1)} t}] + G_{15}^0 e^{-(a'_{15})^{(1)} t} \right)$

$\boxed{T_{13}^0 e^{(R_1)^{(1)} t} \leq T_{13}(t) \leq T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}}$ 287

$\frac{1}{(\mu_1)^{(1)}} T_{13}^0 e^{(R_1)^{(1)} t} \leq T_{13}(t) \leq \frac{1}{(\mu_2)^{(1)}} T_{13}^0 e^{((R_1)^{(1)} + (r_{13})^{(1)})t}$ 288

$\frac{(b_{15})^{(1)} T_{13}^0}{(\mu_1)^{(1)}((R_1)^{(1)} - (b'_{15})^{(1)})} \left[ e^{(R_1)^{(1)} t} - e^{-(b'_{15})^{(1)} t} \right] + T_{15}^0 e^{-(b'_{15})^{(1)} t} \leq T_{15}(t) \leq$ 289

$\frac{(a_{15})^{(1)} T_{13}^0}{(\mu_2)^{(1)}((R_1)^{(1)} + (r_{13})^{(1)} + (R_2)^{(1)})} \left[ e^{((R_1)^{(1)} + (r_{13})^{(1)})t} - e^{-(R_2)^{(1)} t} \right] + T_{15}^0 e^{-(R_2)^{(1)} t}$

**Definition of** $(S_1)^{(1)}, (S_2)^{(1)}, (R_1)^{(1)}, (R_2)^{(1)}$:- 290

Where $(S_1)^{(1)} = (a_{13})^{(1)} (m_2)^{(1)} - (a'_{13})^{(1)}$

$(S_2)^{(1)} = (a_{15})^{(1)} - (p_{15})^{(1)}$

$(R_1)^{(1)} = (b_{13})^{(1)} (\mu_2)^{(1)} - (b'_{13})^{(1)}$

$(R_2)^{(1)} = (b'_{15})^{(1)} - (r_{15})^{(1)}$

**Behavior of the solutions of equation** 291

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ : 292

$(\sigma_1)^{(2)}, (\sigma_2)^{(2)}, (\tau_1)^{(2)}, (\tau_2)^{(2)}$ four constants satisfying

$-(\sigma_2)^{(2)} \leq -(a'_{16})^{(2)} + (a'_{17})^{(2)} - (a''_{16})^{(2)}(T_{17}, t) + (a''_{17})^{(2)}(T_{17}, t) \leq -(\sigma_1)^{(2)}$ 293

$-(\tau_2)^{(2)} \leq -(b'_{16})^{(2)} + (b'_{17})^{(2)} - (b''_{16})^{(2)}\big((G_{19}), t\big) - (b''_{17})^{(2)}\big((G_{19}), t\big) \leq -(\tau_1)^{(2)}$ 294

**Definition of** $(\nu_1)^{(2)}, (\nu_2)^{(2)}, (u_1)^{(2)}, (u_2)^{(2)}$ : 295

By $(\nu_1)^{(2)} > 0$ , $(\nu_2)^{(2)} < 0$ and respectively $(u_1)^{(2)} > 0$ , $(u_2)^{(2)} < 0$ the roots 296

of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ 297

and $(b_{14})^{(2)}\big(u^{(2)}\big)^2 + (\tau_1)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ and 298

**Definition of** $(\bar{\nu}_1)^{(2)}, , (\bar{\nu}_2)^{(2)}, (\bar{u}_1)^{(2)}, (\bar{u}_2)^{(2)}$ : 299

By $(\bar{\nu}_1)^{(2)} > 0$ , $(\bar{\nu}_2)^{(2)} < 0$ and respectively $(\bar{u}_1)^{(2)} > 0$ , $(\bar{u}_2)^{(2)} < 0$ the 300

roots of the equations $(a_{17})^{(2)}\big(\nu^{(2)}\big)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)} = 0$ 301

and $(b_{17})^{(2)}\big(u^{(2)}\big)^2 + (\tau_2)^{(2)}u^{(2)} - (b_{16})^{(2)} = 0$ 302

**Definition of** $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ :- 303

If we define $(m_1)^{(2)}, (m_2)^{(2)}, (\mu_1)^{(2)}, (\mu_2)^{(2)}$ by 304

$(m_2)^{(2)} = (\nu_0)^{(2)}, (m_1)^{(2)} = (\nu_1)^{(2)}$, $\boldsymbol{if}\ (\nu_0)^{(2)} < (\nu_1)^{(2)}$ 305

$(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\bar{\nu}_1)^{(2)}$, $\boldsymbol{if}\ (\nu_1)^{(2)} < (\nu_0)^{(2)} < (\bar{\nu}_1)^{(2)}$, 306

and $\boxed{(\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}}$

$(m_2)^{(2)} = (\nu_1)^{(2)}, (m_1)^{(2)} = (\nu_0)^{(2)}$, $\boldsymbol{if}\ (\bar{\nu}_1)^{(2)} < (\nu_0)^{(2)}$ 307

and analogously 308

$(\mu_2)^{(2)} = (u_0)^{(2)}, (\mu_1)^{(2)} = (u_1)^{(2)}$, $\boldsymbol{if}\ (u_0)^{(2)} < (u_1)^{(2)}$

$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (\bar{u}_1)^{(2)}$, $\boldsymbol{if}\ (u_1)^{(2)} < (u_0)^{(2)} < (\bar{u}_1)^{(2)}$,

and $\boxed{(u_0)^{(2)} = \frac{T_{16}^0}{T_{17}^0}}$

$(\mu_2)^{(2)} = (u_1)^{(2)}, (\mu_1)^{(2)} = (u_0)^{(2)}$, $\boldsymbol{if}\ (\bar{u}_1)^{(2)} < (u_0)^{(2)}$ 309

Then the solution of global equations satisfies the inequalities 310

$G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \leq G_{16}(t) \leq G_{16}^0 e^{(S_1)^{(2)}t}$

$(p_i)^{(2)}$ is defined by equation

$\frac{1}{(m_1)^{(2)}} G_{16}^0 e^{((S_1)^{(2)} - (p_{16})^{(2)})t} \leq G_{17}(t) \leq \frac{1}{(m_2)^{(2)}} G_{16}^0 e^{(S_1)^{(2)}t}$ 311

$\Big(\frac{(a_{18})^{(2)} G_{16}^0}{(m_1)^{(2)}\big((S_1)^{(2)} - (p_{16})^{(2)} - (S_2)^{(2)}\big)}\Big[e^{((S_1)^{(2)} - (p_{16})^{(2)})t} - e^{-(S_2)^{(2)}t}\Big] + G_{18}^0 e^{-(S_2)^{(2)}t} \leq G_{18}(t) \leq$ 312

$\frac{(a_{18})^{(2)}G_{16}^{0}}{(m_2)^{(2)}((S_1)^{(2)}-(a'_{18})^{(2)})}[e^{(S_1)^{(2)}t} - e^{-(a'_{18})^{(2)}t}] + G_{18}^{0}e^{-(a'_{18})^{(2)}t})$

$\boxed{T_{16}^{0}e^{(R_1)^{(2)}t} \leq T_{16}(t) \leq T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}}$ 313

$\frac{1}{(\mu_1)^{(2)}}T_{16}^{0}e^{(R_1)^{(2)}t} \leq T_{16}(t) \leq \frac{1}{(\mu_2)^{(2)}}T_{16}^{0}e^{((R_1)^{(2)}+(r_{16})^{(2)})t}$ 314

$\frac{(b_{18})^{(2)}T_{16}^{0}}{(\mu_1)^{(2)}((R_1)^{(2)}-(b'_{18})^{(2)})}\left[e^{(R_1)^{(2)}t} - e^{-(b'_{18})^{(2)}t}\right] + T_{18}^{0}e^{-(b'_{18})^{(2)}t} \leq T_{18}(t) \leq$ 315

$\frac{(a_{18})^{(2)}T_{16}^{0}}{(\mu_2)^{(2)}((R_1)^{(2)}+(r_{16})^{(2)}+(R_2)^{(2)})}\left[e^{((R_1)^{(2)}+(r_{16})^{(2)})t} - e^{-(R_2)^{(2)}t}\right] + T_{18}^{0}e^{-(R_2)^{(2)}t}$

<u>**Definition of**</u> $(S_1)^{(2)}, (S_2)^{(2)}, (R_1)^{(2)}, (R_2)^{(2)}$:- 316

Where $(S_1)^{(2)} = (a_{16})^{(2)}(m_2)^{(2)} - (a'_{16})^{(2)}$ 317

$(S_2)^{(2)} = (a_{18})^{(2)} - (p_{18})^{(2)}$

$(R_1)^{(2)} = (b_{16})^{(2)}(\mu_2)^{(1)} - (b'_{16})^{(2)}$ 318

$(R_2)^{(2)} = (b'_{18})^{(2)} - (r_{18})^{(2)}$

<u>**Behavior of the solutions**</u> 319

<u>**Theorem 3:**</u> If we denote and define

<u>**Definition of**</u> $(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ :

$(\sigma_1)^{(3)}, (\sigma_2)^{(3)}, (\tau_1)^{(3)}, (\tau_2)^{(3)}$ four constants satisfying
$-(\sigma_2)^{(3)} \leq -(a'_{20})^{(3)} + (a'_{21})^{(3)} - (a''_{20})^{(3)}(T_{21}, t) + (a''_{21})^{(3)}(T_{21}, t) \leq -(\sigma_1)^{(3)}$

$-(\tau_2)^{(3)} \leq -(b'_{20})^{(3)} + (b'_{21})^{(3)} - (b''_{20})^{(3)}(G_{23}, t) - (b''_{21})^{(3)}((G_{23}), t) \leq -(\tau_1)^{(3)}$

<u>**Definition of**</u> $(\nu_1)^{(3)}, (\nu_2)^{(3)}, (u_1)^{(3)}, (u_2)^{(3)}$ : 320

By $(\nu_1)^{(3)} > 0, (\nu_2)^{(3)} < 0$ and respectively $(u_1)^{(3)} > 0, (u_2)^{(3)} < 0$ the roots of the equations
$(a_{21})^{(3)}(\nu^{(3)})^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$
and $(b_{21})^{(3)}(u^{(3)})^2 + (\tau_1)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$ and

By $(\bar{\nu}_1)^{(3)} > 0, (\bar{\nu}_2)^{(3)} < 0$ and respectively $(\bar{u}_1)^{(3)} > 0, (\bar{u}_2)^{(3)} < 0$ the

roots of the equations $(a_{21})^{(3)}(\nu^{(3)})^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)} = 0$

and $(b_{21})^{(3)}(u^{(3)})^2 + (\tau_2)^{(3)}u^{(3)} - (b_{20})^{(3)} = 0$

<u>**Definition of**</u> $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ :- 321

If we define $(m_1)^{(3)}, (m_2)^{(3)}, (\mu_1)^{(3)}, (\mu_2)^{(3)}$ by
$(m_2)^{(3)} = (\nu_0)^{(3)}, (m_1)^{(3)} = (\nu_1)^{(3)}$, ***if*** $(\nu_0)^{(3)} < (\nu_1)^{(3)}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\bar{\nu}_1)^{(3)}$, ***if*** $(\nu_1)^{(3)} < (\nu_0)^{(3)} < (\bar{\nu}_1)^{(3)}$, and $\boxed{(\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}}$

$(m_2)^{(3)} = (\nu_1)^{(3)}, (m_1)^{(3)} = (\nu_0)^{(3)}, \ \boldsymbol{if} \ (\bar{\nu}_1)^{(3)} < (\nu_0)^{(3)}$

and analogously 322

$(\mu_2)^{(3)} = (u_0)^{(3)}, (\mu_1)^{(3)} = (u_1)^{(3)}, \ \boldsymbol{if} \ (u_0)^{(3)} < (u_1)^{(3)}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (\bar{u}_1)^{(3)}, \boldsymbol{if} \ (u_1)^{(3)} < (u_0)^{(3)} < (\bar{u}_1)^{(3)},$ and $\boxed{(u_0)^{(3)} = \frac{T_{20}^0}{T_{21}^0}}$

$(\mu_2)^{(3)} = (u_1)^{(3)}, (\mu_1)^{(3)} = (u_0)^{(3)}, \ \boldsymbol{if} \ (\bar{u}_1)^{(3)} < (u_0)^{(3)}$

Then the solution of global equations satisfies the inequalities

$G_{20}^0 e^{\left((S_1)^{(3)}-(p_{20})^{(3)}\right)t} \leq G_{20}(t) \leq G_{20}^0 e^{(S_1)^{(3)}t}$

$(p_i)^{(3)}$ is defined by equation

$\frac{1}{(m_1)^{(3)}} G_{20}^0 e^{\left((S_1)^{(3)}-(p_{20})^{(3)}\right)t} \leq G_{21}(t) \leq \frac{1}{(m_2)^{(3)}} G_{20}^0 e^{(S_1)^{(3)}t}$ 323

$\left(\frac{(a_{22})^{(3)}G_{20}^0}{(m_1)^{(3)}\left((S_1)^{(3)}-(p_{20})^{(3)}-(S_2)^{(3)}\right)}\left[e^{\left((S_1)^{(3)}-(p_{20})^{(3)}\right)t} - e^{-(S_2)^{(3)}t}\right] + G_{22}^0 e^{-(S_2)^{(3)}t} \leq G_{22}(t) \leq \right.$ 324
$\left.\frac{(a_{22})^{(3)}G_{20}^0}{(m_2)^{(3)}\left((S_1)^{(3)}-(a_{22}')^{(3)}\right)}\left[e^{(S_1)^{(3)}t} - e^{-(a_{22}')^{(3)}t}\right] + G_{22}^0 e^{-(a_{22}')^{(3)}t}\right)$

$\boxed{T_{20}^0 e^{(R_1)^{(3)}t} \leq T_{20}(t) \leq T_{20}^0 e^{\left((R_1)^{(3)}+(r_{20})^{(3)}\right)t}}$ 325

$\frac{1}{(\mu_1)^{(3)}} T_{20}^0 e^{(R_1)^{(3)}t} \leq T_{20}(t) \leq \frac{1}{(\mu_2)^{(3)}} T_{20}^0 e^{\left((R_1)^{(3)}+(r_{20})^{(3)}\right)t}$ 326

$\frac{(b_{22})^{(3)}T_{20}^0}{(\mu_1)^{(3)}\left((R_1)^{(3)}-(b_{22}')^{(3)}\right)}\left[e^{(R_1)^{(3)}t} - e^{-(b_{22}')^{(3)}t}\right] + T_{22}^0 e^{-(b_{22}')^{(3)}t} \leq T_{22}(t) \leq$ 327

$\frac{(a_{22})^{(3)}T_{20}^0}{(\mu_2)^{(3)}\left((R_1)^{(3)}+(r_{20})^{(3)}+(R_2)^{(3)}\right)}\left[e^{\left((R_1)^{(3)}+(r_{20})^{(3)}\right)t} - e^{-(R_2)^{(3)}t}\right] + T_{22}^0 e^{-(R_2)^{(3)}t}$

<u>**Definition of**</u> $(S_1)^{(3)}, (S_2)^{(3)}, (R_1)^{(3)}, (R_2)^{(3)}$:- 328

Where $(S_1)^{(3)} = (a_{20})^{(3)}(m_2)^{(3)} - (a_{20}')^{(3)}$

$(S_2)^{(3)} = (a_{22})^{(3)} - (p_{22})^{(3)}$

$(R_1)^{(3)} = (b_{20})^{(3)}(\mu_2)^{(3)} - (b_{20}')^{(3)}$

$(R_2)^{(3)} = (b_{22}')^{(3)} - (r_{22})^{(3)}$

<u>**Behavior of the solutions of equation**</u>

<u>**Theorem:**</u> If we denote and define

<u>**Definition of**</u> $(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ :

$(\sigma_1)^{(4)}, (\sigma_2)^{(4)}, (\tau_1)^{(4)}, (\tau_2)^{(4)}$ four constants satisfying

$$-(\sigma_2)^{(4)} \leq -(a'_{24})^{(4)} + (a'_{25})^{(4)} - (a''_{24})^{(4)}(T_{25}, t) + (a''_{25})^{(4)}(T_{25}, t) \leq -(\sigma_1)^{(4)}$$

$$-(\tau_2)^{(4)} \leq -(b'_{24})^{(4)} + (b'_{25})^{(4)} - (b''_{24})^{(4)}\big((G_{27}), t\big) - (b''_{25})^{(4)}\big((G_{27}), t\big) \leq -(\tau_1)^{(4)}$$

**Definition of** $(\nu_1)^{(4)}, (\nu_2)^{(4)}, (u_1)^{(4)}, (u_2)^{(4)}, \nu^{(4)}, u^{(4)}$ : 329

By $(\nu_1)^{(4)} > 0, (\nu_2)^{(4)} < 0$ and respectively $(u_1)^{(4)} > 0, (u_2)^{(4)} < 0$ the roots of the equations $(a_{25})^{(4)}\big(\nu^{(4)}\big)^2 + (\sigma_1)^{(4)}\nu^{(4)} - (a_{24})^{(4)} = 0$

and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_1)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(4)},, (\bar{\nu}_2)^{(4)}, (\bar{u}_1)^{(4)}, (\bar{u}_2)^{(4)}$ : 330

By $(\bar{\nu}_1)^{(4)} > 0, (\bar{\nu}_2)^{(4)} < 0$ and respectively $(\bar{u}_1)^{(4)} > 0, (\bar{u}_2)^{(4)} < 0$ the

roots of the equations $(a_{25})^{(4)}\big(\nu^{(4)}\big)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)} = 0$

and $(b_{25})^{(4)}\big(u^{(4)}\big)^2 + (\tau_2)^{(4)}u^{(4)} - (b_{24})^{(4)} = 0$

**Definition of** $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}, (\nu_0)^{(4)}$ :-

If we define $(m_1)^{(4)}, (m_2)^{(4)}, (\mu_1)^{(4)}, (\mu_2)^{(4)}$ by

$(m_2)^{(4)} = (\nu_0)^{(4)}, (m_1)^{(4)} = (\nu_1)^{(4)},$ ***if*** $(\nu_0)^{(4)} < (\nu_1)^{(4)}$

$(m_2)^{(4)} = (\nu_1)^{(4)}, (m_1)^{(4)} = (\bar{\nu}_1)^{(4)},$ ***if*** $(\nu_4)^{(4)} < (\nu_0)^{(4)} < (\bar{\nu}_1)^{(4)}$,

and $\boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$

$(m_2)^{(4)} = (\nu_4)^{(4)}, (m_1)^{(4)} = (\nu_0)^{(4)},$ ***if*** $(\bar{\nu}_4)^{(4)} < (\nu_0)^{(4)}$

and analogously

$(\mu_2)^{(4)} = (u_0)^{(4)}, (\mu_1)^{(4)} = (u_1)^{(4)},$ ***if*** $(u_0)^{(4)} < (u_1)^{(4)}$ 331

$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (\bar{u}_1)^{(4)},$ ***if*** $(u_1)^{(4)} < (u_0)^{(4)} < (\bar{u}_1)^{(4)}$,

and $\boxed{(u_0)^{(4)} = \frac{T_{24}^0}{T_{25}^0}}$

$(\mu_2)^{(4)} = (u_1)^{(4)}, (\mu_1)^{(4)} = (u_0)^{(4)},$ ***if*** $(\bar{u}_1)^{(4)} < (u_0)^{(4)}$ where $(u_1)^{(4)}, (\bar{u}_1)^{(4)}$

Then the solution of global equations satisfies the inequalities

$$G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \leq G_{24}(t) \leq G_{24}^0 e^{(S_1)^{(4)}t}$$ 332

where $(p_i)^{(4)}$ is defined by equation

$$\frac{1}{(m_1)^{(4)}} G_{24}^0 e^{((S_1)^{(4)}-(p_{24})^{(4)})t} \leq G_{25}(t) \leq \frac{1}{(m_2)^{(4)}} G_{24}^0 e^{(S_1)^{(4)}t}$$

333

$$\left(\frac{(a_{26})^{(4)} G_{24}^0}{(m_1)^{(4)}((S_1)^{(4)}-(p_{24})^{(4)}-(S_2)^{(4)})}\left[e^{((S_1)^{(4)}-(p_{24})^{(4)})t} - e^{-(S_2)^{(4)}t}\right] + G_{26}^0 e^{-(S_2)^{(4)}t} \leq G_{26}(t) \leq \frac{(a_{26})^{(4)} G_{24}^0}{(m_2)^{(4)}((S_1)^{(4)}-(a'_{26})^{(4)})}\left[e^{(S_1)^{(4)}t} - e^{-(a'_{26})^{(4)}t}\right] + G_{26}^0 e^{-(a'_{26})^{(4)}t}\right)$$ 334

$$\boxed{T_{24}^0 e^{(R_1)^{(4)}t} \leq T_{24}(t) \leq T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}}$$

$$\frac{1}{(\mu_1)^{(4)}} T_{24}^0 e^{(R_1)^{(4)}t} \leq T_{24}(t) \leq \frac{1}{(\mu_2)^{(4)}} T_{24}^0 e^{((R_1)^{(4)}+(r_{24})^{(4)})t}$$ 335

$$\frac{(b_{26})^{(4)} T_{24}^0}{(\mu_1)^{(4)}((R_1)^{(4)}-(b'_{26})^{(4)})}\left[e^{(R_1)^{(4)}t} - e^{-(b'_{26})^{(4)}t}\right] + T_{26}^0 e^{-(b'_{26})^{(4)}t} \leq T_{26}(t) \leq$$ 336

$$\frac{(a_{26})^{(4)} T_{24}^0}{(\mu_2)^{(4)}((R_1)^{(4)}+(r_{24})^{(4)}+(R_2)^{(4)})}\left[e^{((R_1)^{(4)}+(r_{24})^{(4)})t} - e^{-(R_2)^{(4)}t}\right] + T_{26}^0 e^{-(R_2)^{(4)}t}$$

<u>**Definition of**</u> $(S_1)^{(4)}, (S_2)^{(4)}, (R_1)^{(4)}, (R_2)^{(4)}$:- 337

Where $(S_1)^{(4)} = (a_{24})^{(4)}(m_2)^{(4)} - (a'_{24})^{(4)}$

$(S_2)^{(4)} = (a_{26})^{(4)} - (p_{26})^{(4)}$

$(R_1)^{(4)} = (b_{24})^{(4)}(\mu_2)^{(4)} - (b'_{24})^{(4)}$

$(R_2)^{(4)} = (b'_{26})^{(4)} - (r_{26})^{(4)}$

<u>**Behavior of the solutions of equation**</u> 338

<u>**Theorem 2:**</u> If we denote and define

<u>**Definition of**</u> $(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ :

$(\sigma_1)^{(5)}, (\sigma_2)^{(5)}, (\tau_1)^{(5)}, (\tau_2)^{(5)}$ four constants satisfying
$-(\sigma_2)^{(5)} \leq -(a'_{28})^{(5)} + (a'_{29})^{(5)} - (a''_{28})^{(5)}(T_{29}, t) + (a''_{29})^{(5)}(T_{29}, t) \leq -(\sigma_1)^{(5)}$

$-(\tau_2)^{(5)} \leq -(b'_{28})^{(5)} + (b'_{29})^{(5)} - (b''_{28})^{(5)}((G_{31}), t) - (b''_{29})^{(5)}((G_{31}), t) \leq -(\tau_1)^{(5)}$

<u>**Definition of**</u> $(\nu_1)^{(5)}, (\nu_2)^{(5)}, (u_1)^{(5)}, (u_2)^{(5)}, \nu^{(5)}, u^{(5)}$ : 339

By $(\nu_1)^{(5)} > 0, (\nu_2)^{(5)} < 0$ and respectively $(u_1)^{(5)} > 0, (u_2)^{(5)} < 0$ the roots of the equations
$(a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)} = 0$

and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_1)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(5)},, (\bar{\nu}_2)^{(5)}, (\bar{u}_1)^{(5)}, (\bar{u}_2)^{(5)}$ : 340

By $(\bar{\nu}_1)^{(5)} > 0$ , $(\bar{\nu}_2)^{(5)} < 0$ and respectively $(\bar{u}_1)^{(5)} > 0$ , $(\bar{u}_2)^{(5)} < 0$ the roots of the equations $(a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)} = 0$

and $(b_{29})^{(5)}\left(u^{(5)}\right)^2 + (\tau_2)^{(5)}u^{(5)} - (b_{28})^{(5)} = 0$

**Definition of** $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)}, (\mu_2)^{(5)}, (\nu_0)^{(5)}$ :-

If we define $(m_1)^{(5)}$ , $(m_2)^{(5)}$ , $(\mu_1)^{(5)}, (\mu_2)^{(5)}$ by

$(m_2)^{(5)} = (\nu_0)^{(5)}, (m_1)^{(5)} = (\nu_1)^{(5)},$ ***if*** $(\nu_0)^{(5)} < (\nu_1)^{(5)}$

$(m_2)^{(5)} = (\nu_1)^{(5)}, (m_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ , ***if*** $(\nu_1)^{(5)} < (\nu_0)^{(5)} < (\bar{\nu}_1)^{(5)},$

and $\boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$

$(m_2)^{(5)} = (\nu_1)^{(5)}, (m_1)^{(5)} = (\nu_0)^{(5)},$ ***if*** $(\bar{\nu}_1)^{(5)} < (\nu_0)^{(5)}$

and analogously 341

$(\mu_2)^{(5)} = (u_0)^{(5)}, (\mu_1)^{(5)} = (u_1)^{(5)},$ ***if*** $(u_0)^{(5)} < (u_1)^{(5)}$

$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (\bar{u}_1)^{(5)}$ , ***if*** $(u_1)^{(5)} < (u_0)^{(5)} < (\bar{u}_1)^{(5)},$

and $\boxed{(u_0)^{(5)} = \frac{T_{28}^0}{T_{29}^0}}$

$(\mu_2)^{(5)} = (u_1)^{(5)}, (\mu_1)^{(5)} = (u_0)^{(5)},$ ***if*** $(\bar{u}_1)^{(5)} < (u_0)^{(5)}$ where $(u_1)^{(5)}, (\bar{u}_1)^{(5)}$

Then the solution of global equations satisfies the inequalities 342

$$G_{28}^0 e^{\left((S_1)^{(5)}-(p_{28})^{(5)}\right)t} \leq G_{28}(t) \leq G_{28}^0 e^{(S_1)^{(5)}t}$$

where $(p_i)^{(5)}$ is defined by equation

$$\frac{1}{(m_5)^{(5)}} G_{28}^0 e^{\left((S_1)^{(5)}-(p_{28})^{(5)}\right)t} \leq G_{29}(t) \leq \frac{1}{(m_2)^{(5)}} G_{28}^0 e^{(S_1)^{(5)}t}$$ 343

$$\left(\frac{(a_{30})^{(5)}G_{28}^0}{(m_1)^{(5)}\left((S_1)^{(5)}-(p_{28})^{(5)}-(S_2)^{(5)}\right)}\left[e^{\left((S_1)^{(5)}-(p_{28})^{(5)}\right)t} - e^{-(S_2)^{(5)}t}\right] + G_{30}^0 e^{-(S_2)^{(5)}t} \leq G_{30}(t) \leq \frac{(a_{30})^{(5)}G_{28}^0}{(m_2)^{(5)}\left((S_1)^{(5)}-(a'_{30})^{(5)}\right)}\left[e^{(S_1)^{(5)}t} - e^{-(a'_{30})^{(5)}t}\right] + G_{30}^0 e^{-(a'_{30})^{(5)}t}\right)$$ 344

$$\boxed{T_{28}^0 e^{(R_1)^{(5)}t} \leq T_{28}(t) \leq T_{28}^0 e^{\left((R_1)^{(5)}+(r_{28})^{(5)}\right)t}}$$ 345

$$\frac{1}{(\mu_1)^{(5)}} T_{28}^0 e^{(R_1)^{(5)}t} \leq T_{28}(t) \leq \frac{1}{(\mu_2)^{(5)}} T_{28}^0 e^{((R_1)^{(5)}+(r_{28})^{(5)})t}$$ 346

$$\frac{(b_{30})^{(5)}T_{28}^0}{(\mu_1)^{(5)}((R_1)^{(5)}-(b'_{30})^{(5)})}\left[e^{(R_1)^{(5)}t} - e^{-(b'_{30})^{(5)}t}\right] + T_{30}^0 e^{-(b'_{30})^{(5)}t} \leq T_{30}(t) \leq$$ 347

$$\frac{(a_{30})^{(5)}T_{28}^0}{(\mu_2)^{(5)}((R_1)^{(5)}+(r_{28})^{(5)}+(R_2)^{(5)})}\left[e^{((R_1)^{(5)}+(r_{28})^{(5)})t} - e^{-(R_2)^{(5)}t}\right] + T_{30}^0 e^{-(R_2)^{(5)}t}$$

Definition of $(S_1)^{(5)}, (S_2)^{(5)}, (R_1)^{(5)}, (R_2)^{(5)}$:- 348

Where $(S_1)^{(5)} = (a_{28})^{(5)}(m_2)^{(5)} - (a'_{28})^{(5)}$

$(S_2)^{(5)} = (a_{30})^{(5)} - (p_{30})^{(5)}$

$(R_1)^{(5)} = (b_{28})^{(5)}(\mu_2)^{(5)} - (b'_{28})^{(5)}$

$(R_2)^{(5)} = (b'_{30})^{(5)} - (r_{30})^{(5)}$

Behavior of the solutions of equation 349

Theorem 2: If we denote and define

Definition of $(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ :

$(\sigma_1)^{(6)}, (\sigma_2)^{(6)}, (\tau_1)^{(6)}, (\tau_2)^{(6)}$ four constants satisfying

$$-(\sigma_2)^{(6)} \leq -(a'_{32})^{(6)} + (a'_{33})^{(6)} - (a''_{32})^{(6)}(T_{33}, t) + (a''_{33})^{(6)}(T_{33}, t) \leq -(\sigma_1)^{(6)}$$

$$-(\tau_2)^{(6)} \leq -(b'_{32})^{(6)} + (b'_{33})^{(6)} - (b''_{32})^{(6)}((G_{35}), t) - (b''_{33})^{(6)}((G_{35}), t) \leq -(\tau_1)^{(6)}$$

Definition of $(\nu_1)^{(6)}, (\nu_2)^{(6)}, (u_1)^{(6)}, (u_2)^{(6)}, \nu^{(6)}, u^{(6)}$ : 350

By $(\nu_1)^{(6)} > 0, (\nu_2)^{(6)} < 0$ and respectively $(u_1)^{(6)} > 0, (u_2)^{(6)} < 0$ the roots of the equations $(a_{33})^{(6)}(\nu^{(6)})^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$
and $(b_{33})^{(6)}(u^{(6)})^2 + (\tau_1)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$ and

Definition of $(\bar{\nu}_1)^{(6)},, (\bar{\nu}_2)^{(6)}, (\bar{u}_1)^{(6)}, (\bar{u}_2)^{(6)}$ : 351

By $(\bar{\nu}_1)^{(6)} > 0, (\bar{\nu}_2)^{(6)} < 0$ and respectively $(\bar{u}_1)^{(6)} > 0, (\bar{u}_2)^{(6)} < 0$ the

roots of the equations $(a_{33})^{(6)}(\nu^{(6)})^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)} = 0$

and $(b_{33})^{(6)}(u^{(6)})^2 + (\tau_2)^{(6)}u^{(6)} - (b_{32})^{(6)} = 0$

Definition of $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}, (\nu_0)^{(6)}$ :-

If we define $(m_1)^{(6)}, (m_2)^{(6)}, (\mu_1)^{(6)}, (\mu_2)^{(6)}$ by

$(m_2)^{(6)} = (\nu_0)^{(6)}, (m_1)^{(6)} = (\nu_1)^{(6)},\ \boldsymbol{if}\ (\nu_0)^{(6)} < (\nu_1)^{(6)}$

$(m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\bar{\nu}_6)^{(6)}\ , \boldsymbol{if}\ (\nu_1)^{(6)} < (\nu_0)^{(6)} < (\bar{\nu}_1)^{(6)},$

and $\boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$

$(\ m_2)^{(6)} = (\nu_1)^{(6)}, (m_1)^{(6)} = (\nu_0)^{(6)},\ \boldsymbol{if}\ (\bar{\nu}_1)^{(6)} < (\nu_0)^{(6)}$

and analogously 352

$(\mu_2)^{(6)} = (u_0)^{(6)}, (\mu_1)^{(6)} = (u_1)^{(6)},\ \boldsymbol{if}\ (u_0)^{(6)} < (u_1)^{(6)}$

$(\mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (\bar{u}_1)^{(6)}\ , \boldsymbol{if}\ (u_1)^{(6)} < (u_0)^{(6)} < (\bar{u}_1)^{(6)},$

and $\boxed{(u_0)^{(6)} = \frac{T_{32}^0}{T_{33}^0}}$

$(\ \mu_2)^{(6)} = (u_1)^{(6)}, (\mu_1)^{(6)} = (u_0)^{(6)}, \boldsymbol{if}\ (\bar{u}_1)^{(6)} < (u_0)^{(6)}$ where $(u_1)^{(6)}, (\bar{u}_1)^{(6)}$

Then the solution of global equations satisfies the inequalities 353

$G_{32}^0 e^{((S_1)^{(6)}-(p_{32})^{(6)})t} \leq G_{32}(t) \leq G_{32}^0 e^{(S_1)^{(6)}t}$

where $(p_i)^{(6)}$ is defined by equation

$\frac{1}{(m_1)^{(6)}} G_{32}^0 e^{((S_1)^{(6)}-(p_{32})^{(6)})t} \leq G_{33}(t) \leq \frac{1}{(m_2)^{(6)}} G_{32}^0 e^{(S_1)^{(6)}t}$ 354

$\left(\frac{(a_{34})^{(6)} G_{32}^0}{(m_1)^{(6)}((S_1)^{(6)}-(p_{32})^{(6)}-(S_2)^{(6)})}\left[e^{((S_1)^{(6)}-(p_{32})^{(6)})t} - e^{-(S_2)^{(6)}t}\right] + G_{34}^0 e^{-(S_2)^{(6)}t} \leq G_{34}(t) \leq\right.$ 355
$\left.\frac{(a_{34})^{(6)} G_{32}^0}{(m_2)^{(6)}((S_1)^{(6)}-(a'_{34})^{(6)})}\left[e^{(S_1)^{(6)}t} - e^{-(a'_{34})^{(6)}t}\right] + G_{34}^0 e^{-(a'_{34})^{(6)}t}\right)$

$\boxed{T_{32}^0 e^{(R_1)^{(6)}t} \leq T_{32}(t) \leq T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}}$ 356

$\frac{1}{(\mu_1)^{(6)}} T_{32}^0 e^{(R_1)^{(6)}t} \leq T_{32}(t) \leq \frac{1}{(\mu_2)^{(6)}} T_{32}^0 e^{((R_1)^{(6)}+(r_{32})^{(6)})t}$ 357

$\frac{(b_{34})^{(6)} T_{32}^0}{(\mu_1)^{(6)}((R_1)^{(6)}-(b'_{34})^{(6)})}\left[e^{(R_1)^{(6)}t} - e^{-(b'_{34})^{(6)}t}\right] + T_{34}^0 e^{-(b'_{34})^{(6)}t} \leq T_{34}(t) \leq$ 358

$\frac{(a_{34})^{(6)} T_{32}^0}{(\mu_2)^{(6)}((R_1)^{(6)}+(r_{32})^{(6)}+(R_2)^{(6)})}\left[e^{((R_1)^{(6)}+(r_{32})^{(6)})t} - e^{-(R_2)^{(6)}t}\right] + T_{34}^0 e^{-(R_2)^{(6)}t}$

<u>Definition of</u> $(S_1)^{(6)}, (S_2)^{(6)}, (R_1)^{(6)}, (R_2)^{(6)}$:- 359

Where $(S_1)^{(6)} = (a_{32})^{(6)}(m_2)^{(6)} - (a'_{32})^{(6)}$

$(S_2)^{(6)} = (a_{34})^{(6)} - (p_{34})^{(6)}$

$(R_1)^{(6)} = (b_{32})^{(6)}(\mu_2)^{(6)} - (b'_{32})^{(6)}$

$(R_2)^{(6)} = (b'_{34})^{(6)} - (r_{34})^{(6)}$

360

**Behavior of the solutions of equation**

**Theorem 2:** If we denote and define

**Definition of** $(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ :

$(\sigma_1)^{(7)}, (\sigma_2)^{(7)}, (\tau_1)^{(7)}, (\tau_2)^{(7)}$ four constants satisfying

$-(\sigma_2)^{(7)} \le -(a'_{36})^{(7)} + (a'_{37})^{(7)} - (a''_{36})^{(7)}(T_{37}, t) + (a''_{37})^{(7)}(T_{37}, t) \le -(\sigma_1)^{(7)}$

$-(\tau_2)^{(7)} \le -(b'_{36})^{(7)} + (b'_{37})^{(7)} - (b''_{36})^{(7)}\big((G_{39}), t\big) - (b''_{37})^{(7)}\big((G_{39}), t\big) \le -(\tau_1)^{(7)}$

**Definition of** $(\nu_1)^{(7)}, (\nu_2)^{(7)}, (u_1)^{(7)}, (u_2)^{(7)}, \nu^{(7)}, u^{(7)}$ : 361

By $(\nu_1)^{(7)} > 0, (\nu_2)^{(7)} < 0$ and respectively $(u_1)^{(7)} > 0, (u_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(\nu^{(7)}\big)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$
and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_1)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$ and

**Definition of** $(\bar{\nu}_1)^{(7)},, (\bar{\nu}_2)^{(7)}, (\bar{u}_1)^{(7)}, (\bar{u}_2)^{(7)}$ : 362

By $(\bar{\nu}_1)^{(7)} > 0, (\bar{\nu}_2)^{(7)} < 0$ and respectively $(\bar{u}_1)^{(7)} > 0, (\bar{u}_2)^{(7)} < 0$ the roots of the equations $(a_{37})^{(7)}\big(\nu^{(7)}\big)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)} = 0$
and $(b_{37})^{(7)}\big(u^{(7)}\big)^2 + (\tau_2)^{(7)}u^{(7)} - (b_{36})^{(7)} = 0$

**Definition of** $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}, (\nu_0)^{(7)}$ :-

If we define $(m_1)^{(7)}, (m_2)^{(7)}, (\mu_1)^{(7)}, (\mu_2)^{(7)}$ by

$(m_2)^{(7)} = (\nu_0)^{(7)}, (m_1)^{(7)} = (\nu_1)^{(7)}, \; \boldsymbol{if} \; (\nu_0)^{(7)} < (\nu_1)^{(7)}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\bar{\nu}_1)^{(7)}, \boldsymbol{if} \; (\nu_1)^{(7)} < (\nu_0)^{(7)} < (\bar{\nu}_1)^{(7)}$,
and $\boxed{(\nu_0)^{(7)} = \frac{G^0_{36}}{G^0_{37}}}$

$(m_2)^{(7)} = (\nu_1)^{(7)}, (m_1)^{(7)} = (\nu_0)^{(7)}, \; \boldsymbol{if} \; (\bar{\nu}_1)^{(7)} < (\nu_0)^{(7)}$

and analogously 363

$(\mu_2)^{(7)} = (u_0)^{(7)}, (\mu_1)^{(7)} = (u_1)^{(7)}, \; \boldsymbol{if} \; (u_0)^{(7)} < (u_1)^{(7)}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (\bar{u}_1)^{(7)}, \boldsymbol{if} \; (u_1)^{(7)} < (u_0)^{(7)} < (\bar{u}_1)^{(7)}$,
and $\boxed{(u_0)^{(7)} = \frac{T^0_{36}}{T^0_{37}}}$

$(\mu_2)^{(7)} = (u_1)^{(7)}, (\mu_1)^{(7)} = (u_0)^{(7)}, \boldsymbol{if} \; (\bar{u}_1)^{(7)} < (u_0)^{(7)}$ where $(u_1)^{(7)}, (\bar{u}_1)^{(7)}$

Then the solution of global equations satisfies the inequalities 364

$G_{36}^0 e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \leq G_{36}(t) \leq G_{36}^0 e^{(S_1)^{(7)}t}$

where $(p_i)^{(7)}$ is defined by equation

$\frac{1}{(m_7)^{(7)}} G_{36}^0 e^{((S_1)^{(7)}-(p_{36})^{(7)})t} \leq G_{37}(t) \leq \frac{1}{(m_2)^{(7)}} G_{36}^0 e^{(S_1)^{(7)}t}$ 365

$\Big(\frac{(a_{38})^{(7)} G_{36}^0}{(m_1)^{(7)}((S_1)^{(7)}-(p_{36})^{(7)}-(S_2)^{(7)})}\left[e^{((S_1)^{(7)}-(p_{36})^{(7)})t} - e^{-(S_2)^{(7)}t}\right] + G_{38}^0 e^{-(S_2)^{(7)}t} \leq G_{38}(t) \leq$ 366
$\frac{(a_{38})^{(7)} G_{36}^0}{(m_2)^{(7)}((S_1)^{(7)}-(a'_{38})^{(7)})}\left[e^{(S_1)^{(7)}t} - e^{-(a'_{38})^{(7)}t}\right] + G_{38}^0 e^{-(a'_{38})^{(7)}t}\Big)$

$\boxed{T_{36}^0 e^{(R_1)^{(7)}t} \leq T_{36}(t) \leq T_{36}^0 e^{((R_1)^{(7)}+(r_{36})^{(7)})t}}$ 367

$\frac{1}{(\mu_1)^{(7)}} T_{36}^0 e^{(R_1)^{(7)}t} \leq T_{36}(t) \leq \frac{1}{(\mu_2)^{(7)}} T_{36}^0 e^{((R_1)^{(7)}+(r_{36})^{(7)})t}$ 368

$\frac{(b_{38})^{(7)} T_{36}^0}{(\mu_1)^{(7)}((R_1)^{(7)}-(b'_{38})^{(7)})}\left[e^{(R_1)^{(7)}t} - e^{-(b'_{38})^{(7)}t}\right] + T_{38}^0 e^{-(b'_{38})^{(7)}t} \leq T_{38}(t) \leq$ 369

$\frac{(a_{38})^{(7)} T_{36}^0}{(\mu_2)^{(7)}((R_1)^{(7)}+(r_{36})^{(7)}+(R_2)^{(7)})}\left[e^{((R_1)^{(7)}+(r_{36})^{(7)})t} - e^{-(R_2)^{(7)}t}\right] + T_{38}^0 e^{-(R_2)^{(7)}t}$

<u>**Definition of**</u> $(S_1)^{(7)}, (S_2)^{(7)}, (R_1)^{(7)}, (R_2)^{(7)}$:- 370

Where $(S_1)^{(7)} = (a_{36})^{(7)}(m_2)^{(7)} - (a'_{36})^{(7)}$

$(S_2)^{(7)} = (a_{38})^{(7)} - (p_{38})^{(7)}$

$(R_1)^{(7)} = (b_{36})^{(7)}(\mu_2)^{(7)} - (b'_{36})^{(7)}$

$(R_2)^{(7)} = (b'_{38})^{(7)} - (r_{38})^{(7)}$

<u>**Behavior of the solutions of equation**</u> 371

<u>**Theorem 2:**</u> If we denote and define

<u>**Definition of**</u> $(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ :

$(\sigma_1)^{(8)}, (\sigma_2)^{(8)}, (\tau_1)^{(8)}, (\tau_2)^{(8)}$ four constants satisfying

$-(\sigma_2)^{(8)} \leq -(a'_{40})^{(8)} + (a'_{41})^{(8)} - (a''_{40})^{(8)}(T_{41}, t) + (a''_{41})^{(8)}(T_{41}, t) \leq -(\sigma_1)^{(8)}$

$-(\tau_2)^{(8)} \leq -(b'_{40})^{(8)} + (b'_{41})^{(8)} - (b''_{40})^{(8)}\big((G_{43}), t\big) - (b''_{41})^{(8)}\big((G_{43}), t\big) \leq -(\tau_1)^{(8)}$

<u>**Definition of**</u> $(\nu_1)^{(8)}, (\nu_2)^{(8)}, (u_1)^{(8)}, (u_2)^{(8)}, \nu^{(8)}, u^{(8)}$ : 372

By $(\nu_1)^{(8)} > 0, (\nu_2)^{(8)} < 0$ and respectively $(u_1)^{(8)} > 0, (u_2)^{(8)} < 0$ the roots of the equations $(a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$
and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_1)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$ and

<u>**Definition of**</u> $(\bar{\nu}_1)^{(8)},, (\bar{\nu}_2)^{(8)}, (\bar{u}_1)^{(8)}, (\bar{u}_2)^{(8)}$ :

By $(\bar{\nu}_1)^{(8)} > 0$ , $(\bar{\nu}_2)^{(8)} < 0$ and respectively $(\bar{u}_1)^{(8)} > 0$ , $(\bar{u}_2)^{(8)} < 0$ the

roots of the equations $(a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)} = 0$

and $(b_{41})^{(8)}\left(u^{(8)}\right)^2 + (\tau_2)^{(8)}u^{(8)} - (b_{40})^{(8)} = 0$

<u>**Definition of**</u> $(m_1)^{(8)}$ , $(m_2)^{(8)}$ , $(\mu_1)^{(8)}, (\mu_2)^{(8)}, (\nu_0)^{(8)}$ :-

If we define $(m_1)^{(8)}$ , $(m_2)^{(8)}$ , $(\mu_1)^{(8)}, (\mu_2)^{(8)}$ by

$(m_2)^{(8)} = (\nu_0)^{(8)}, (m_1)^{(8)} = (\nu_1)^{(8)},$ ***if*** $(\nu_0)^{(8)} < (\nu_1)^{(8)}$

$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ , ***if*** $(\nu_1)^{(8)} < (\nu_0)^{(8)} < (\bar{\nu}_1)^{(8)}$,
and $\boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$

$(m_2)^{(8)} = (\nu_1)^{(8)}, (m_1)^{(8)} = (\nu_0)^{(8)},$ ***if*** $(\bar{\nu}_1)^{(8)} < (\nu_0)^{(8)}$

and analogously 374

$(\mu_2)^{(8)} = (u_0)^{(8)}, (\mu_1)^{(8)} = (u_1)^{(8)},$ ***if*** $(u_0)^{(8)} < (u_1)^{(8)}$

$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (\bar{u}_1)^{(8)}$ , ***if*** $(u_1)^{(8)} < (u_0)^{(8)} < (\bar{u}_1)^{(8)}$,
and $\boxed{(u_0)^{(8)} = \frac{T_{40}^0}{T_{41}^0}}$

$(\mu_2)^{(8)} = (u_1)^{(8)}, (\mu_1)^{(8)} = (u_0)^{(8)},$ ***if*** $(\bar{u}_1)^{(8)} < (u_0)^{(8)}$ where $(u_1)^{(8)}, (\bar{u}_1)^{(8)}$

Then the solution of global equations satisfies the inequalities 375

$$G_{40}^0 e^{((S_1)^{(8)}-(p_{40})^{(8)})t} \leq G_{40}(t) \leq G_{40}^0 e^{(S_1)^{(8)}t}$$

where $(p_i)^{(8)}$ is defined by equation

$$\frac{1}{(m_1)^{(8)}} G_{40}^0 e^{((S_1)^{(8)}-(p_{40})^{(8)})t} \leq G_{41}(t) \leq \frac{1}{(m_2)^{(8)}} G_{40}^0 e^{(S_1)^{(8)}t}$$ 376

$$\left(\frac{(a_{42})^{(8)}G_{40}^0}{(m_1)^{(8)}((S_1)^{(8)}-(p_{40})^{(8)}-(S_2)^{(8)})}\left[e^{((S_1)^{(8)}-(p_{40})^{(8)})t} - e^{-(S_2)^{(8)}t}\right] + G_{42}^0 e^{-(S_2)^{(8)}t} \leq G_{42}(t) \leq \frac{(a_{42})^{(8)}G_{40}^0}{(m_2)^{(8)}((S_1)^{(8)}-(a'_{42})^{(8)})}\left[e^{(S_1)^{(8)}t} - e^{-(a'_{42})^{(8)}t}\right] + G_{42}^0 e^{-(a'_{42})^{(8)}t}\right)$$ 377

$$\boxed{T_{40}^0 e^{(R_1)^{(8)}t} \leq T_{40}(t) \leq T_{40}^0 e^{((R_1)^{(8)}+(r_{40})^{(8)})t}}$$ 378

$$\frac{1}{(\mu_1)^{(8)}} T_{40}^0 e^{(R_1)^{(8)}t} \leq T_{40}(t) \leq \frac{1}{(\mu_2)^{(8)}} T_{40}^0 e^{((R_1)^{(8)}+(r_{40})^{(8)})t}$$ 379

$$\frac{(b_{42})^{(8)}T_{40}^0}{(\mu_1)^{(8)}((R_1)^{(8)}-(b'_{42})^{(8)})}\left[e^{(R_1)^{(8)}t} - e^{-(b'_{42})^{(8)}t}\right] + T_{42}^0 e^{-(b'_{42})^{(8)}t} \leq T_{42}(t) \leq$$ 380

$$\frac{(a_{42})^{(8)}T_{40}^0}{(\mu_2)^{(8)}((R_1)^{(8)}+(r_{40})^{(8)}+(R_2)^{(8)})}\left[e^{((R_1)^{(8)}+(r_{40})^{(8)})t} - e^{-(R_2)^{(8)}t}\right] + T_{42}^0 e^{-(R_2)^{(8)}t}$$

<u>**Definition of**</u> $(S_1)^{(8)}, (S_2)^{(8)}, (R_1)^{(8)}, (R_2)^{(8)}$:- 381

Where $(S_1)^{(8)} = (a_{40})^{(8)}(m_2)^{(8)} - (a'_{40})^{(8)}$

$(S_2)^{(8)} = (a_{42})^{(8)} - (p_{42})^{(8)}$

$(R_1)^{(8)} = (b_{40})^{(8)}(\mu_2)^{(8)} - (b'_{40})^{(8)}$

$(R_2)^{(8)} = (b'_{42})^{(8)} - (r_{42})^{(8)}$

<u>**Behavior of the solutions of equation 37 to 92**</u> 382

<u>**Theorem 2:**</u> If we denote and define

<u>**Definition of**</u> $(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ :

$(\sigma_1)^{(9)}, (\sigma_2)^{(9)}, (\tau_1)^{(9)}, (\tau_2)^{(9)}$ four constants satisfying

$-(\sigma_2)^{(9)} \leq -(a'_{44})^{(9)} + (a'_{45})^{(9)} - (a''_{44})^{(9)}(T_{45}, t) + (a''_{45})^{(9)}(T_{45}, t) \leq -(\sigma_1)^{(9)}$

$-(\tau_2)^{(9)} \leq -(b'_{44})^{(9)} + (b'_{45})^{(9)} - (b''_{44})^{(9)}\big((G_{47}), t\big) - (b''_{45})^{(9)}\big((G_{47}), t\big) \leq -(\tau_1)^{(9)}$

<u>**Definition of**</u> $(\nu_1)^{(9)}, (\nu_2)^{(9)}, (u_1)^{(9)}, (u_2)^{(9)}, \nu^{(9)}, u^{(9)}$ :

By $(\nu_1)^{(9)} > 0, (\nu_2)^{(9)} < 0$ and respectively $(u_1)^{(9)} > 0, (u_2)^{(9)} < 0$ the roots of the equations $(a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$

and $(b_{45})^{(9)}\left(u^{(9)}\right)^2 + (\tau_1)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$ and

<u>**Definition of**</u> $(\bar{\nu}_1)^{(9)},, (\bar{\nu}_2)^{(9)}, (\bar{u}_1)^{(9)}, (\bar{u}_2)^{(9)}$ :

By $(\bar{\nu}_1)^{(9)} > 0, (\bar{\nu}_2)^{(9)} < 0$ and respectively $(\bar{u}_1)^{(9)} > 0, (\bar{u}_2)^{(9)} < 0$ the

roots of the equations $(a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)} = 0$

and $(b_{45})^{(9)}\left(u^{(9)}\right)^2 + (\tau_2)^{(9)}u^{(9)} - (b_{44})^{(9)} = 0$

<u>**Definition of**</u> $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}, (\nu_0)^{(9)}$ :-

If we define $(m_1)^{(9)}, (m_2)^{(9)}, (\mu_1)^{(9)}, (\mu_2)^{(9)}$ by

$(m_2)^{(9)} = (\nu_0)^{(9)}, (m_1)^{(9)} = (\nu_1)^{(9)}, \ \boldsymbol{if} \ (\nu_0)^{(9)} < (\nu_1)^{(9)}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\bar{\nu}_1)^{(9)}, \boldsymbol{if} \ (\nu_1)^{(9)} < (\nu_0)^{(9)} < (\bar{\nu}_1)^{(9)},$

and $\boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$

$(m_2)^{(9)} = (\nu_1)^{(9)}, (m_1)^{(9)} = (\nu_0)^{(9)}, \ \boldsymbol{if} \ (\bar{\nu}_1)^{(9)} < (\nu_0)^{(9)}$

and analogously

$(\mu_2)^{(9)} = (u_0)^{(9)}, (\mu_1)^{(9)} = (u_1)^{(9)},\ \boldsymbol{if}\ (u_0)^{(9)} < (u_1)^{(9)}$

$(\mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (\bar{u}_1)^{(9)}\ , \boldsymbol{if}\ (u_1)^{(9)} < (u_0)^{(9)} < (\bar{u}_1)^{(9)},$

and $\boxed{(u_0)^{(9)} = \frac{T_{44}^0}{T_{45}^0}}$

$(\ \mu_2)^{(9)} = (u_1)^{(9)}, (\mu_1)^{(9)} = (u_0)^{(9)}, \boldsymbol{if}\ (\bar{u}_1)^{(9)} < (u_0)^{(9)}$ where $(u_1)^{(9)}, (\bar{u}_1)^{(9)}$

are defined by 59 and 69 respectively

Then the solution of 19,20,21,22,23 and 24 satisfies the inequalities

$$G_{44}^0 e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \leq G_{44}(t) \leq G_{44}^0 e^{(S_1)^{(9)}t}$$

where $(p_i)^{(9)}$ is defined by equation 45

$$\frac{1}{(m_9)^{(9)}} G_{44}^0 e^{((S_1)^{(9)}-(p_{44})^{(9)})t} \leq G_{45}(t) \leq \frac{1}{(m_2)^{(9)}} G_{44}^0 e^{(S_1)^{(9)}t}$$

$$\left( \frac{(a_{46})^{(9)} G_{44}^0}{(m_1)^{(9)}((S_1)^{(9)}-(p_{44})^{(9)}-(S_2)^{(9)})} \left[ e^{((S_1)^{(9)}-(p_{44})^{(9)})t} - e^{-(S_2)^{(9)}t} \right] + G_{46}^0 e^{-(S_2)^{(9)}t} \leq G_{46}(t) \leq \frac{(a_{46})^{(9)} G_{44}^0}{(m_2)^{(9)}((S_1)^{(9)}-(a_{46}')^{(9)})} \left[ e^{(S_1)^{(9)}t} - e^{-(a_{46}')^{(9)}t} \right] + G_{46}^0 e^{-(a_{46}')^{(9)}t} \right)$$

$$\boxed{T_{44}^0 e^{(R_1)^{(9)}t} \leq T_{44}(t) \leq T_{44}^0 e^{((R_1)^{(9)}+(r_{44})^{(9)})t}}$$

$$\frac{1}{(\mu_1)^{(9)}} T_{44}^0 e^{(R_1)^{(9)}t} \leq T_{44}(t) \leq \frac{1}{(\mu_2)^{(9)}} T_{44}^0 e^{((R_1)^{(9)}+(r_{44})^{(9)})t}$$

$$\frac{(b_{46})^{(9)} T_{44}^0}{(\mu_1)^{(9)}((R_1)^{(9)}-(b_{46}')^{(9)})} \left[ e^{(R_1)^{(9)}t} - e^{-(b_{46}')^{(9)}t} \right] + T_{46}^0 e^{-(b_{46}')^{(9)}t} \leq T_{46}(t) \leq$$

$$\frac{(a_{46})^{(9)} T_{44}^0}{(\mu_2)^{(9)}((R_1)^{(9)}+(r_{44})^{(9)}+(R_2)^{(9)})} \left[ e^{((R_1)^{(9)}+(r_{44})^{(9)})t} - e^{-(R_2)^{(9)}t} \right] + T_{46}^0 e^{-(R_2)^{(9)}t}$$

<u>**Definition of**</u> $(S_1)^{(9)}, (S_2)^{(9)}, (R_1)^{(9)}, (R_2)^{(9)}$:-

Where $(S_1)^{(9)} = (a_{44})^{(9)}(m_2)^{(9)} - (a_{44}')^{(9)}$

$(S_2)^{(9)} = (a_{46})^{(9)} - (p_{46})^{(9)}$

$(R_1)^{(9)} = (b_{44})^{(9)}(\mu_2)^{(9)} - (b_{44}')^{(9)}$

$(R_2)^{(9)} = (b_{46}')^{(9)} - (r_{46})^{(9)}$

<u>**Proof :**</u> From global equations we obtain 383

$$\frac{dv^{(1)}}{dt} = (a_{13})^{(1)} - \left( (a_{13}')^{(1)} - (a_{14}')^{(1)} + (a_{13}'')^{(1)}(T_{14}, t) \right) - (a_{14}'')^{(1)}(T_{14}, t) v^{(1)} - (a_{14})^{(1)} v^{(1)}$$

<u>**Definition of**</u> $v^{(1)}$ :- $\boxed{v^{(1)} = \frac{G_{13}}{G_{14}}}$

It follows

$$-\left((a_{14})^{(1)}\left(\nu^{(1)}\right)^2+(\sigma_2)^{(1)}\nu^{(1)}-(a_{13})^{(1)}\right)\leq\frac{d\nu^{(1)}}{dt}\leq-\left((a_{14})^{(1)}\left(\nu^{(1)}\right)^2+(\sigma_1)^{(1)}\nu^{(1)}-(a_{13})^{(1)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(1)},(\nu_0)^{(1)}$ :-

For $0<\boxed{(\nu_0)^{(1)}=\frac{G_{13}^0}{G_{14}^0}}<(\nu_1)^{(1)}<(\bar{\nu}_1)^{(1)}$

$$\nu^{(1)}(t)\geq\frac{(\nu_1)^{(1)}+(C)^{(1)}(\nu_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_0)^{(1)}\right)t\right]}}{1+(C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_0)^{(1)}\right)t\right]}}\quad,\quad\boxed{(C)^{(1)}=\frac{(\nu_1)^{(1)}-(\nu_0)^{(1)}}{(\nu_0)^{(1)}-(\nu_2)^{(1)}}}$$

it follows $(\nu_0)^{(1)}\leq\nu^{(1)}(t)\leq(\nu_1)^{(1)}$

In the same manner , we get 384

$$\nu^{(1)}(t)\leq\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}\quad,\quad\boxed{(\bar{C})^{(1)}=\frac{(\bar{\nu}_1)^{(1)}-(\nu_0)^{(1)}}{(\nu_0)^{(1)}-(\bar{\nu}_2)^{(1)}}}$$

From which we deduce $(\nu_0)^{(1)}\leq\nu^{(1)}(t)\leq(\bar{\nu}_1)^{(1)}$

If $0<(\nu_1)^{(1)}<(\nu_0)^{(1)}=\frac{G_{13}^0}{G_{14}^0}<(\bar{\nu}_1)^{(1)}$ we find like in the previous case, 385

$$(\nu_1)^{(1)}\leq\frac{(\nu_1)^{(1)}+(C)^{(1)}(\nu_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_2)^{(1)}\right)t\right]}}{1+(C)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\nu_1)^{(1)}-(\nu_2)^{(1)}\right)t\right]}}\leq\nu^{(1)}(t)\leq$$

$$\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}\leq(\bar{\nu}_1)^{(1)}$$

If $0<(\nu_1)^{(1)}\leq(\bar{\nu}_1)^{(1)}\leq\boxed{(\nu_0)^{(1)}=\frac{G_{13}^0}{G_{14}^0}}$ , we obtain 386

$$(\nu_1)^{(1)}\leq\nu^{(1)}(t)\leq\frac{(\bar{\nu}_1)^{(1)}+(\bar{C})^{(1)}(\bar{\nu}_2)^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}{1+(\bar{C})^{(1)}e^{\left[-(a_{14})^{(1)}\left((\bar{\nu}_1)^{(1)}-(\bar{\nu}_2)^{(1)}\right)t\right]}}\leq(\nu_0)^{(1)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(1)}(t)$ :-

$$(m_2)^{(1)}\leq\nu^{(1)}(t)\leq(m_1)^{(1)},\quad\boxed{\nu^{(1)}(t)=\frac{G_{13}(t)}{G_{14}(t)}}$$

In a completely analogous way, we obtain

**Definition of** $u^{(1)}(t)$ :-

$$(\mu_2)^{(1)}\leq u^{(1)}(t)\leq(\mu_1)^{(1)},\quad\boxed{u^{(1)}(t)=\frac{T_{13}(t)}{T_{14}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

Particular case :

If $(a''_{13})^{(1)} = (a''_{14})^{(1)}, then\ (\sigma_1)^{(1)} = (\sigma_2)^{(1)}$ and in this case $(\nu_1)^{(1)} = (\bar{\nu}_1)^{(1)}$ if in addition $(\nu_0)^{(1)} = (\nu_1)^{(1)}$ then $\nu^{(1)}(t) = (\nu_0)^{(1)}$ and as a consequence $G_{13}(t) = (\nu_0)^{(1)} G_{14}(t)$ this also defines $(\nu_0)^{(1)}$ for the special case

Analogously if $(b''_{13})^{(1)} = (b''_{14})^{(1)}, then\ (\tau_1)^{(1)} = (\tau_2)^{(1)}$ and then

$(u_1)^{(1)} = (\bar{u}_1)^{(1)}$ if in addition $(u_0)^{(1)} = (u_1)^{(1)}$ then $T_{13}(t) = (u_0)^{(1)} T_{14}(t)$ This is an important consequence of the relation between $(\nu_1)^{(1)}$ and $(\bar{\nu}_1)^{(1)}$, and definition of $(u_0)^{(1)}$.

**Proof** : From global equations we obtain 387

$$\frac{d\nu^{(2)}}{dt} = (a_{16})^{(2)} - \left((a'_{16})^{(2)} - (a'_{17})^{(2)} + (a''_{16})^{(2)}(T_{17},t)\right) - (a''_{17})^{(2)}(T_{17},t)\nu^{(2)} - (a_{17})^{(2)}\nu^{(2)}$$

**Definition of** $\nu^{(2)}$ :- $\boxed{\nu^{(2)} = \frac{G_{16}}{G_{17}}}$ 388

It follows 389

$$-\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_2)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right) \le \frac{d\nu^{(2)}}{dt} \le -\left((a_{17})^{(2)}\left(\nu^{(2)}\right)^2 + (\sigma_1)^{(2)}\nu^{(2)} - (a_{16})^{(2)}\right)$$

From which one obtains 390

**Definition of** $(\bar{\nu}_1)^{(2)}, (\nu_0)^{(2)}$ :-

For $0 < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\nu_1)^{(2)} < (\bar{\nu}_1)^{(2)}$

$$\nu^{(2)}(t) \ge \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_0)^{(2)}\right)t\right]}} \quad , \quad \boxed{(C)^{(2)} = \frac{(\nu_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\nu_2)^{(2)}}}$$

it follows $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\nu_1)^{(2)}$

In the same manner , we get 391

$$\nu^{(2)}(t) \le \frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(2)} = \frac{(\bar{\nu}_1)^{(2)} - (\nu_0)^{(2)}}{(\nu_0)^{(2)} - (\bar{\nu}_2)^{(2)}}}$$

From which we deduce $(\nu_0)^{(2)} \le \nu^{(2)}(t) \le (\bar{\nu}_1)^{(2)}$ 392

If $0 < (\nu_1)^{(2)} < (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0} < (\bar{\nu}_1)^{(2)}$ we find like in the previous case, 393

$$(\nu_1)^{(2)} \le \frac{(\nu_1)^{(2)} + (C)^{(2)}(\nu_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}}{1 + (C)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\nu_1)^{(2)} - (\nu_2)^{(2)}\right)t\right]}} \le \nu^{(2)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(2)} + (\bar{C})^{(2)}(\bar{\nu}_2)^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}}{1 + (\bar{C})^{(2)} e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)} - (\bar{\nu}_2)^{(2)}\right)t\right]}} \le (\bar{\nu}_1)^{(2)}$$

If $0 < (\nu_1)^{(2)} \le (\bar{\nu}_1)^{(2)} \le (\nu_0)^{(2)} = \frac{G_{16}^0}{G_{17}^0}$ , we obtain 394

$(\nu_1)^{(2)} \leq \nu^{(2)}(t) \leq \frac{(\bar{\nu}_1)^{(2)}+(\bar{C})^{(2)}(\bar{\nu}_2)^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}}{1+(\bar{C})^{(2)}e^{\left[-(a_{17})^{(2)}\left((\bar{\nu}_1)^{(2)}-(\bar{\nu}_2)^{(2)}\right)t\right]}} \leq (\nu_0)^{(2)}$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(2)}(t)$ :- 395

$(m_2)^{(2)} \leq \nu^{(2)}(t) \leq (m_1)^{(2)}$, $\boxed{\nu^{(2)}(t) = \frac{G_{16}(t)}{G_{17}(t)}}$

In a completely analogous way, we obtain 396

**Definition of** $u^{(2)}(t)$ :-

$(\mu_2)^{(2)} \leq u^{(2)}(t) \leq (\mu_1)^{(2)}$, $\boxed{u^{(2)}(t) = \frac{T_{16}(t)}{T_{17}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :** 397

If $(a''_{16})^{(2)} = (a''_{17})^{(2)}, then\ (\sigma_1)^{(2)} = (\sigma_2)^{(2)}$ and in this case $(\nu_1)^{(2)} = (\bar{\nu}_1)^{(2)}$ if in addition $(\nu_0)^{(2)} = (\nu_1)^{(2)}$ then $\nu^{(2)}(t) = (\nu_0)^{(2)}$ and as a consequence $G_{16}(t) = (\nu_0)^{(2)} G_{17}(t)$

Analogously if $(b''_{16})^{(2)} = (b''_{17})^{(2)}, then\ (\tau_1)^{(2)} = (\tau_2)^{(2)}$ and then

$(u_1)^{(2)} = (\bar{u}_1)^{(2)}$if in addition $(u_0)^{(2)} = (u_1)^{(2)}$ then $T_{16}(t) = (u_0)^{(2)} T_{17}(t)$ This is an important consequence of the relation between $(\nu_1)^{(2)}$ and $(\bar{\nu}_1)^{(2)}$

**Proof** : From global equations we obtain 398

$\frac{d\nu^{(3)}}{dt} = (a_{20})^{(3)} - \left((a'_{20})^{(3)} - (a'_{21})^{(3)} + (a''_{20})^{(3)}(T_{21},t)\right) - (a''_{21})^{(3)}(T_{21},t)\nu^{(3)} - (a_{21})^{(3)}\nu^{(3)}$

**Definition of** $\nu^{(3)}$ :- $\boxed{\nu^{(3)} = \frac{G_{20}}{G_{21}}}$ 399

It follows

$-\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_2)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right) \leq \frac{d\nu^{(3)}}{dt} \leq -\left((a_{21})^{(3)}\left(\nu^{(3)}\right)^2 + (\sigma_1)^{(3)}\nu^{(3)} - (a_{20})^{(3)}\right)$

400

From which one obtains

For $0 < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\nu_1)^{(3)} < (\bar{\nu}_1)^{(3)}$

$\nu^{(3)}(t) \geq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_0)^{(3)}\right)t\right]}}$ , $\boxed{(C)^{(3)} = \frac{(\nu_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\nu_2)^{(3)}}}$

it follows $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\nu_1)^{(3)}$

In the same manner , we get 401

$$\nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(3)} = \frac{(\bar{\nu}_1)^{(3)}-(\nu_0)^{(3)}}{(\nu_0)^{(3)}-(\bar{\nu}_2)^{(3)}}}$$

**<u>Definition of</u>** $(\bar{\nu}_1)^{(3)}$ :-

From which we deduce $(\nu_0)^{(3)} \leq \nu^{(3)}(t) \leq (\bar{\nu}_1)^{(3)}$

If $0 < (\nu_1)^{(3)} < (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0} < (\bar{\nu}_1)^{(3)}$ we find like in the previous case, 402

$$(\nu_1)^{(3)} \leq \frac{(\nu_1)^{(3)}+(C)^{(3)}(\nu_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}}{1+(C)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\nu_1)^{(3)}-(\nu_2)^{(3)}\right)t\right]}} \leq \nu^{(3)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\bar{\nu}_1)^{(3)}$$

If $0 < (\nu_1)^{(3)} \leq (\bar{\nu}_1)^{(3)} \leq (\nu_0)^{(3)} = \frac{G_{20}^0}{G_{21}^0}$ , we obtain 403

$$(\nu_1)^{(3)} \leq \nu^{(3)}(t) \leq \frac{(\bar{\nu}_1)^{(3)}+(\bar{C})^{(3)}(\bar{\nu}_2)^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}}{1+(\bar{C})^{(3)}e^{\left[-(a_{21})^{(3)}\left((\bar{\nu}_1)^{(3)}-(\bar{\nu}_2)^{(3)}\right)t\right]}} \leq (\nu_0)^{(3)}$$

And so with the notation of the first part of condition (c) , we have

**<u>Definition of</u>** $\nu^{(3)}(t)$ :-

$$(m_2)^{(3)} \leq \nu^{(3)}(t) \leq (m_1)^{(3)}, \quad \boxed{\nu^{(3)}(t) = \frac{G_{20}(t)}{G_{21}(t)}}$$

In a completely analogous way, we obtain

**<u>Definition of</u>** $u^{(3)}(t)$ :-

$$(\mu_2)^{(3)} \leq u^{(3)}(t) \leq (\mu_1)^{(3)}, \quad \boxed{u^{(3)}(t) = \frac{T_{20}(t)}{T_{21}(t)}}$$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

<u>Particular case :</u>

If $(a''_{20})^{(3)} = (a''_{21})^{(3)}, then\ (\sigma_1)^{(3)} = (\sigma_2)^{(3)}$ and in this case $(\nu_1)^{(3)} = (\bar{\nu}_1)^{(3)}$ if in addition $(\nu_0)^{(3)} = (\nu_1)^{(3)}$ then $\nu^{(3)}(t) = (\nu_0)^{(3)}$ and as a consequence $G_{20}(t) = (\nu_0)^{(3)}G_{21}(t)$

Analogously if $(b''_{20})^{(3)} = (b''_{21})^{(3)}, then\ (\tau_1)^{(3)} = (\tau_2)^{(3)}$ and then

$(u_1)^{(3)} = (\bar{u}_1)^{(3)}$if in addition $(u_0)^{(3)} = (u_1)^{(3)}$ then $T_{20}(t) = (u_0)^{(3)}T_{21}(t)$ This is an important consequence of the relation between $(\nu_1)^{(3)}$ and $(\bar{\nu}_1)^{(3)}$

**<u>Proof</u>** : From global equations we obtain 404

$$\frac{d\nu^{(4)}}{dt} = (a_{24})^{(4)} - \left((a'_{24})^{(4)} - (a'_{25})^{(4)} + (a''_{24})^{(4)}(T_{25},t)\right) - (a''_{25})^{(4)}(T_{25},t)\nu^{(4)} - (a_{25})^{(4)}\nu^{(4)}$$

<u>**Definition of**</u> $\nu^{(4)}$ :- $\boxed{\nu^{(4)} = \frac{G_{24}}{G_{25}}}$

It follows

$$-\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_2)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right) \leq \frac{d\nu^{(4)}}{dt} \leq -\left((a_{25})^{(4)}\left(\nu^{(4)}\right)^2 + (\sigma_4)^{(4)}\nu^{(4)} - (a_{24})^{(4)}\right)$$

From which one obtains

<u>**Definition of**</u> $(\bar{\nu}_1)^{(4)}, (\nu_0)^{(4)}$ :-

For $0 < \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}} < (\nu_1)^{(4)} < (\bar{\nu}_1)^{(4)}$

$$\nu^{(4)}(t) \geq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}}{4 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_0)^{(4)}\right)t\right]}} \quad , \quad \boxed{(C)^{(4)} = \frac{(\nu_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\nu_2)^{(4)}}}$$

it follows $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\nu_1)^{(4)}$

In the same manner , we get 405

$$\nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{4 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(4)} = \frac{(\bar{\nu}_1)^{(4)} - (\nu_0)^{(4)}}{(\nu_0)^{(4)} - (\bar{\nu}_2)^{(4)}}}$$

From which we deduce $(\nu_0)^{(4)} \leq \nu^{(4)}(t) \leq (\bar{\nu}_1)^{(4)}$

If $0 < (\nu_1)^{(4)} < (\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0} < (\bar{\nu}_1)^{(4)}$ we find like in the previous case, 406

$$(\nu_1)^{(4)} \leq \frac{(\nu_1)^{(4)} + (C)^{(4)}(\nu_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}}{1 + (C)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\nu_1)^{(4)} - (\nu_2)^{(4)}\right)t\right]}} \leq \nu^{(4)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\bar{\nu}_1)^{(4)}$$

If $0 < (\nu_1)^{(4)} \leq (\bar{\nu}_1)^{(4)} \leq \boxed{(\nu_0)^{(4)} = \frac{G_{24}^0}{G_{25}^0}}$ , we obtain 407

$$(\nu_1)^{(4)} \leq \nu^{(4)}(t) \leq \frac{(\bar{\nu}_1)^{(4)} + (\bar{C})^{(4)}(\bar{\nu}_2)^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}}{1 + (\bar{C})^{(4)} e^{\left[-(a_{25})^{(4)}\left((\bar{\nu}_1)^{(4)} - (\bar{\nu}_2)^{(4)}\right)t\right]}} \leq (\nu_0)^{(4)}$$

And so with the notation of the first part of condition (c) , we have

<u>**Definition of**</u> $\nu^{(4)}(t)$ :-

$(m_2)^{(4)} \leq \; \nu^{(4)}(t) \leq (m_1)^{(4)}, \; \boxed{\nu^{(4)}(t) = \frac{G_{24}(t)}{G_{25}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(4)}(t)$ :-

$(\mu_2)^{(4)} \leq \; u^{(4)}(t) \leq (\mu_1)^{(4)}, \; \boxed{u^{(4)}(t) = \frac{T_{24}(t)}{T_{25}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a''_{24})^{(4)} = (a''_{25})^{(4)}, then\ (\sigma_1)^{(4)} = (\sigma_2)^{(4)}$ and in this case $(\nu_1)^{(4)} = (\bar{\nu}_1)^{(4)}$ if in addition $(\nu_0)^{(4)} = (\nu_1)^{(4)}$ then $\nu^{(4)}(t) = (\nu_0)^{(4)}$ and as a consequence $G_{24}(t) = (\nu_0)^{(4)} G_{25}(t)$ **this also defines** $(\nu_0)^{(4)}$ **for the special case .**

Analogously if $(b''_{24})^{(4)} = (b''_{25})^{(4)}, then\ (\tau_1)^{(4)} = (\tau_2)^{(4)}$ and then

$(u_1)^{(4)} = (\bar{u}_4)^{(4)}$if in addition $(u_0)^{(4)} = (u_1)^{(4)}$ then $T_{24}(t) = (u_0)^{(4)} T_{25}(t)$ This is an important consequence of the relation between $(\nu_1)^{(4)}$ and $(\bar{\nu}_1)^{(4)}$, **and definition of** $(u_0)^{(4)}$.

**Proof** : From global equations we obtain 408

$$\frac{d\nu^{(5)}}{dt} = (a_{28})^{(5)} - \left((a'_{28})^{(5)} - (a'_{29})^{(5)} + (a''_{28})^{(5)}(T_{29}, t)\right) - (a''_{29})^{(5)}(T_{29}, t)\nu^{(5)} - (a_{29})^{(5)}\nu^{(5)}$$

**Definition of** $\nu^{(5)}$ :- $\boxed{\nu^{(5)} = \frac{G_{28}}{G_{29}}}$

It follows

$$-\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_2)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right) \leq \frac{d\nu^{(5)}}{dt} \leq -\left((a_{29})^{(5)}\left(\nu^{(5)}\right)^2 + (\sigma_1)^{(5)}\nu^{(5)} - (a_{28})^{(5)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(5)}, (\nu_0)^{(5)}$ :-

For $0 < \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}} < (\nu_1)^{(5)} < (\bar{\nu}_1)^{(5)}$

$$\nu^{(5)}(t) \geq \frac{(\nu_1)^{(5)} + (C)^{(5)}(\nu_2)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}}{5 + (C)^{(5)} e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)} - (\nu_0)^{(5)}\right)t\right]}} \quad , \quad \boxed{(C)^{(5)} = \frac{(\nu_1)^{(5)} - (\nu_0)^{(5)}}{(\nu_0)^{(5)} - (\nu_2)^{(5)}}}$$

it follows $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\nu_1)^{(5)}$

In the same manner , we get 409

$$\nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{5+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(5)} = \frac{(\bar{\nu}_1)^{(5)}-(\nu_0)^{(5)}}{(\nu_0)^{(5)}-(\bar{\nu}_2)^{(5)}}}$$

From which we deduce $(\nu_0)^{(5)} \leq \nu^{(5)}(t) \leq (\bar{\nu}_5)^{(5)}$

If $0 < (\nu_1)^{(5)} < (\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0} < (\bar{\nu}_1)^{(5)}$ we find like in the previous case, 410

$$(\nu_1)^{(5)} \leq \frac{(\nu_1)^{(5)}+(C)^{(5)}(\nu_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)}-(\nu_2)^{(5)}\right)t\right]}}{1+(C)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\nu_1)^{(5)}-(\nu_2)^{(5)}\right)t\right]}} \leq \nu^{(5)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\bar{\nu}_1)^{(5)}$$

If $0 < (\nu_1)^{(5)} \leq (\bar{\nu}_1)^{(5)} \leq \boxed{(\nu_0)^{(5)} = \frac{G_{28}^0}{G_{29}^0}}$ , we obtain 411

$$(\nu_1)^{(5)} \leq \nu^{(5)}(t) \leq \frac{(\bar{\nu}_1)^{(5)}+(\bar{C})^{(5)}(\bar{\nu}_2)^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}}{1+(\bar{C})^{(5)}e^{\left[-(a_{29})^{(5)}\left((\bar{\nu}_1)^{(5)}-(\bar{\nu}_2)^{(5)}\right)t\right]}} \leq (\nu_0)^{(5)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(5)}(t)$ :-

$(m_2)^{(5)} \leq \nu^{(5)}(t) \leq (m_1)^{(5)}$, $\boxed{\nu^{(5)}(t) = \frac{G_{28}(t)}{G_{29}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(5)}(t)$ :-

$(\mu_2)^{(5)} \leq u^{(5)}(t) \leq (\mu_1)^{(5)}$, $\boxed{u^{(5)}(t) = \frac{T_{28}(t)}{T_{29}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a''_{28})^{(5)} = (a''_{29})^{(5)}, then\ (\sigma_1)^{(5)} = (\sigma_2)^{(5)}$ and in this case $(\nu_1)^{(5)} = (\bar{\nu}_1)^{(5)}$ if in addition $(\nu_0)^{(5)} = (\nu_5)^{(5)}$ then $\nu^{(5)}(t) = (\nu_0)^{(5)}$ and as a consequence $G_{28}(t) = (\nu_0)^{(5)}G_{29}(t)$ **this also defines** $(\nu_0)^{(5)}$ **for the special case .**

Analogously if $(b''_{28})^{(5)} = (b''_{29})^{(5)}, then\ (\tau_1)^{(5)} = (\tau_2)^{(5)}$ and then

$(u_1)^{(5)} = (\bar{u}_1)^{(5)}$if in addition $(u_0)^{(5)} = (u_1)^{(5)}$ then $T_{28}(t) = (u_0)^{(5)}T_{29}(t)$ This is an important consequence of the relation between $(\nu_1)^{(5)}$ and $(\bar{\nu}_1)^{(5)}$, **and definition of** $(u_0)^{(5)}$.

**Proof** : From global equations we obtain 412

$$\frac{d\nu^{(6)}}{dt} = (a_{32})^{(6)} - \left((a'_{32})^{(6)} - (a'_{33})^{(6)} + (a''_{32})^{(6)}(T_{33},t)\right) - (a''_{33})^{(6)}(T_{33},t)\nu^{(6)} - (a_{33})^{(6)}\nu^{(6)}$$

**Definition of** $\nu^{(6)}$ :- $\boxed{\nu^{(6)} = \frac{G_{32}}{G_{33}}}$

It follows

$$-\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_2)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right) \le \frac{d\nu^{(6)}}{dt} \le -\left((a_{33})^{(6)}\left(\nu^{(6)}\right)^2 + (\sigma_1)^{(6)}\nu^{(6)} - (a_{32})^{(6)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(6)}, (\nu_0)^{(6)}$ :-

For $0 < \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}} < (\nu_1)^{(6)} < (\bar{\nu}_1)^{(6)}$

$$\nu^{(6)}(t) \ge \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}}{1 + (C)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_0)^{(6)}\right)t\right]}} \quad , \quad \boxed{(C)^{(6)} = \frac{(\nu_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\nu_2)^{(6)}}}$$

it follows $(\nu_0)^{(6)} \le \nu^{(6)}(t) \le (\nu_1)^{(6)}$

In the same manner , we get 413

$$\nu^{(6)}(t) \le \frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(6)} = \frac{(\bar{\nu}_1)^{(6)} - (\nu_0)^{(6)}}{(\nu_0)^{(6)} - (\bar{\nu}_2)^{(6)}}}$$

From which we deduce $(\nu_0)^{(6)} \le \nu^{(6)}(t) \le (\bar{\nu}_1)^{(6)}$

If $0 < (\nu_1)^{(6)} < (\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0} < (\bar{\nu}_1)^{(6)}$ we find like in the previous case, 414

$$(\nu_1)^{(6)} \le \frac{(\nu_1)^{(6)} + (C)^{(6)}(\nu_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}}{1 + (C)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\nu_1)^{(6)} - (\nu_2)^{(6)}\right)t\right]}} \le \nu^{(6)}(t) \le$$

$$\frac{(\bar{\nu}_1)^{(6)} + (\bar{C})^{(6)}(\bar{\nu}_2)^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}}{1 + (\bar{C})^{(6)} e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)} - (\bar{\nu}_2)^{(6)}\right)t\right]}} \le (\bar{\nu}_1)^{(6)}$$

If $0 < (\nu_1)^{(6)} \le (\bar{\nu}_1)^{(6)} \le \boxed{(\nu_0)^{(6)} = \frac{G_{32}^0}{G_{33}^0}}$ , we obtain 415

$$(\nu_1)^{(6)} \leq \nu^{(6)}(t) \leq \frac{(\bar{\nu}_1)^{(6)}+(\bar{C})^{(6)}(\bar{\nu}_2)^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}}{1+(\bar{C})^{(6)}e^{\left[-(a_{33})^{(6)}\left((\bar{\nu}_1)^{(6)}-(\bar{\nu}_2)^{(6)}\right)t\right]}} \leq (\nu_0)^{(6)}$$

And so with the notation of the first part of condition (c) , we have

**Definition of** $\nu^{(6)}(t)$ :-

$(m_2)^{(6)} \leq \nu^{(6)}(t) \leq (m_1)^{(6)}$, $\boxed{\nu^{(6)}(t) = \frac{G_{32}(t)}{G_{33}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(6)}(t)$ :-

$(\mu_2)^{(6)} \leq u^{(6)}(t) \leq (\mu_1)^{(6)}$, $\boxed{u^{(6)}(t) = \frac{T_{32}(t)}{T_{33}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a''_{32})^{(6)} = (a''_{33})^{(6)}, then\ (\sigma_1)^{(6)} = (\sigma_2)^{(6)}$ and in this case $(\nu_1)^{(6)} = (\bar{\nu}_1)^{(6)}$ if in addition $(\nu_0)^{(6)} = (\nu_1)^{(6)}$ then $\nu^{(6)}(t) = (\nu_0)^{(6)}$ and as a consequence $G_{32}(t) = (\nu_0)^{(6)}G_{33}(t)$ **this also defines** $(\nu_0)^{(6)}$ **for the special case .**

Analogously if $(b''_{32})^{(6)} = (b''_{33})^{(6)}, then\ (\tau_1)^{(6)} = (\tau_2)^{(6)}$ and then

$(u_1)^{(6)} = (\bar{u}_1)^{(6)}$if in addition $(u_0)^{(6)} = (u_1)^{(6)}$ then $T_{32}(t) = (u_0)^{(6)}T_{33}(t)$ This is an important consequence of the relation between $(\nu_1)^{(6)}$ and $(\bar{\nu}_1)^{(6)}$, **and definition of** $(u_0)^{(6)}$.

416

**Proof** : From global equations we obtain

$$\frac{d\nu^{(7)}}{dt} = (a_{36})^{(7)} - \left((a'_{36})^{(7)} - (a'_{37})^{(7)} + (a''_{36})^{(7)}(T_{37},t)\right) - (a''_{37})^{(7)}(T_{37},t)\nu^{(7)} - (a_{37})^{(7)}\nu^{(7)}$$

**Definition of** $\nu^{(7)}$ :- $\boxed{\nu^{(7)} = \frac{G_{36}}{G_{37}}}$

It follows

$$-\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_2)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right) \leq \frac{d\nu^{(7)}}{dt} \leq -\left((a_{37})^{(7)}\left(\nu^{(7)}\right)^2 + (\sigma_1)^{(7)}\nu^{(7)} - (a_{36})^{(7)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(7)}, (\nu_0)^{(7)}$ :-

For $0 < \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}} < (\nu_1)^{(7)} < (\bar{\nu}_1)^{(7)}$

$$\nu^{(7)}(t) \geq \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_0)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_0)^{(7)}\right)t\right]}} \quad , \quad \boxed{(C)^{(7)} = \frac{(\nu_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\nu_2)^{(7)}}}$$

it follows $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\nu_1)^{(7)}$

In the same manner , we get 417

$$\nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(7)} = \frac{(\bar{\nu}_1)^{(7)}-(\nu_0)^{(7)}}{(\nu_0)^{(7)}-(\bar{\nu}_2)^{(7)}}}$$

From which we deduce $(\nu_0)^{(7)} \leq \nu^{(7)}(t) \leq (\bar{\nu}_1)^{(7)}$

If $0 < (\nu_1)^{(7)} < (\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0} < (\bar{\nu}_1)^{(7)}$ we find like in the previous case, 418

$$(\nu_1)^{(7)} \leq \frac{(\nu_1)^{(7)}+(C)^{(7)}(\nu_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}}{1+(C)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\nu_1)^{(7)}-(\nu_2)^{(7)}\right)t\right]}} \leq \nu^{(7)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\bar{\nu}_1)^{(7)}$$

If $0 < (\nu_1)^{(7)} \leq (\bar{\nu}_1)^{(7)} \leq \boxed{(\nu_0)^{(7)} = \frac{G_{36}^0}{G_{37}^0}}$ , we obtain 419

$$(\nu_1)^{(7)} \leq \nu^{(7)}(t) \leq \frac{(\bar{\nu}_1)^{(7)}+(\bar{C})^{(7)}(\bar{\nu}_2)^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}}{1+(\bar{C})^{(7)}e^{\left[-(a_{37})^{(7)}\left((\bar{\nu}_1)^{(7)}-(\bar{\nu}_2)^{(7)}\right)t\right]}} \leq (\nu_0)^{(7)}$$

And so with the notation of the first part of condition (c) , we have
**Definition of** $\nu^{(7)}(t)$ :-

$(m_2)^{(7)} \leq \nu^{(7)}(t) \leq (m_1)^{(7)}$, $\boxed{\nu^{(7)}(t) = \frac{G_{36}(t)}{G_{37}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(7)}(t)$ :- 420

$(\mu_2)^{(7)} \leq u^{(7)}(t) \leq (\mu_1)^{(7)}$, $\boxed{u^{(7)}(t) = \frac{T_{36}(t)}{T_{37}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a''_{36})^{(7)} = (a''_{37})^{(7)}, then\ (\sigma_1)^{(7)} = (\sigma_2)^{(7)}$ and in this case $(\nu_1)^{(7)} = (\bar{\nu}_1)^{(7)}$ if in addition $(\nu_0)^{(7)} = (\nu_1)^{(7)}$ then $\nu^{(7)}(t) = (\nu_0)^{(7)}$ and as a consequence $G_{36}(t) = (\nu_0)^{(7)}G_{37}(t)$ **this also defines** $(\nu_0)^{(7)}$ **for the special case .**

Analogously if $(b''_{36})^{(7)} = (b''_{37})^{(7)}, then\ (\tau_1)^{(7)} = (\tau_2)^{(7)}$ and then $(u_1)^{(7)} = (\bar{u}_1)^{(7)}$if in addition

$(u_0)^{(7)} = (u_1)^{(7)}$ then $T_{36}(t) = (u_0)^{(7)} T_{37}(t)$ This is an important consequence of the relation between $(\nu_1)^{(7)}$ and $(\bar{\nu}_1)^{(7)}$, **and definition of** $(u_0)^{(7)}$.

**Proof** : From global equations we obtain 421

$$\frac{d\nu^{(8)}}{dt} = (a_{40})^{(8)} - \left((a'_{40})^{(8)} - (a'_{41})^{(8)} + (a''_{40})^{(8)}(T_{41}, t)\right) - (a''_{41})^{(8)}(T_{41}, t)\nu^{(8)} - (a_{41})^{(8)}\nu^{(8)}$$

**Definition of** $\nu^{(8)}$ :- $\boxed{\nu^{(8)} = \frac{G_{40}}{G_{41}}}$

It follows

$$-\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_2)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right) \leq \frac{d\nu^{(8)}}{dt} \leq -\left((a_{41})^{(8)}\left(\nu^{(8)}\right)^2 + (\sigma_1)^{(8)}\nu^{(8)} - (a_{40})^{(8)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(8)}, (\nu_0)^{(8)}$ :-

For $0 < \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}} < (\nu_1)^{(8)} < (\bar{\nu}_1)^{(8)}$

$$\nu^{(8)}(t) \geq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_0)^{(8)}\right)t\right]}} \quad , \quad \boxed{(C)^{(8)} = \frac{(\nu_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\nu_2)^{(8)}}}$$

it follows $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\nu_1)^{(8)}$

In the same manner , we get 422

$$\nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(8)} = \frac{(\bar{\nu}_1)^{(8)} - (\nu_0)^{(8)}}{(\nu_0)^{(8)} - (\bar{\nu}_2)^{(8)}}}$$

From which we deduce $(\nu_0)^{(8)} \leq \nu^{(8)}(t) \leq (\bar{\nu}_8)^{(8)}$

If $0 < (\nu_1)^{(8)} < (\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0} < (\bar{\nu}_1)^{(8)}$ we find like in the previous case, 423

$$(\nu_1)^{(8)} \leq \frac{(\nu_1)^{(8)} + (C)^{(8)}(\nu_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_2)^{(8)}\right)t\right]}}{1 + (C)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\nu_1)^{(8)} - (\nu_2)^{(8)}\right)t\right]}} \leq \nu^{(8)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\bar{\nu}_1)^{(8)}$$

If $0 < (\nu_1)^{(8)} \leq (\bar{\nu}_1)^{(8)} \leq \boxed{(\nu_0)^{(8)} = \frac{G_{40}^0}{G_{41}^0}}$ , we obtain 424

$$(\nu_1)^{(8)} \leq \nu^{(8)}(t) \leq \frac{(\bar{\nu}_1)^{(8)} + (\bar{C})^{(8)}(\bar{\nu}_2)^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}}{1 + (\bar{C})^{(8)} e^{\left[-(a_{41})^{(8)}\left((\bar{\nu}_1)^{(8)} - (\bar{\nu}_2)^{(8)}\right)t\right]}} \leq (\nu_0)^{(8)}$$

And so with the notation of the first part of condition (c) , we have
**Definition of** $\nu^{(8)}(t)$ :-

$(m_2)^{(8)} \leq \nu^{(8)}(t) \leq (m_1)^{(8)}$, $\boxed{\nu^{(8)}(t) = \frac{G_{40}(t)}{G_{41}(t)}}$

In a completely analogous way, we obtain

**Definition of** $u^{(8)}(t)$ :-

$(\mu_2)^{(8)} \leq u^{(8)}(t) \leq (\mu_1)^{(8)}$, $\boxed{u^{(8)}(t) = \frac{T_{40}(t)}{T_{41}(t)}}$

Now, using this result and replacing it in global equations we get easily the result stated in the theorem.

**Particular case :**

If $(a''_{40})^{(8)} = (a''_{41})^{(8)}, then\ (\sigma_1)^{(8)} = (\sigma_2)^{(8)}$ and in this case $(\nu_1)^{(8)} = (\bar{\nu}_1)^{(8)}$ if in addition $(\nu_0)^{(8)} = (\nu_1)^{(8)}$ then $\nu^{(8)}(t) = (\nu_0)^{(8)}$ and as a consequence $G_{40}(t) = (\nu_0)^{(8)} G_{41}(t)$ **this also defines** $(\nu_0)^{(8)}$ **for the special case .**

Analogously if $(b''_{40})^{(8)} = (b''_{41})^{(8)}, then\ (\tau_1)^{(8)} = (\tau_2)^{(8)}$ and then $(u_1)^{(8)} = (\bar{u}_1)^{(8)}$if in addition $(u_0)^{(8)} = (u_1)^{(8)}$ then $T_{40}(t) = (u_0)^{(8)} T_{41}(t)$ This is an important consequence of the relation between $(\nu_1)^{(8)}$ and $(\bar{\nu}_1)^{(8)}$, **and definition of** $(u_0)^{(8)}$.

424 A

**Proof** : From 99,20,44,22,23,44 we obtain

$$\frac{d\nu^{(9)}}{dt} = (a_{44})^{(9)} - \left((a'_{44})^{(9)} - (a'_{45})^{(9)} + (a''_{44})^{(9)}(T_{45}, t)\right) - (a''_{45})^{(9)}(T_{45}, t)\nu^{(9)} - (a_{45})^{(9)}\nu^{(9)}$$

**Definition of** $\nu^{(9)}$ :- $\boxed{\nu^{(9)} = \frac{G_{44}}{G_{45}}}$

It follows

$$-\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_2)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right) \leq \frac{d\nu^{(9)}}{dt} \leq -\left((a_{45})^{(9)}\left(\nu^{(9)}\right)^2 + (\sigma_1)^{(9)}\nu^{(9)} - (a_{44})^{(9)}\right)$$

From which one obtains

**Definition of** $(\bar{\nu}_1)^{(9)}, (\nu_0)^{(9)}$ :-

For $0 < \boxed{(\nu_0)^{(9)} = \frac{G^0_{44}}{G^0_{45}}} < (\nu_1)^{(9)} < (\bar{\nu}_1)^{(9)}$

$$\nu^{(9)}(t) \geq \frac{(\nu_1)^{(9)} + (C)^{(9)}(\nu_2)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}}{1 + (C)^{(9)} e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)} - (\nu_0)^{(9)}\right)t\right]}} \quad , \quad \boxed{(C)^{(9)} = \frac{(\nu_1)^{(9)} - (\nu_0)^{(9)}}{(\nu_0)^{(9)} - (\nu_2)^{(9)}}}$$

it follows $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\nu_9)^{(9)}$

In the same manner , we get

$$\nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \quad , \quad \boxed{(\bar{C})^{(9)} = \frac{(\bar{\nu}_1)^{(9)}-(\nu_0)^{(9)}}{(\nu_0)^{(9)}-(\bar{\nu}_2)^{(9)}}}$$

From which we deduce $(\nu_0)^{(9)} \leq \nu^{(9)}(t) \leq (\bar{\nu}_1)^{(9)}$

If $0 < (\nu_1)^{(9)} < (\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0} < (\bar{\nu}_1)^{(9)}$ we find like in the previous case,

$$(\nu_1)^{(9)} \leq \frac{(\nu_1)^{(9)}+(C)^{(9)}(\nu_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}}{1+(C)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\nu_1)^{(9)}-(\nu_2)^{(9)}\right)t\right]}} \leq \nu^{(9)}(t) \leq$$

$$\frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\bar{\nu}_1)^{(9)}$$

If $0 < (\nu_1)^{(9)} \leq (\bar{\nu}_1)^{(9)} \leq \boxed{(\nu_0)^{(9)} = \frac{G_{44}^0}{G_{45}^0}}$ , we obtain

$$(\nu_1)^{(9)} \leq \nu^{(9)}(t) \leq \frac{(\bar{\nu}_1)^{(9)}+(\bar{C})^{(9)}(\bar{\nu}_2)^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}}{1+(\bar{C})^{(9)}e^{\left[-(a_{45})^{(9)}\left((\bar{\nu}_1)^{(9)}-(\bar{\nu}_2)^{(9)}\right)t\right]}} \leq (\nu_0)^{(9)}$$

And so with the notation of the first part of condition (c) , we have

<u>Definition of</u> $\nu^{(9)}(t)$ :-

$$(m_2)^{(9)} \leq \nu^{(9)}(t) \leq (m_1)^{(9)}, \quad \boxed{\nu^{(9)}(t) = \frac{G_{44}(t)}{G_{45}(t)}}$$

In a completely analogous way, we obtain

<u>Definition of</u> $u^{(9)}(t)$ :-

$$(\mu_2)^{(9)} \leq u^{(9)}(t) \leq (\mu_1)^{(9)}, \quad \boxed{u^{(9)}(t) = \frac{T_{44}(t)}{T_{45}(t)}}$$

Now, using this result and replacing it in 99, 20,44,22,23, and 44 we get easily the result stated in the theorem.

<u>Particular case :</u>

If $(a''_{44})^{(9)} = (a''_{45})^{(9)}, then\ (\sigma_1)^{(9)} = (\sigma_2)^{(9)}$ and in this case $(\nu_1)^{(9)} = (\bar{\nu}_1)^{(9)}$ if in addition $(\nu_0)^{(9)} = (\nu_1)^{(9)}$ then $\nu^{(9)}(t) = (\nu_0)^{(9)}$ and as a consequence $G_{44}(t) = (\nu_0)^{(9)} G_{45}(t)$ **this also defines** $(\nu_0)^{(9)}$ **for the special case .**

Analogously if $(b''_{44})^{(9)} = (b''_{45})^{(9)}, then\ (\tau_1)^{(9)} = (\tau_2)^{(9)}$ and then

$(u_1)^{(9)} = (\bar{u}_1)^{(9)}$if in addition $(u_0)^{(9)} = (u_1)^{(9)}$ then $T_{44}(t) = (u_0)^{(9)} T_{45}(t)$ This is an important consequence of the relation between $(\nu_1)^{(9)}$ and $(\bar{\nu}_1)^{(9)}$, **and definition of** $(u_0)^{(9)}$.

We can prove the following 425

**Theorem** : If $(a_i'')^{(1)} and\ (b_i'')^{(1)}$ are independent on $t$ , and the conditions with the notations

$(a_{13}')^{(1)}(a_{14}')^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} < 0$

$(a_{13}')^{(1)}(a_{14}')^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a_{13})^{(1)}(p_{13})^{(1)} + (a_{14}')^{(1)}(p_{14})^{(1)} + (p_{13})^{(1)}(p_{14})^{(1)} > 0$

$(b_{13}')^{(1)}(b_{14}')^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} > 0$ ,

$(b_{13}')^{(1)}(b_{14}')^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} - (b_{13}')^{(1)}(r_{14})^{(1)} - (b_{14}')^{(1)}(r_{14})^{(1)} + (r_{13})^{(1)}(r_{14})^{(1)} < 0$

$with\ \ (p_{13})^{(1)}, (r_{14})^{(1)}$ as defined by equation are satisfied , then the system

**Theorem** : If $(a_i'')^{(2)} and\ (b_i'')^{(2)}$ are independent on t , and the conditions with the notations 426

$(a_{16}')^{(2)}(a_{17}')^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} < 0$ 427

$(a_{16}')^{(2)}(a_{17}')^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a_{16})^{(2)}(p_{16})^{(2)} + (a_{17}')^{(2)}(p_{17})^{(2)} + (p_{16})^{(2)}(p_{17})^{(2)} > 0$ 428

$(b_{16}')^{(2)}(b_{17}')^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} > 0$ , 429

$(b_{16}')^{(2)}(b_{17}')^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} - (b_{16}')^{(2)}(r_{17})^{(2)} - (b_{17}')^{(2)}(r_{17})^{(2)} + (r_{16})^{(2)}(r_{17})^{(2)} < 0$ 430

$with\ \ (p_{16})^{(2)}, (r_{17})^{(2)}$ as defined by equation are satisfied , then the system

**Theorem** : If $(a_i'')^{(3)} and\ (b_i'')^{(3)}$ are independent on $t$ , and the conditions with the notations 431

$(a_{20}')^{(3)}(a_{21}')^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} < 0$

$(a_{20}')^{(3)}(a_{21}')^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a_{20})^{(3)}(p_{20})^{(3)} + (a_{21}')^{(3)}(p_{21})^{(3)} + (p_{20})^{(3)}(p_{21})^{(3)} > 0$

$(b_{20}')^{(3)}(b_{21}')^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} > 0$ ,

$(b_{20}')^{(3)}(b_{21}')^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} - (b_{20}')^{(3)}(r_{21})^{(3)} - (b_{21}')^{(3)}(r_{21})^{(3)} + (r_{20})^{(3)}(r_{21})^{(3)} < 0$

$with\ \ (p_{20})^{(3)}, (r_{21})^{(3)}$ as defined by equation are satisfied , then the system

We can prove the following 432

**Theorem** : If $(a_i'')^{(4)} and\ (b_i'')^{(4)}$ are independent on $t$ , and the conditions with the notations

$(a_{24}')^{(4)}(a_{25}')^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} < 0$

$(a_{24}')^{(4)}(a_{25}')^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a_{24})^{(4)}(p_{24})^{(4)} + (a_{25}')^{(4)}(p_{25})^{(4)} + (p_{24})^{(4)}(p_{25})^{(4)} > 0$

$(b_{24}')^{(4)}(b_{25}')^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} > 0$ ,

$(b_{24}')^{(4)}(b_{25}')^{(4)} - (b_{24})^{(4)}(b_{25})^{(4)} - (b_{24}')^{(4)}(r_{25})^{(4)} - (b_{25}')^{(4)}(r_{25})^{(4)} + (r_{24})^{(4)}(r_{25})^{(4)} < 0$

$with\ \ (p_{24})^{(4)}, (r_{25})^{(4)}$ as defined by equation are satisfied , then the system

**Theorem** : If $(a_i'')^{(5)} and\ (b_i'')^{(5)}$ are independent on $t$ , and the conditions with the notations 433

$(a_{28}')^{(5)}(a_{29}')^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} < 0$

$(a'_{28})^{(5)}(a'_{29})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a_{28})^{(5)}(p_{28})^{(5)} + (a'_{29})^{(5)}(p_{29})^{(5)} + (p_{28})^{(5)}(p_{29})^{(5)} > 0$

$(b'_{28})^{(5)}(b'_{29})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} > 0$ ,

$(b'_{28})^{(5)}(b'_{29})^{(5)} - (b_{28})^{(5)}(b_{29})^{(5)} - (b'_{28})^{(5)}(r_{29})^{(5)} - (b'_{29})^{(5)}(r_{29})^{(5)} + (r_{28})^{(5)}(r_{29})^{(5)} < 0$

$with\ \ (p_{28})^{(5)}, (r_{29})^{(5)}$ as defined by equation are satisfied , then the system

**Theorem** If $(a''_i)^{(6)} and\ (b''_i)^{(6)}$ are independent on $t$ , and the conditions with the notations 434

$(a'_{32})^{(6)}(a'_{33})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} < 0$

$(a'_{32})^{(6)}(a'_{33})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a_{32})^{(6)}(p_{32})^{(6)} + (a'_{33})^{(6)}(p_{33})^{(6)} + (p_{32})^{(6)}(p_{33})^{(6)} > 0$

$(b'_{32})^{(6)}(b'_{33})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} > 0$ ,

$(b'_{32})^{(6)}(b'_{33})^{(6)} - (b_{32})^{(6)}(b_{33})^{(6)} - (b'_{32})^{(6)}(r_{33})^{(6)} - (b'_{33})^{(6)}(r_{33})^{(6)} + (r_{32})^{(6)}(r_{33})^{(6)} < 0$

$with\ \ (p_{32})^{(6)}, (r_{33})^{(6)}$ as defined by equation are satisfied , then the system

**Theorem** : If $(a''_i)^{(7)} and\ (b''_i)^{(7)}$ are independent on $t$ , and the conditions with the notations 435

$(a'_{36})^{(7)}(a'_{37})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} < 0$

$(a'_{36})^{(7)}(a'_{37})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a_{36})^{(7)}(p_{36})^{(7)} + (a'_{37})^{(7)}(p_{37})^{(7)} + (p_{36})^{(7)}(p_{37})^{(7)} > 0$

$(b'_{36})^{(7)}(b'_{37})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} > 0$ ,

$(b'_{36})^{(7)}(b'_{37})^{(7)} - (b_{36})^{(7)}(b_{37})^{(7)} - (b'_{36})^{(7)}(r_{37})^{(7)} - (b'_{37})^{(7)}(r_{37})^{(7)} + (r_{36})^{(7)}(r_{37})^{(7)} < 0$

$with\ \ (p_{36})^{(7)}, (r_{37})^{(7)}$ as defined by equation are satisfied , then the system

**Theorem** : If $(a''_i)^{(8)} and\ (b''_i)^{(8)}$ are independent on $t$ , and the conditions with the notations 436

$(a'_{40})^{(8)}(a'_{41})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} < 0$

$(a'_{40})^{(8)}(a'_{41})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a_{40})^{(8)}(p_{40})^{(8)} + (a'_{41})^{(8)}(p_{41})^{(8)} + (p_{40})^{(8)}(p_{41})^{(8)} > 0$

$(b'_{40})^{(8)}(b'_{41})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} > 0$ ,

$(b'_{40})^{(8)}(b'_{41})^{(8)} - (b_{40})^{(8)}(b_{41})^{(8)} - (b'_{40})^{(8)}(r_{41})^{(8)} - (b'_{41})^{(8)}(r_{41})^{(8)} + (r_{40})^{(8)}(r_{41})^{(8)} < 0$

$with\ \ (p_{40})^{(8)}, (r_{41})^{(8)}$ as defined by equation are satisfied , then the system

**Theorem** : If $(a''_i)^{(9)} and\ (b''_i)^{(9)}$ are independent on $t$ , and the conditions (with the notations 45,46,27,28) 436 A

$(a'_{44})^{(9)}(a'_{45})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} < 0$

$(a'_{44})^{(9)}(a'_{45})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a_{44})^{(9)}(p_{44})^{(9)} + (a'_{45})^{(9)}(p_{45})^{(9)} + (p_{44})^{(9)}(p_{45})^{(9)} > 0$

$(b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} > 0$ ,

$(b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} - (b'_{44})^{(9)}(r_{45})^{(9)} - (b'_{45})^{(9)}(r_{45})^{(9)} + (r_{44})^{(9)}(r_{45})^{(9)} < 0$

$with\ \ (p_{44})^{(9)}, (r_{45})^{(9)}$ as defined by equation 45 are satisfied , then the system

$(a_{13})^{(1)}G_{14} - \left[(a'_{13})^{(1)} + (a''_{13})^{(1)}(T_{14})\right]G_{13} = 0$ 437

$(a_{14})^{(1)}G_{13} - \left[(a'_{14})^{(1)} + (a''_{14})^{(1)}(T_{14})\right]G_{14} = 0$ 438

$(a_{15})^{(1)}G_{14} - \left[(a'_{15})^{(1)} + (a''_{15})^{(1)}(T_{14})\right]G_{15} = 0$ 439

$(b_{13})^{(1)}T_{14} - [(b'_{13})^{(1)} - (b''_{13})^{(1)}(G)\ ]T_{13} = 0$ 440

$(b_{14})^{(1)}T_{13} - [(b'_{14})^{(1)} - (b''_{14})^{(1)}(G)\ ]T_{14} = 0$ 441

$(b_{15})^{(1)}T_{14} - [(b'_{15})^{(1)} - (b''_{15})^{(1)}(G)\ ]T_{15} = 0$ 442

has a unique positive solution , which is an equilibrium solution for the system

$(a_{16})^{(2)}G_{17} - \left[(a'_{16})^{(2)} + (a''_{16})^{(2)}(T_{17})\right]G_{16} = 0$ 443

$(a_{17})^{(2)}G_{16} - \left[(a'_{17})^{(2)} + (a''_{17})^{(2)}(T_{17})\right]G_{17} = 0$ 444

$(a_{18})^{(2)}G_{17} - \left[(a'_{18})^{(2)} + (a''_{18})^{(2)}(T_{17})\right]G_{18} = 0$ 445

$(b_{16})^{(2)}T_{17} - [(b'_{16})^{(2)} - (b''_{16})^{(2)}(G_{19})\ ]T_{16} = 0$ 446

$(b_{17})^{(2)}T_{16} - [(b'_{17})^{(2)} - (b''_{17})^{(2)}(G_{19})\ ]T_{17} = 0$ 447

$(b_{18})^{(2)}T_{17} - [(b'_{18})^{(2)} - (b''_{18})^{(2)}(G_{19})\ ]T_{18} = 0$ 448

has a unique positive solution , which is an equilibrium solution

$(a_{20})^{(3)}G_{21} - \left[(a'_{20})^{(3)} + (a''_{20})^{(3)}(T_{21})\right]G_{20} = 0$ 449

$(a_{21})^{(3)}G_{20} - \left[(a'_{21})^{(3)} + (a''_{21})^{(3)}(T_{21})\right]G_{21} = 0$ 450

$(a_{22})^{(3)}G_{21} - \left[(a'_{22})^{(3)} + (a''_{22})^{(3)}(T_{21})\right]G_{22} = 0$ 451

$(b_{20})^{(3)}T_{21} - [(b'_{20})^{(3)} - (b''_{20})^{(3)}(G_{23})\ ]T_{20} = 0$ 452

$(b_{21})^{(3)}T_{20} - [(b'_{21})^{(3)} - (b''_{21})^{(3)}(G_{23})\ ]T_{21} = 0$ 453

$(b_{22})^{(3)}T_{21} - [(b'_{22})^{(3)} - (b''_{22})^{(3)}(G_{23})\ ]T_{22} = 0$ 454

has a unique positive solution , which is an equilibrium solution

$(a_{24})^{(4)}G_{25} - \left[(a'_{24})^{(4)} + (a''_{24})^{(4)}(T_{25})\right]G_{24} = 0$ 455

$(a_{25})^{(4)}G_{24} - \left[(a'_{25})^{(4)} + (a''_{25})^{(4)}(T_{25})\right]G_{25} = 0$ 456

$(a_{26})^{(4)}G_{25} - \left[(a'_{26})^{(4)} + (a''_{26})^{(4)}(T_{25})\right]G_{26} = 0$ 457

$(b_{24})^{(4)}T_{25} - [(b'_{24})^{(4)} - (b''_{24})^{(4)}\left((G_{27})\right)]T_{24} = 0$ 458

$(b_{25})^{(4)}T_{24} - [(b'_{25})^{(4)} - (b''_{25})^{(4)}\left((G_{27})\right)]T_{25} = 0$ 459

$(b_{26})^{(4)}T_{25} - [(b'_{26})^{(4)} - (b''_{26})^{(4)}\left((G_{27})\right)]T_{26} = 0$ 460

has a unique positive solution , which is an equilibrium solution

$(a_{28})^{(5)}G_{29} - \left[(a'_{28})^{(5)} + (a''_{28})^{(5)}(T_{29})\right]G_{28} = 0$ 461

$(a_{29})^{(5)}G_{28} - \left[(a'_{29})^{(5)} + (a''_{29})^{(5)}(T_{29})\right]G_{29} = 0$ 462

$(a_{30})^{(5)}G_{29} - \left[(a'_{30})^{(5)} + (a''_{30})^{(5)}(T_{29})\right]G_{30} = 0$ 463

$(b_{28})^{(5)}T_{29} - [(b'_{28})^{(5)} - (b''_{28})^{(5)}(G_{31})]T_{28} = 0$ 464

$(b_{29})^{(5)}T_{28} - [(b'_{29})^{(5)} - (b''_{29})^{(5)}(G_{31})]T_{29} = 0$ 465

$(b_{30})^{(5)}T_{29} - [(b'_{30})^{(5)} - (b''_{30})^{(5)}(G_{31})]T_{30} = 0$ 466

has a unique positive solution , which is an equilibrium solution

$(a_{32})^{(6)}G_{33} - \left[(a'_{32})^{(6)} + (a''_{32})^{(6)}(T_{33})\right]G_{32} = 0$ 467

$(a_{33})^{(6)}G_{32} - \left[(a'_{33})^{(6)} + (a''_{33})^{(6)}(T_{33})\right]G_{33} = 0$ 468

$(a_{34})^{(6)}G_{33} - \left[(a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33})\right]G_{34} = 0$ 469

$(b_{32})^{(6)}T_{33} - [(b'_{32})^{(6)} - (b''_{32})^{(6)}(G_{35})]T_{32} = 0$ 470

$(b_{33})^{(6)}T_{32} - [(b'_{33})^{(6)} - (b''_{33})^{(6)}(G_{35})]T_{33} = 0$ 471

$(b_{34})^{(6)}T_{33} - [(b'_{34})^{(6)} - (b''_{34})^{(6)}(G_{35})]T_{34} = 0$ 472

has a unique positive solution , which is an equilibrium solution

$(a_{36})^{(7)}G_{37} - \left[(a'_{36})^{(7)} + (a''_{36})^{(7)}(T_{37})\right]G_{36} = 0$ 473

$(a_{37})^{(7)}G_{36} - \left[(a'_{37})^{(7)} + (a''_{37})^{(7)}(T_{37})\right]G_{37} = 0$ 474

$(a_{38})^{(7)}G_{37} - \left[(a'_{38})^{(7)} + (a''_{38})^{(7)}(T_{37})\right]G_{38} = 0$ 475

$(b_{36})^{(7)}T_{37} - [(b'_{36})^{(7)} - (b''_{36})^{(7)}(G_{39})]T_{36} = 0$ 476

$(b_{37})^{(7)}T_{36} - [(b'_{37})^{(7)} - (b''_{37})^{(7)}(G_{39})\ ]T_{37} = 0$ 477

$(b_{38})^{(7)}T_{37} - [(b'_{38})^{(7)} - (b''_{38})^{(7)}(G_{39})\ ]T_{38} = 0$ 478

$(a_{40})^{(8)}G_{41} - \left[(a'_{40})^{(8)} + (a''_{40})^{(8)}(T_{41})\right]G_{40} = 0$ 479

$(a_{41})^{(8)}G_{40} - \left[(a'_{41})^{(8)} + (a''_{41})^{(8)}(T_{41})\right]G_{41} = 0$ 480

$(a_{42})^{(8)}G_{41} - \left[(a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41})\right]G_{42} = 0$ 481

$(b_{40})^{(8)}T_{41} - [(b'_{40})^{(8)} - (b''_{40})^{(8)}(G_{43})\ ]T_{40} = 0$ 482

$(b_{41})^{(8)}T_{40} - [(b'_{41})^{(8)} - (b''_{41})^{(8)}(G_{43})\ ]T_{41} = 0$ 483

$(b_{42})^{(8)}T_{41} - [(b'_{42})^{(8)} - (b''_{42})^{(8)}(G_{43})\ ]T_{42} = 0$ 484

$(a_{44})^{(9)}G_{45} - \left[(a'_{44})^{(9)} + (a''_{44})^{(9)}(T_{45})\right]G_{44} = 0$ 84A

$(a_{45})^{(9)}G_{44} - \left[(a'_{45})^{(9)} + (a''_{45})^{(9)}(T_{45})\right]G_{45} = 0$

$(a_{46})^{(9)}G_{45} - \left[(a'_{46})^{(9)} + (a''_{46})^{(9)}(T_{45})\right]G_{46} = 0$

$(b_{44})^{(9)}T_{45} - [(b'_{44})^{(9)} - (b''_{44})^{(9)}(G_{47})\ ]T_{44} = 0$

$(b_{45})^{(9)}T_{44} - [(b'_{45})^{(9)} - (b''_{45})^{(9)}(G_{47})\ ]T_{45} = 0$

$(b_{46})^{(9)}T_{45} - [(b'_{46})^{(9)} - (b''_{46})^{(9)}(G_{47})\ ]T_{46} = 0$

**Proof:** 485

(a) Indeed the first two equations have a nontrivial solution $G_{13}, G_{14}$ if

$F(T) = (a'_{13})^{(1)}(a'_{14})^{(1)} - (a_{13})^{(1)}(a_{14})^{(1)} + (a'_{13})^{(1)}(a''_{14})^{(1)}(T_{14}) + (a'_{14})^{(1)}(a''_{13})^{(1)}(T_{14}) + (a''_{13})^{(1)}(T_{14})(a''_{14})^{(1)}(T_{14}) = 0$

**Proof:** 486

(a) Indeed the first two equations have a nontrivial solution $G_{16}, G_{17}$ if

$\mathrm{F}(T_{19}) = (a'_{16})^{(2)}(a'_{17})^{(2)} - (a_{16})^{(2)}(a_{17})^{(2)} + (a'_{16})^{(2)}(a''_{17})^{(2)}(T_{17}) + (a'_{17})^{(2)}(a''_{16})^{(2)}(T_{17}) +$

$(a''_{16})^{(2)}(T_{17})(a''_{17})^{(2)}(T_{17}) = 0$

**Proof:** 487

(a) Indeed the first two equations have a nontrivial solution $G_{20}, G_{21}$ if

$F(T_{23}) = (a'_{20})^{(3)}(a'_{21})^{(3)} - (a_{20})^{(3)}(a_{21})^{(3)} + (a'_{20})^{(3)}(a''_{21})^{(3)}(T_{21}) + (a'_{21})^{(3)}(a''_{20})^{(3)}(T_{21}) + (a''_{20})^{(3)}(T_{21})(a''_{21})^{(3)}(T_{21}) = 0$

**Proof:** 488

(a) Indeed the first two equations have a nontrivial solution $G_{24}, G_{25}$ if

$F(T_{27}) = (a'_{24})^{(4)}(a'_{25})^{(4)} - (a_{24})^{(4)}(a_{25})^{(4)} + (a'_{24})^{(4)}(a''_{25})^{(4)}(T_{25}) + (a'_{25})^{(4)}(a''_{24})^{(4)}(T_{25}) + (a''_{24})^{(4)}(T_{25})(a''_{25})^{(4)}(T_{25}) = 0$

**Proof:** 489

(a) Indeed the first two equations have a nontrivial solution $G_{28}, G_{29}$ if

$F(T_{31}) = (a'_{28})^{(5)}(a'_{29})^{(5)} - (a_{28})^{(5)}(a_{29})^{(5)} + (a'_{28})^{(5)}(a''_{29})^{(5)}(T_{29}) + (a'_{29})^{(5)}(a''_{28})^{(5)}(T_{29}) + (a''_{28})^{(5)}(T_{29})(a''_{29})^{(5)}(T_{29}) = 0$

**Proof:** 490

(a) Indeed the first two equations have a nontrivial solution $G_{32}, G_{33}$ if

$F(T_{35}) = (a'_{32})^{(6)}(a'_{33})^{(6)} - (a_{32})^{(6)}(a_{33})^{(6)} + (a'_{32})^{(6)}(a''_{33})^{(6)}(T_{33}) + (a'_{33})^{(6)}(a''_{32})^{(6)}(T_{33}) + (a''_{32})^{(6)}(T_{33})(a''_{33})^{(6)}(T_{33}) = 0$

**Proof:** 491

(a) Indeed the first two equations have a nontrivial solution $G_{36}, G_{37}$ if

$F(T_{39}) = (a'_{36})^{(7)}(a'_{37})^{(7)} - (a_{36})^{(7)}(a_{37})^{(7)} + (a'_{36})^{(7)}(a''_{37})^{(7)}(T_{37}) + (a'_{37})^{(7)}(a''_{36})^{(7)}(T_{37}) + (a''_{36})^{(7)}(T_{37})(a''_{37})^{(7)}(T_{37}) = 0$

**Proof:** 492

(a) Indeed the first two equations have a nontrivial solution $G_{40}, G_{41}$ if

$F(T_{43}) = (a'_{40})^{(8)}(a'_{41})^{(8)} - (a_{40})^{(8)}(a_{41})^{(8)} + (a'_{40})^{(8)}(a''_{41})^{(8)}(T_{41}) + (a'_{41})^{(8)}(a''_{40})^{(8)}(T_{41}) + (a''_{40})^{(8)}(T_{41})(a''_{41})^{(8)}(T_{41}) = 0$

**Proof:** 492 A

(a) Indeed the first two equations have a nontrivial solution $G_{44}, G_{45}$ if

$F(T_{47}) = (a'_{44})^{(9)}(a'_{45})^{(9)} - (a_{44})^{(9)}(a_{45})^{(9)} + (a'_{44})^{(9)}(a''_{45})^{(9)}(T_{45}) + (a'_{45})^{(9)}(a''_{44})^{(9)}(T_{45}) + (a''_{44})^{(9)}(T_{45})(a''_{45})^{(9)}(T_{45}) = 0$

**Definition and uniqueness of** $T_{14}^{*}$ :- 493

After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i'')^{(1)}(T_{14})$ being increasing, it follows that there exists a unique $T_{14}^{*}$ for which $f(T_{14}^{*}) = 0$. With this value , we obtain from the three first equations

$$G_{13} = \frac{(a_{13})^{(1)}G_{14}}{[(a_{13}')^{(1)}+(a_{13}'')^{(1)}(T_{14}^{*})]} \quad , \quad G_{15} = \frac{(a_{15})^{(1)}G_{14}}{[(a_{15}')^{(1)}+(a_{15}'')^{(1)}(T_{14}^{*})]}$$

**Definition and uniqueness of** $T_{17}^{*}$ :- 494

After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i'')^{(2)}(T_{17})$ being increasing, it follows that there exists a unique $T_{17}^{*}$ for which $f(T_{17}^{*}) = 0$. With this value , we obtain from the three first equations

$$G_{16} = \frac{(a_{16})^{(2)}G_{17}}{[(a_{16}')^{(2)}+(a_{16}'')^{(2)}(T_{17}^{*})]} \quad , \quad G_{18} = \frac{(a_{18})^{(2)}G_{17}}{[(a_{18}')^{(2)}+(a_{18}'')^{(2)}(T_{17}^{*})]}$$ 495

**Definition and uniqueness of** $T_{21}^{*}$ :- 496

After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i'')^{(1)}(T_{21})$ being increasing, it follows that there exists a unique $T_{21}^{*}$ for which $f(T_{21}^{*}) = 0$. With this value , we obtain from the three first equations

$$G_{20} = \frac{(a_{20})^{(3)}G_{21}}{[(a_{20}')^{(3)}+(a_{20}'')^{(3)}(T_{21}^{*})]} \quad , \quad G_{22} = \frac{(a_{22})^{(3)}G_{21}}{[(a_{22}')^{(3)}+(a_{22}'')^{(3)}(T_{21}^{*})]}$$

**Definition and uniqueness of** $T_{25}^{*}$ :- 497

After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i'')^{(4)}(T_{25})$ being increasing, it follows that there exists a unique $T_{25}^{*}$ for which $f(T_{25}^{*}) = 0$. With this value , we obtain from the three first equations

$$G_{24} = \frac{(a_{24})^{(4)}G_{25}}{[(a_{24}')^{(4)}+(a_{24}'')^{(4)}(T_{25}^{*})]} \quad , \quad G_{26} = \frac{(a_{26})^{(4)}G_{25}}{[(a_{26}')^{(4)}+(a_{26}'')^{(4)}(T_{25}^{*})]}$$

**Definition and uniqueness of** $T_{29}^{*}$ :- 498

After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i'')^{(5)}(T_{29})$ being increasing, it follows that there exists a unique $T_{29}^{*}$ for which $f(T_{29}^{*}) = 0$. With this value , we obtain from the three first equations

$$G_{28} = \frac{(a_{28})^{(5)}G_{29}}{[(a_{28}')^{(5)}+(a_{28}'')^{(5)}(T_{29}^{*})]} \quad , \quad G_{30} = \frac{(a_{30})^{(5)}G_{29}}{[(a_{30}')^{(5)}+(a_{30}'')^{(5)}(T_{29}^{*})]}$$

**Definition and uniqueness of** $T_{33}^{*}$ :- 499

After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i'')^{(6)}(T_{33})$ being increasing, it follows that there exists a unique $T_{33}^{*}$ for which $f(T_{33}^{*}) = 0$. With this value , we obtain from the three first equations

$$G_{32} = \frac{(a_{32})^{(6)}G_{33}}{[(a_{32}')^{(6)}+(a_{32}'')^{(6)}(T_{33}^{*})]} \quad , \quad G_{34} = \frac{(a_{34})^{(6)}G_{33}}{[(a_{34}')^{(6)}+(a_{34}'')^{(6)}(T_{33}^{*})]}$$

**Definition and uniqueness of** $T_{37}^{*}$ :- 500

After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i'')^{(7)}(T_{37})$ being increasing, it follows that there exists a unique $T_{37}^{*}$ for which $f(T_{37}^{*}) = 0$. With this value , we obtain from the three first equations

$$G_{36} = \frac{(a_{36})^{(7)} G_{37}}{[(a_{36}')^{(7)} + (a_{36}'')^{(7)}(T_{37}^{*})]} \quad , \quad G_{38} = \frac{(a_{38})^{(7)} G_{37}}{[(a_{38}')^{(7)} + (a_{38}'')^{(7)}(T_{37}^{*})]}$$

**Definition and uniqueness of** $T_{41}^{*}$ :- 501

After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i'')^{(8)}(T_{41})$ being increasing, it follows that there exists a unique $T_{41}^{*}$ for which $f(T_{41}^{*}) = 0$. With this value , we obtain from the three first equations

$$G_{40} = \frac{(a_{40})^{(8)} G_{41}}{[(a_{40}')^{(8)} + (a_{40}'')^{(8)}(T_{41}^{*})]} \quad , \quad G_{42} = \frac{(a_{42})^{(8)} G_{41}}{[(a_{42}')^{(8)} + (a_{42}'')^{(8)}(T_{41}^{*})]}$$

**Definition and uniqueness of** $T_{45}^{*}$ :- 501 A

After hypothesis $f(0) < 0, f(\infty) > 0$ and the functions $(a_i'')^{(9)}(T_{45})$ being increasing, it follows that there exists a unique $T_{45}^{*}$ for which $f(T_{45}^{*}) = 0$. With this value , we obtain from the three first equations

$$G_{44} = \frac{(a_{44})^{(9)} G_{45}}{[(a_{44}')^{(9)} + (a_{44}'')^{(9)}(T_{45}^{*})]} \quad , \quad G_{46} = \frac{(a_{46})^{(9)} G_{45}}{[(a_{46}')^{(9)} + (a_{46}'')^{(9)}(T_{45}^{*})]}$$

By the same argument, the equations admit solutions $G_{13}, G_{14}$ if 502

$$\varphi(G) = (b_{13}')^{(1)}(b_{14}')^{(1)} - (b_{13})^{(1)}(b_{14})^{(1)} -$$

$$\left[(b_{13}')^{(1)}(b_{14}'')^{(1)}(G) + (b_{14}')^{(1)}(b_{13}'')^{(1)}(G)\right] + (b_{13}'')^{(1)}(G)(b_{14}'')^{(1)}(G) = 0$$

Where in $G(G_{13}, G_{14}, G_{15}), G_{13}, G_{15}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{14}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi(G^{*}) = 0$

By the same argument, the equations admit solutions $G_{16}, G_{17}$ if 503

$$\varphi(G_{19}) = (b_{16}')^{(2)}(b_{17}')^{(2)} - (b_{16})^{(2)}(b_{17})^{(2)} -$$

$$\left[(b_{16}')^{(2)}(b_{17}'')^{(2)}(G_{19}) + (b_{17}')^{(2)}(b_{16}'')^{(2)}(G_{19})\right] + (b_{16}'')^{(2)}(G_{19})(b_{17}'')^{(2)}(G_{19}) = 0$$

Where in $(G_{19})(G_{16}, G_{17}, G_{18}), G_{16}, G_{18}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{17}$ taking into account the hypothesis $\varphi(0) > 0$ , $\varphi(\infty) < 0$ it follows that there exists a unique $G_{14}^{*}$ such that $\varphi((G_{19})^{*}) = 0$ 504

By the same argument, the equations admit solutions $G_{20}, G_{21}$ if 505

$$\varphi(G_{23}) = (b_{20}')^{(3)}(b_{21}')^{(3)} - (b_{20})^{(3)}(b_{21})^{(3)} -$$

$\left[(b'_{20})^{(3)}(b''_{21})^{(3)}(G_{23})+(b'_{21})^{(3)}(b''_{20})^{(3)}(G_{23})\right]+(b''_{20})^{(3)}(G_{23})(b''_{21})^{(3)}(G_{23})=0$

Where in $G_{23}(G_{20},G_{21},G_{22}),G_{20},G_{22}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{21}$ taking into account the hypothesis $\varphi(0)>0\,,\varphi(\infty)\;<0$ it follows that there exists a unique $G_{21}^{*}$ such that $\varphi((G_{23})^{*})=0$

By the same argument, the equations admit solutions $G_{24},G_{25}$ if 506

$\varphi(G_{27})=(b'_{24})^{(4)}(b'_{25})^{(4)}-(b_{24})^{(4)}(b_{25})^{(4)}-$

$\left[(b'_{24})^{(4)}(b''_{25})^{(4)}(G_{27})+(b'_{25})^{(4)}(b''_{24})^{(4)}(G_{27})\right]+(b''_{24})^{(4)}(G_{27})(b''_{25})^{(4)}(G_{27})=0$

Where in $(G_{27})(G_{24},G_{25},G_{26}),G_{24},G_{26}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{25}$ taking into account the hypothesis $\varphi(0)>0\,,\varphi(\infty)\;<0$ it follows that there exists a unique $G_{25}^{*}$ such that $\varphi((G_{27})^{*})=0$

By the same argument, the equations admit solutions $G_{28},G_{29}$ if 507

$\varphi(G_{31})=(b'_{28})^{(5)}(b'_{29})^{(5)}-(b_{28})^{(5)}(b_{29})^{(5)}-$

$\left[(b'_{28})^{(5)}(b''_{29})^{(5)}(G_{31})+(b'_{29})^{(5)}(b''_{28})^{(5)}(G_{31})\right]+(b''_{28})^{(5)}(G_{31})(b''_{29})^{(5)}(G_{31})=0$

Where in $(G_{31})(G_{28},G_{29},G_{30}),G_{28},G_{30}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{29}$ taking into account the hypothesis $\varphi(0)>0\,,\varphi(\infty)\;<0$ it follows that there exists a unique $G_{29}^{*}$ such that $\varphi((G_{31})^{*})=0$

By the same argument, the equations admit solutions $G_{32},G_{33}$ if 508

$\varphi(G_{35})=(b'_{32})^{(6)}(b'_{33})^{(6)}-(b_{32})^{(6)}(b_{33})^{(6)}-$

$\left[(b'_{32})^{(6)}(b''_{33})^{(6)}(G_{35})+(b'_{33})^{(6)}(b''_{32})^{(6)}(G_{35})\right]+(b''_{32})^{(6)}(G_{35})(b''_{33})^{(6)}(G_{35})=0$

Where in $(G_{35})(G_{32},G_{33},G_{34}),G_{32},G_{34}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{33}$ taking into account the hypothesis $\varphi(0)>0\,,\varphi(\infty)\;<0$ it follows that there exists a unique $G_{33}^{*}$ such that $\varphi(G^{*})=0$

By the same argument, the equations admit solutions $G_{36},G_{37}$ if 509

$\varphi(G_{39})=(b'_{36})^{(7)}(b'_{37})^{(7)}-(b_{36})^{(7)}(b_{37})^{(7)}-$
$\left[(b'_{36})^{(7)}(b''_{37})^{(7)}(G_{39})+(b'_{37})^{(7)}(b''_{36})^{(7)}(G_{39})\right]+(b''_{36})^{(7)}(G_{39})(b''_{37})^{(7)}(G_{39})=0$

Where in $(G_{39})(G_{36},G_{37},G_{38}),G_{36},G_{38}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{37}$ taking into account the hypothesis $\varphi(0)>0\,,\varphi(\infty)\;<0$ it follows that there exists a unique $G_{37}^{*}$ such that $\varphi(G^{*})=0$

By the same argument, the equations admit solutions $G_{40},G_{41}$ if 510

$\varphi(G_{43})=(b'_{40})^{(8)}(b'_{41})^{(8)}-(b_{40})^{(8)}(b_{41})^{(8)}-$
$\left[(b'_{40})^{(8)}(b''_{41})^{(8)}(G_{43})+(b'_{41})^{(8)}(b''_{40})^{(8)}(G_{43})\right]+(b''_{40})^{(8)}(G_{43})(b''_{41})^{(8)}(G_{43})=0$

Where in $(G_{43})(G_{40}, G_{41}, G_{42}), G_{40}, G_{42}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{41}$ taking into account the hypothesis $\varphi(0) > 0, \varphi(\infty) < 0$ it follows that there exists a unique $G_{41}^*$ such that $\varphi(G^*) = 0$

By the same argument, the equations 92,93 admit solutions $G_{44}, G_{45}$ if

$$\varphi(G_{47}) = (b'_{44})^{(9)}(b'_{45})^{(9)} - (b_{44})^{(9)}(b_{45})^{(9)} -$$

$$\left[(b'_{44})^{(9)}(b''_{45})^{(9)}(G_{47}) + (b'_{45})^{(9)}(b''_{44})^{(9)}(G_{47})\right]+(b''_{44})^{(9)}(G_{47})(b''_{45})^{(9)}(G_{47}) = 0$$

Where in $(G_{47})(G_{44}, G_{45}, G_{46}), G_{44}, G_{46}$ must be replaced by their values from 96. It is easy to see that φ is a decreasing function in $G_{45}$ taking into account the hypothesis $\varphi(0) > 0, \varphi(\infty) < 0$ it follows that there exists a unique $G_{45}^*$ such that $\varphi((G_{47})^*) = 0$

Finally we obtain the unique solution 511

$G_{14}^*$ given by $\varphi(G^*) = 0$ , $T_{14}^*$ given by $f(T_{14}^*) = 0$ and

$$G_{13}^* = \frac{(a_{13})^{(1)}G_{14}^*}{[(a'_{13})^{(1)}+(a''_{13})^{(1)}(T_{14}^*)]} \quad , \quad G_{15}^* = \frac{(a_{15})^{(1)}G_{14}^*}{[(a'_{15})^{(1)}+(a''_{15})^{(1)}(T_{14}^*)]}$$

$$T_{13}^* = \frac{(b_{13})^{(1)}T_{14}^*}{[(b'_{13})^{(1)}-(b''_{13})^{(1)}(G^*)]} \quad , \quad T_{15}^* = \frac{(b_{15})^{(1)}T_{14}^*}{[(b'_{15})^{(1)}-(b''_{15})^{(1)}(G^*)]}$$

Obviously, these values represent an equilibrium solution

Finally we obtain the unique solution

$G_{17}^*$ given by $\varphi((G_{19})^*) = 0$ , $T_{17}^*$ given by $f(T_{17}^*) = 0$ and 512

$$G_{16}^* = \frac{(a_{16})^{(2)}G_{17}^*}{[(a'_{16})^{(2)}+(a''_{16})^{(2)}(T_{17}^*)]} \quad , \quad G_{18}^* = \frac{(a_{18})^{(2)}G_{17}^*}{[(a'_{18})^{(2)}+(a''_{18})^{(2)}(T_{17}^*)]}$$ 513

$$T_{16}^* = \frac{(b_{16})^{(2)}T_{17}^*}{[(b'_{16})^{(2)}-(b''_{16})^{(2)}((G_{19})^*)]} \quad , \quad T_{18}^* = \frac{(b_{18})^{(2)}T_{17}^*}{[(b'_{18})^{(2)}-(b''_{18})^{(2)}((G_{19})^*)]}$$ 514

Obviously, these values represent an equilibrium solution

Finally we obtain the unique solution 515

$G_{21}^*$ given by $\varphi((G_{23})^*) = 0$ , $T_{21}^*$ given by $f(T_{21}^*) = 0$ and

$$G_{20}^* = \frac{(a_{20})^{(3)}G_{21}^*}{[(a'_{20})^{(3)}+(a''_{20})^{(3)}(T_{21}^*)]} \quad , \quad G_{22}^* = \frac{(a_{22})^{(3)}G_{21}^*}{[(a'_{22})^{(3)}+(a''_{22})^{(3)}(T_{21}^*)]}$$

$$T_{20}^* = \frac{(b_{20})^{(3)}T_{21}^*}{[(b'_{20})^{(3)}-(b''_{20})^{(3)}(G_{23}{}^*)]} \quad , \quad T_{22}^* = \frac{(b_{22})^{(3)}T_{21}^*}{[(b'_{22})^{(3)}-(b''_{22})^{(3)}(G_{23}{}^*)]}$$

Obviously, these values represent an equilibrium solution of global equations

Finally we obtain the unique solution 516

$G_{25}^*$ given by $\varphi(G_{27}) = 0$ , $T_{25}^*$ given by $f(T_{25}^*) = 0$ and

$G_{24}^* = \frac{(a_{24})^{(4)} G_{25}^*}{[(a'_{24})^{(4)} + (a''_{24})^{(4)}(T_{25}^*)]}$ , $G_{26}^* = \frac{(a_{26})^{(4)} G_{25}^*}{[(a'_{26})^{(4)} + (a''_{26})^{(4)}(T_{25}^*)]}$

$T_{24}^* = \frac{(b_{24})^{(4)} T_{25}^*}{[(b'_{24})^{(4)} - (b''_{24})^{(4)}((G_{27})^*)]}$ , $T_{26}^* = \frac{(b_{26})^{(4)} T_{25}^*}{[(b'_{26})^{(4)} - (b''_{26})^{(4)}((G_{27})^*)]}$ 517

Obviously, these values represent an equilibrium solution of global equations

Finally we obtain the unique solution 518

$G_{29}^*$ given by $\varphi((G_{31})^*) = 0$ , $T_{29}^*$ given by $f(T_{29}^*) = 0$ and

$G_{28}^* = \frac{(a_{28})^{(5)} G_{29}^*}{[(a'_{28})^{(5)} + (a''_{28})^{(5)}(T_{29}^*)]}$ , $G_{30}^* = \frac{(a_{30})^{(5)} G_{29}^*}{[(a'_{30})^{(5)} + (a''_{30})^{(5)}(T_{29}^*)]}$

$T_{28}^* = \frac{(b_{28})^{(5)} T_{29}^*}{[(b'_{28})^{(5)} - (b''_{28})^{(5)}((G_{31})^*)]}$ , $T_{30}^* = \frac{(b_{30})^{(5)} T_{29}^*}{[(b'_{30})^{(5)} - (b''_{30})^{(5)}((G_{31})^*)]}$ 519

Obviously, these values represent an equilibrium solution of global equations

Finally we obtain the unique solution 520

$G_{33}^*$ given by $\varphi((G_{35})^*) = 0$ , $T_{33}^*$ given by $f(T_{33}^*) = 0$ and

$G_{32}^* = \frac{(a_{32})^{(6)} G_{33}^*}{[(a'_{32})^{(6)} + (a''_{32})^{(6)}(T_{33}^*)]}$ , $G_{34}^* = \frac{(a_{34})^{(6)} G_{33}^*}{[(a'_{34})^{(6)} + (a''_{34})^{(6)}(T_{33}^*)]}$

$T_{32}^* = \frac{(b_{32})^{(6)} T_{33}^*}{[(b'_{32})^{(6)} - (b''_{32})^{(6)}((G_{35})^*)]}$ , $T_{34}^* = \frac{(b_{34})^{(6)} T_{33}^*}{[(b'_{34})^{(6)} - (b''_{34})^{(6)}((G_{35})^*)]}$ 521

Obviously, these values represent an equilibrium solution of global equations

Finally we obtain the unique solution 522

$G_{37}^*$ given by $\varphi((G_{39})^*) = 0$ , $T_{37}^*$ given by $f(T_{37}^*) = 0$ and

$G_{36}^* = \frac{(a_{36})^{(7)} G_{37}^*}{[(a'_{36})^{(7)} + (a''_{36})^{(7)}(T_{37}^*)]}$ , $G_{38}^* = \frac{(a_{38})^{(7)} G_{37}^*}{[(a'_{38})^{(7)} + (a''_{38})^{(7)}(T_{37}^*)]}$

$T_{36}^* = \frac{(b_{36})^{(7)} T_{37}^*}{[(b'_{36})^{(7)} - (b''_{36})^{(7)}((G_{39})^*)]}$ , $T_{38}^* = \frac{(b_{38})^{(7)} T_{37}^*}{[(b'_{38})^{(7)} - (b''_{38})^{(7)}((G_{39})^*)]}$

Finally we obtain the unique solution 523

$G_{41}^*$ given by $\varphi((G_{43})^*) = 0$ , $T_{41}^*$ given by $f(T_{41}^*) = 0$ and

$G_{40}^* = \frac{(a_{40})^{(8)} G_{41}^*}{[(a'_{40})^{(8)} + (a''_{40})^{(8)}(T_{41}^*)]}$ , $G_{42}^* = \frac{(a_{42})^{(8)} G_{41}^*}{[(a'_{42})^{(8)} + (a''_{42})^{(8)}(T_{41}^*)]}$

$T_{40}^* = \frac{(b_{40})^{(8)} T_{41}^*}{[(b'_{40})^{(8)} - (b''_{40})^{(8)}((G_{43})^*)]}$ , $T_{42}^* = \frac{(b_{42})^{(8)} T_{41}^*}{[(b'_{42})^{(8)} - (b''_{42})^{(8)}((G_{43})^*)]}$

Finally we obtain the unique solution of 89 to 99 523 A

$G_{45}^*$ given by $\varphi((G_{47})^*) = 0$ , $T_{45}^*$ given by $f(T_{45}^*) = 0$ and

$$G_{44}^* = \frac{(a_{44})^{(9)} G_{45}^*}{[(a_{44}')^{(9)} + (a_{44}'')^{(9)}(T_{45}^*)]} \quad , \quad G_{46}^* = \frac{(a_{46})^{(9)} G_{45}^*}{[(a_{46}')^{(9)} + (a_{46}'')^{(9)}(T_{45}^*)]}$$

$$T_{44}^* = \frac{(b_{44})^{(9)} T_{45}^*}{[(b_{44}')^{(9)} - (b_{44}'')^{(9)}((G_{47})^*)]} \quad , \quad T_{46}^* = \frac{(b_{46})^{(9)} T_{45}^*}{[(b_{46}')^{(9)} - (b_{46}'')^{(9)}((G_{47})^*)]}$$

**ASYMPTOTIC STABILITY ANALYSIS** 524

**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i'')^{(1)}$ *and* $(b_i'')^{(1)}$ Belong to $C^{(1)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

**Proof:** Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-

$$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{14}'')^{(1)}}{\partial T_{14}}(T_{14}^*) = (q_{14})^{(1)} \quad , \frac{\partial (b_i'')^{(1)}}{\partial G_j}(G^*) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain

$$\frac{d\mathbb{G}_{13}}{dt} = -\left((a_{13}')^{(1)} + (p_{13})^{(1)}\right)\mathbb{G}_{13} + (a_{13})^{(1)}\mathbb{G}_{14} - (q_{13})^{(1)} G_{13}^* \mathbb{T}_{14}$$ 525

$$\frac{d\mathbb{G}_{14}}{dt} = -\left((a_{14}')^{(1)} + (p_{14})^{(1)}\right)\mathbb{G}_{14} + (a_{14})^{(1)}\mathbb{G}_{13} - (q_{14})^{(1)} G_{14}^* \mathbb{T}_{14}$$ 526

$$\frac{d\mathbb{G}_{15}}{dt} = -\left((a_{15}')^{(1)} + (p_{15})^{(1)}\right)\mathbb{G}_{15} + (a_{15})^{(1)}\mathbb{G}_{14} - (q_{15})^{(1)} G_{15}^* \mathbb{T}_{14}$$ 527

$$\frac{d\mathbb{T}_{13}}{dt} = -\left((b_{13}')^{(1)} - (r_{13})^{(1)}\right)\mathbb{T}_{13} + (b_{13})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(13)(j)} T_{13}^* \mathbb{G}_j\right)$$ 528

$$\frac{d\mathbb{T}_{14}}{dt} = -\left((b_{14}')^{(1)} - (r_{14})^{(1)}\right)\mathbb{T}_{14} + (b_{14})^{(1)}\mathbb{T}_{13} + \sum_{j=13}^{15}\left(s_{(14)(j)} T_{14}^* \mathbb{G}_j\right)$$ 529

$$\frac{d\mathbb{T}_{15}}{dt} = -\left((b_{15}')^{(1)} - (r_{15})^{(1)}\right)\mathbb{T}_{15} + (b_{15})^{(1)}\mathbb{T}_{14} + \sum_{j=13}^{15}\left(s_{(15)(j)} T_{15}^* \mathbb{G}_j\right)$$ 530

**ASYMPTOTIC STABILITY ANALYSIS** 531

**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a_i'')^{(2)}$ and $(b_i'')^{(2)}$ Belong to $C^{(2)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-

$$G_i = G_i^* + \mathbb{G}_i \quad , T_i = T_i^* + \mathbb{T}_i$$ 532

$$\frac{\partial (a_{17}'')^{(2)}}{\partial T_{17}}(T_{17}^*) = (q_{17})^{(2)} \quad , \frac{\partial (b_i'')^{(2)}}{\partial G_j}((G_{19})^*) = s_{ij}$$ 533

taking into account equations and neglecting the terms of power 2, we obtain

$\frac{d\mathbb{G}_{16}}{dt} = -\left((a'_{16})^{(2)} + (p_{16})^{(2)}\right)\mathbb{G}_{16} + (a_{16})^{(2)}\mathbb{G}_{17} - (q_{16})^{(2)}G_{16}^{*}\mathbb{T}_{17}$ 534

$\frac{d\mathbb{G}_{17}}{dt} = -\left((a'_{17})^{(2)} + (p_{17})^{(2)}\right)\mathbb{G}_{17} + (a_{17})^{(2)}\mathbb{G}_{16} - (q_{17})^{(2)}G_{17}^{*}\mathbb{T}_{17}$ 535

$\frac{d\mathbb{G}_{18}}{dt} = -\left((a'_{18})^{(2)} + (p_{18})^{(2)}\right)\mathbb{G}_{18} + (a_{18})^{(2)}\mathbb{G}_{17} - (q_{18})^{(2)}G_{18}^{*}\mathbb{T}_{17}$ 536

$\frac{d\mathbb{T}_{16}}{dt} = -\left((b'_{16})^{(2)} - (r_{16})^{(2)}\right)\mathbb{T}_{16} + (b_{16})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(16)(j)}T_{16}^{*}\mathbb{G}_{j}\right)$ 537

$\frac{d\mathbb{T}_{17}}{dt} = -\left((b'_{17})^{(2)} - (r_{17})^{(2)}\right)\mathbb{T}_{17} + (b_{17})^{(2)}\mathbb{T}_{16} + \sum_{j=16}^{18}\left(s_{(17)(j)}T_{17}^{*}\mathbb{G}_{j}\right)$ 538

$\frac{d\mathbb{T}_{18}}{dt} = -\left((b'_{18})^{(2)} - (r_{18})^{(2)}\right)\mathbb{T}_{18} + (b_{18})^{(2)}\mathbb{T}_{17} + \sum_{j=16}^{18}\left(s_{(18)(j)}T_{18}^{*}\mathbb{G}_{j}\right)$ 539

**ASYMPTOTIC STABILITY ANALYSIS** 540

**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a''_i)^{(3)}$ *and* $(b''_i)^{(3)}$ Belong to $C^{(3)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-

$$G_i = G_i^* + \mathbb{G}_i \qquad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a''_{21})^{(3)}}{\partial T_{21}}(T_{21}^*) = (q_{21})^{(3)} \quad , \frac{\partial (b''_i)^{(3)}}{\partial G_j}\left((G_{23})^*\right) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain

$\frac{d\mathbb{G}_{20}}{dt} = -\left((a'_{20})^{(3)} + (p_{20})^{(3)}\right)\mathbb{G}_{20} + (a_{20})^{(3)}\mathbb{G}_{21} - (q_{20})^{(3)}G_{20}^{*}\mathbb{T}_{21}$ 541

$\frac{d\mathbb{G}_{21}}{dt} = -\left((a'_{21})^{(3)} + (p_{21})^{(3)}\right)\mathbb{G}_{21} + (a_{21})^{(3)}\mathbb{G}_{20} - (q_{21})^{(3)}G_{21}^{*}\mathbb{T}_{21}$ 542

$\frac{d\mathbb{G}_{22}}{dt} = -\left((a'_{22})^{(3)} + (p_{22})^{(3)}\right)\mathbb{G}_{22} + (a_{22})^{(3)}\mathbb{G}_{21} - (q_{22})^{(3)}G_{22}^{*}\mathbb{T}_{21}$ 543

$\frac{d\mathbb{T}_{20}}{dt} = -\left((b'_{20})^{(3)} - (r_{20})^{(3)}\right)\mathbb{T}_{20} + (b_{20})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(20)(j)}T_{20}^{*}\mathbb{G}_{j}\right)$ 544

$\frac{d\mathbb{T}_{21}}{dt} = -\left((b'_{21})^{(3)} - (r_{21})^{(3)}\right)\mathbb{T}_{21} + (b_{21})^{(3)}\mathbb{T}_{20} + \sum_{j=20}^{22}\left(s_{(21)(j)}T_{21}^{*}\mathbb{G}_{j}\right)$ 545

$\frac{d\mathbb{T}_{22}}{dt} = -\left((b'_{22})^{(3)} - (r_{22})^{(3)}\right)\mathbb{T}_{22} + (b_{22})^{(3)}\mathbb{T}_{21} + \sum_{j=20}^{22}\left(s_{(22)(j)}T_{22}^{*}\mathbb{G}_{j}\right)$ 546

**ASYMPTOTIC STABILITY ANALYSIS** 547

**Theorem 4:** If the conditions of the previous theorem are satisfied and if the functions $(a''_i)^{(4)}$ *and* $(b''_i)^{(4)}$ Belong to $C^{(4)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- 548

$$G_i = G_i^* + \mathbb{G}_i \qquad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{25}'')^{(4)}}{\partial T_{25}}(T_{25}^*) = (q_{25})^{(4)} \quad , \frac{\partial (b_i'')^{(4)}}{\partial G_j}((G_{27})^* \ ) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain

$$\frac{d\mathbb{G}_{24}}{dt} = -\left((a_{24}')^{(4)} + (p_{24})^{(4)}\right)\mathbb{G}_{24} + (a_{24})^{(4)}\mathbb{G}_{25} - (q_{24})^{(4)}G_{24}^*\mathbb{T}_{25}$$ 549

$$\frac{d\mathbb{G}_{25}}{dt} = -\left((a_{25}')^{(4)} + (p_{25})^{(4)}\right)\mathbb{G}_{25} + (a_{25})^{(4)}\mathbb{G}_{24} - (q_{25})^{(4)}G_{25}^*\mathbb{T}_{25}$$ 550

$$\frac{d\mathbb{G}_{26}}{dt} = -\left((a_{26}')^{(4)} + (p_{26})^{(4)}\right)\mathbb{G}_{26} + (a_{26})^{(4)}\mathbb{G}_{25} - (q_{26})^{(4)}G_{26}^*\mathbb{T}_{25}$$ 551

$$\frac{d\mathbb{T}_{24}}{dt} = -\left((b_{24}')^{(4)} - (r_{24})^{(4)}\right)\mathbb{T}_{24} + (b_{24})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(24)(j)}T_{24}^*\mathbb{G}_j\right)$$ 552

$$\frac{d\mathbb{T}_{25}}{dt} = -\left((b_{25}')^{(4)} - (r_{25})^{(4)}\right)\mathbb{T}_{25} + (b_{25})^{(4)}\mathbb{T}_{24} + \sum_{j=24}^{26}\left(s_{(25)(j)}T_{25}^*\mathbb{G}_j\right)$$ 553

$$\frac{d\mathbb{T}_{26}}{dt} = -\left((b_{26}')^{(4)} - (r_{26})^{(4)}\right)\mathbb{T}_{26} + (b_{26})^{(4)}\mathbb{T}_{25} + \sum_{j=24}^{26}\left(s_{(26)(j)}T_{26}^*\mathbb{G}_j\right)$$ 554

**ASYMPTOTIC STABILITY ANALYSIS** 555

**Theorem 5:** If the conditions of the previous theorem are satisfied and if the functions $(a_i'')^{(5)}$ *and* $(b_i'')^{(5)}$ Belong to $C^{(5)}(\ \mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- 556

$$G_i = G_i^* + \mathbb{G}_i \qquad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{29}'')^{(5)}}{\partial T_{29}}(T_{29}^*) = (q_{29})^{(5)} \quad , \frac{\partial (b_i'')^{(5)}}{\partial G_j}(\ (G_{31})^*\ ) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain

$$\frac{d\mathbb{G}_{28}}{dt} = -\left((a_{28}')^{(5)} + (p_{28})^{(5)}\right)\mathbb{G}_{28} + (a_{28})^{(5)}\mathbb{G}_{29} - (q_{28})^{(5)}G_{28}^*\mathbb{T}_{29}$$ 557

$$\frac{d\mathbb{G}_{29}}{dt} = -\left((a_{29}')^{(5)} + (p_{29})^{(5)}\right)\mathbb{G}_{29} + (a_{29})^{(5)}\mathbb{G}_{28} - (q_{29})^{(5)}G_{29}^*\mathbb{T}_{29}$$ 558

$$\frac{d\mathbb{G}_{30}}{dt} = -\left((a_{30}')^{(5)} + (p_{30})^{(5)}\right)\mathbb{G}_{30} + (a_{30})^{(5)}\mathbb{G}_{29} - (q_{30})^{(5)}G_{30}^*\mathbb{T}_{29}$$ 559

$$\frac{d\mathbb{T}_{28}}{dt} = -\left((b_{28}')^{(5)} - (r_{28})^{(5)}\right)\mathbb{T}_{28} + (b_{28})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(28)(j)}T_{28}^*\mathbb{G}_j\right)$$ 560

$$\frac{d\mathbb{T}_{29}}{dt} = -\left((b_{29}')^{(5)} - (r_{29})^{(5)}\right)\mathbb{T}_{29} + (b_{29})^{(5)}\mathbb{T}_{28} + \sum_{j=28}^{30}\left(s_{(29)(j)}T_{29}^*\mathbb{G}_j\right)$$ 561

$$\frac{d\mathbb{T}_{30}}{dt} = -\left((b_{30}')^{(5)} - (r_{30})^{(5)}\right)\mathbb{T}_{30} + (b_{30})^{(5)}\mathbb{T}_{29} + \sum_{j=28}^{30}\left(s_{(30)(j)}T_{30}^*\mathbb{G}_j\right)$$ 562

**ASYMPTOTIC STABILITY ANALYSIS** 563

**Theorem 6:** If the conditions of the previous theorem are satisfied and if the functions $(a_i'')^{(6)}$ *and* $(b_i'')^{(6)}$ Belong to $C^{(6)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- 564

$$G_i = G_i^* + \mathbb{G}_i \qquad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{33}'')^{(6)}}{\partial T_{33}}(T_{33}^*) = (q_{33})^{(6)} \quad , \frac{\partial (b_i'')^{(6)}}{\partial G_j}(\,(G_{35})^*\,) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain

$$\frac{d\mathbb{G}_{32}}{dt} = -\left((a_{32}')^{(6)} + (p_{32})^{(6)}\right)\mathbb{G}_{32} + (a_{32})^{(6)}\mathbb{G}_{33} - (q_{32})^{(6)}G_{32}^*\mathbb{T}_{33}$$ 565

$$\frac{d\mathbb{G}_{33}}{dt} = -\left((a_{33}')^{(6)} + (p_{33})^{(6)}\right)\mathbb{G}_{33} + (a_{33})^{(6)}\mathbb{G}_{32} - (q_{33})^{(6)}G_{33}^*\mathbb{T}_{33}$$ 566

$$\frac{d\mathbb{G}_{34}}{dt} = -\left((a_{34}')^{(6)} + (p_{34})^{(6)}\right)\mathbb{G}_{34} + (a_{34})^{(6)}\mathbb{G}_{33} - (q_{34})^{(6)}G_{34}^*\mathbb{T}_{33}$$ 567

$$\frac{d\mathbb{T}_{32}}{dt} = -\left((b_{32}')^{(6)} - (r_{32})^{(6)}\right)\mathbb{T}_{32} + (b_{32})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(32)(j)}T_{32}^*\mathbb{G}_j\right)$$ 568

$$\frac{d\mathbb{T}_{33}}{dt} = -\left((b_{33}')^{(6)} - (r_{33})^{(6)}\right)\mathbb{T}_{33} + (b_{33})^{(6)}\mathbb{T}_{32} + \sum_{j=32}^{34}\left(s_{(33)(j)}T_{33}^*\mathbb{G}_j\right)$$ 569

$$\frac{d\mathbb{T}_{34}}{dt} = -\left((b_{34}')^{(6)} - (r_{34})^{(6)}\right)\mathbb{T}_{34} + (b_{34})^{(6)}\mathbb{T}_{33} + \sum_{j=32}^{34}\left(s_{(34)(j)}T_{34}^*\mathbb{G}_j\right)$$ 570

**ASYMPTOTIC STABILITY ANALYSIS** 571

**Theorem 7:** If the conditions of the previous theorem are satisfied and if the functions $(a_i'')^{(7)}$ *and* $(b_i'')^{(7)}$ Belong to $C^{(7)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- 572

$$G_i = G_i^* + \mathbb{G}_i \qquad , T_i = T_i^* + \mathbb{T}_i$$

$$\frac{\partial (a_{37}'')^{(7)}}{\partial T_{37}}(T_{37}^*) = (q_{37})^{(7)} \quad , \frac{\partial (b_i'')^{(7)}}{\partial G_j}\left(\,(G_{39})^{**}\,\right) = s_{ij}$$

Then taking into account equations and neglecting the terms of power 2, we obtain from

$$\frac{d\mathbb{G}_{36}}{dt} = -\left((a_{36}')^{(7)} + (p_{36})^{(7)}\right)\mathbb{G}_{36} + (a_{36})^{(7)}\mathbb{G}_{37} - (q_{36})^{(7)}G_{36}^*\mathbb{T}_{37}$$ 573

$$\frac{d\mathbb{G}_{37}}{dt} = -\left((a_{37}')^{(7)} + (p_{37})^{(7)}\right)\mathbb{G}_{37} + (a_{37})^{(7)}\mathbb{G}_{36} - (q_{37})^{(7)}G_{37}^*\mathbb{T}_{37}$$ 574

$\frac{d\mathbb{G}_{38}}{dt} = -\left((a'_{38})^{(7)} + (p_{38})^{(7)}\right)\mathbb{G}_{38} + (a_{38})^{(7)}\mathbb{G}_{37} - (q_{38})^{(7)}G^*_{38}\mathbb{T}_{37}$ 575

$\frac{d\mathbb{T}_{36}}{dt} = -\left((b'_{36})^{(7)} - (r_{36})^{(7)}\right)\mathbb{T}_{36} + (b_{36})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(36)(j)}T^*_{36}\mathbb{G}_j\right)$ 576

$\frac{d\mathbb{T}_{37}}{dt} = -\left((b'_{37})^{(7)} - (r_{37})^{(7)}\right)\mathbb{T}_{37} + (b_{37})^{(7)}\mathbb{T}_{36} + \sum_{j=36}^{38}\left(s_{(37)(j)}T^*_{37}\mathbb{G}_j\right)$ 578

$\frac{d\mathbb{T}_{38}}{dt} = -\left((b'_{38})^{(7)} - (r_{38})^{(7)}\right)\mathbb{T}_{38} + (b_{38})^{(7)}\mathbb{T}_{37} + \sum_{j=36}^{38}\left(s_{(38)(j)}T^*_{38}\mathbb{G}_j\right)$ 579

Obviously, these values represent an equilibrium solution

**ASYMPTOTIC STABILITY ANALYSIS**

**Theorem 8:** If the conditions of the previous theorem are satisfied and if the functions $(a''_i)^{(8)}$ *and* $(b''_i)^{(8)}$ Belong to $C^{(8)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :- 580

$G_i = G_i^* + \mathbb{G}_i \qquad , T_i = T_i^* + \mathbb{T}_i$

$\frac{\partial (a''_{41})^{(8)}}{\partial T_{41}}(T^*_{41}) = (q_{41})^{(8)} \quad , \frac{\partial (b''_i)^{(8)}}{\partial G_j}(\,(G_{43})^*\,) = s_{ij}$

Then taking into account equations and neglecting the terms of power 2, we obtain

$\frac{d\mathbb{G}_{40}}{dt} = -\left((a'_{40})^{(8)} + (p_{40})^{(8)}\right)\mathbb{G}_{40} + (a_{40})^{(8)}\mathbb{G}_{41} - (q_{40})^{(8)}G^*_{40}\mathbb{T}_{41}$ 581

$\frac{d\mathbb{G}_{41}}{dt} = -\left((a'_{41})^{(8)} + (p_{41})^{(8)}\right)\mathbb{G}_{41} + (a_{41})^{(8)}\mathbb{G}_{40} - (q_{41})^{(8)}G^*_{41}\mathbb{T}_{41}$ 582

$\frac{d\mathbb{G}_{42}}{dt} = -\left((a'_{42})^{(8)} + (p_{42})^{(8)}\right)\mathbb{G}_{42} + (a_{42})^{(8)}\mathbb{G}_{41} - (q_{42})^{(8)}G^*_{42}\mathbb{T}_{41}$ 583

$\frac{d\mathbb{T}_{40}}{dt} = -\left((b'_{40})^{(8)} - (r_{40})^{(8)}\right)\mathbb{T}_{40} + (b_{40})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(40)(j)}T^*_{40}\mathbb{G}_j\right)$ 584

$\frac{d\mathbb{T}_{41}}{dt} = -\left((b'_{41})^{(8)} - (r_{41})^{(8)}\right)\mathbb{T}_{41} + (b_{41})^{(8)}\mathbb{T}_{40} + \sum_{j=40}^{42}\left(s_{(41)(j)}T^*_{41}\mathbb{G}_j\right)$ 585

$\frac{d\mathbb{T}_{42}}{dt} = -\left((b'_{42})^{(8)} - (r_{42})^{(8)}\right)\mathbb{T}_{42} + (b_{42})^{(8)}\mathbb{T}_{41} + \sum_{j=40}^{42}\left(s_{(42)(j)}T^*_{42}\mathbb{G}_j\right)$ 586

**ASYMPTOTIC STABILITY ANALYSIS** 586 A

**Theorem 9:** If the conditions of the previous theorem are satisfied and if the functions $(a''_i)^{(9)}$ *and* $(b''_i)^{(9)}$ Belong to $C^{(9)}(\mathbb{R}_+)$ then the above equilibrium point is asymptotically stable.

Proof: Denote

**Definition of** $\mathbb{G}_i, \mathbb{T}_i$ :-

$G_i = G_i^* + \mathbb{G}_i \qquad , T_i = T_i^* + \mathbb{T}_i$

$$\frac{\partial(a''_{45})^{(9)}}{\partial T_{45}}(T^*_{45}) = (q_{45})^{(9)} \quad , \frac{\partial(b''_i)^{(9)}}{\partial G_j}(\,(G_{47})^*\,) = s_{ij}$$

Then taking into account equations 89 to 99 and neglecting the terms of power 2, we obtain from 99 to 44

$$\frac{d\mathbb{G}_{44}}{dt} = -\left((a'_{44})^{(9)} + (p_{44})^{(9)}\right)\mathbb{G}_{44} + (a_{44})^{(9)}\mathbb{G}_{45} - (q_{44})^{(9)}G^*_{44}\mathbb{T}_{45}$$ 586 B

$$\frac{d\mathbb{G}_{45}}{dt} = -\left((a'_{45})^{(9)} + (p_{45})^{(9)}\right)\mathbb{G}_{45} + (a_{45})^{(9)}\mathbb{G}_{44} - (q_{45})^{(9)}G^*_{45}\mathbb{T}_{45}$$ 586 C

$$\frac{d\mathbb{G}_{46}}{dt} = -\left((a'_{46})^{(9)} + (p_{46})^{(9)}\right)\mathbb{G}_{46} + (a_{46})^{(9)}\mathbb{G}_{45} - (q_{46})^{(9)}G^*_{46}\mathbb{T}_{45}$$ 586 D

$$\frac{d\mathbb{T}_{44}}{dt} = -\left((b'_{44})^{(9)} - (r_{44})^{(9)}\right)\mathbb{T}_{44} + (b_{44})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(44)(j)}T^*_{44}\mathbb{G}_j\right)$$ 586 E

$$\frac{d\mathbb{T}_{45}}{dt} = -\left((b'_{45})^{(9)} - (r_{45})^{(9)}\right)\mathbb{T}_{45} + (b_{45})^{(9)}\mathbb{T}_{44} + \sum_{j=44}^{46}\left(s_{(45)(j)}T^*_{45}\mathbb{G}_j\right)$$ 586 F

$$\frac{d\mathbb{T}_{46}}{dt} = -\left((b'_{46})^{(9)} - (r_{46})^{(9)}\right)\mathbb{T}_{46} + (b_{46})^{(9)}\mathbb{T}_{45} + \sum_{j=44}^{46}\left(s_{(46)(j)}T^*_{46}\mathbb{G}_j\right)$$ 586 G

**The characteristic equation of this system is** 587

$$\left((\lambda)^{(1)} + (b'_{15})^{(1)} - (r_{15})^{(1)}\right)\{\left((\lambda)^{(1)} + (a'_{15})^{(1)} + (p_{15})^{(1)}\right)$$

$$\left[\left(\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(q_{13})^{(1)}G^*_{13}\right)\right]$$

$$\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(14)}T^*_{14} + (b_{14})^{(1)}s_{(13),(14)}T^*_{14}\right)$$

$$+\left(\left((\lambda)^{(1)} + (a'_{14})^{(1)} + (p_{14})^{(1)}\right)(q_{13})^{(1)}G^*_{13} + (a_{13})^{(1)}(q_{14})^{(1)}G^*_{14}\right)$$

$$\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(13)}T^*_{14} + (b_{14})^{(1)}s_{(13),(13)}T^*_{13}\right)$$

$$\left(\left((\lambda)^{(1)}\right)^2 + \left(\,(a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)$$

$$\left(\left((\lambda)^{(1)}\right)^2 + \left(\,(b'_{13})^{(1)} + (b'_{14})^{(1)} - (r_{13})^{(1)} + (r_{14})^{(1)}\right)(\lambda)^{(1)}\right)$$

$$+\left(\left((\lambda)^{(1)}\right)^2 + \left(\,(a'_{13})^{(1)} + (a'_{14})^{(1)} + (p_{13})^{(1)} + (p_{14})^{(1)}\right)(\lambda)^{(1)}\right)(q_{15})^{(1)}G_{15}$$

$$+\left((\lambda)^{(1)} + (a'_{13})^{(1)} + (p_{13})^{(1)}\right)\left((a_{15})^{(1)}(q_{14})^{(1)}G^*_{14} + (a_{14})^{(1)}(a_{15})^{(1)}(q_{13})^{(1)}G^*_{13}\right)$$

$$\left(\left((\lambda)^{(1)} + (b'_{13})^{(1)} - (r_{13})^{(1)}\right)s_{(14),(15)}T^*_{14} + (b_{14})^{(1)}s_{(13),(15)}T^*_{13}\right)\} = 0$$

+

$\left((\lambda)^{(2)}+(b_{18}')^{(2)}-(r_{18})^{(2)}\right)\{\left((\lambda)^{(2)}+(a_{18}')^{(2)}+(p_{18})^{(2)}\right)$

$\left[\left(\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)\right]$

$\left(\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(17)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(17)}\mathrm{T}_{17}^{*}\right)$

$+\left(\left((\lambda)^{(2)}+(a_{17}')^{(2)}+(p_{17})^{(2)}\right)(q_{16})^{(2)}\mathrm{G}_{16}^{*}+(a_{16})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}\right)$

$\left(\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(16)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(16)}\mathrm{T}_{16}^{*}\right)$

$\left(\left((\lambda)^{(2)}\right)^{2}+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)$

$\left(\left((\lambda)^{(2)}\right)^{2}+\left((b_{16}')^{(2)}+(b_{17}')^{(2)}-(r_{16})^{(2)}+(r_{17})^{(2)}\right)(\lambda)^{(2)}\right)$

$+\left(\left((\lambda)^{(2)}\right)^{2}+\left((a_{16}')^{(2)}+(a_{17}')^{(2)}+(p_{16})^{(2)}+(p_{17})^{(2)}\right)(\lambda)^{(2)}\right)(q_{18})^{(2)}\mathrm{G}_{18}$

$+\left((\lambda)^{(2)}+(a_{16}')^{(2)}+(p_{16})^{(2)}\right)\left((a_{18})^{(2)}(q_{17})^{(2)}\mathrm{G}_{17}^{*}+(a_{17})^{(2)}(a_{18})^{(2)}(q_{16})^{(2)}\mathrm{G}_{16}^{*}\right)$

$\left(\left((\lambda)^{(2)}+(b_{16}')^{(2)}-(r_{16})^{(2)}\right)s_{(17),(18)}\mathrm{T}_{17}^{*}+(b_{17})^{(2)}s_{(16),(18)}\mathrm{T}_{16}^{*}\right)\}=0$

$+$

$\left((\lambda)^{(3)}+(b_{22}')^{(3)}-(r_{22})^{(3)}\right)\{\left((\lambda)^{(3)}+(a_{22}')^{(3)}+(p_{22})^{(3)}\right)$

$\left[\left(\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)\right]$

$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(21)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(21)}T_{21}^{*}\right)$

$+\left(\left((\lambda)^{(3)}+(a_{21}')^{(3)}+(p_{21})^{(3)}\right)(q_{20})^{(3)}G_{20}^{*}+(a_{20})^{(3)}(q_{21})^{(1)}G_{21}^{*}\right)$

$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(20)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(20)}T_{20}^{*}\right)$

$\left(\left((\lambda)^{(3)}\right)^{2}+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)$

$\left(\left((\lambda)^{(3)}\right)^{2}+\left((b_{20}')^{(3)}+(b_{21}')^{(3)}-(r_{20})^{(3)}+(r_{21})^{(3)}\right)(\lambda)^{(3)}\right)$

$+\left(\left((\lambda)^{(3)}\right)^{2}+\left((a_{20}')^{(3)}+(a_{21}')^{(3)}+(p_{20})^{(3)}+(p_{21})^{(3)}\right)(\lambda)^{(3)}\right)(q_{22})^{(3)}G_{22}$

$+\left((\lambda)^{(3)}+(a_{20}')^{(3)}+(p_{20})^{(3)}\right)\left((a_{22})^{(3)}(q_{21})^{(3)}G_{21}^{*}+(a_{21})^{(3)}(a_{22})^{(3)}(q_{20})^{(3)}G_{20}^{*}\right)$

$\left(\left((\lambda)^{(3)}+(b_{20}')^{(3)}-(r_{20})^{(3)}\right)s_{(21),(22)}T_{21}^{*}+(b_{21})^{(3)}s_{(20),(22)}T_{20}^{*}\right)\}=0$

$+$

$\left((\lambda)^{(4)}+(b_{26}')^{(4)}-(r_{26})^{(4)}\right)\{\left((\lambda)^{(4)}+(a_{26}')^{(4)}+(p_{26})^{(4)}\right)$

$$\left[\left(\left((\lambda)^{(4)}+(a_{24}')^{(4)}+(p_{24})^{(4)}\right)(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(25)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(25)}T_{25}^{*}\right)$$

$$+\left(\left((\lambda)^{(4)}+(a_{25}')^{(4)}+(p_{25})^{(4)}\right)(q_{24})^{(4)}G_{24}^{*}+(a_{24})^{(4)}(q_{25})^{(4)}G_{25}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(24)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(24)}T_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}')^{(4)}+(a_{25}')^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$\left(\left((\lambda)^{(4)}\right)^{2}+\left((b_{24}')^{(4)}+(b_{25}')^{(4)}-(r_{24})^{(4)}+(r_{25})^{(4)}\right)(\lambda)^{(4)}\right)$$

$$+\left(\left((\lambda)^{(4)}\right)^{2}+\left((a_{24}')^{(4)}+(a_{25}')^{(4)}+(p_{24})^{(4)}+(p_{25})^{(4)}\right)(\lambda)^{(4)}\right)(q_{26})^{(4)}G_{26}$$

$$+\left((\lambda)^{(4)}+(a_{24}')^{(4)}+(p_{24})^{(4)}\right)\left((a_{26})^{(4)}(q_{25})^{(4)}G_{25}^{*}+(a_{25})^{(4)}(a_{26})^{(4)}(q_{24})^{(4)}G_{24}^{*}\right)$$

$$\left(\left((\lambda)^{(4)}+(b_{24}')^{(4)}-(r_{24})^{(4)}\right)s_{(25),(26)}T_{25}^{*}+(b_{25})^{(4)}s_{(24),(26)}T_{24}^{*}\right)\}=0$$

+

$$\left((\lambda)^{(5)}+(b_{30}')^{(5)}-(r_{30})^{(5)}\right)\{\left((\lambda)^{(5)}+(a_{30}')^{(5)}+(p_{30})^{(5)}\right)$$

$$\left[\left(\left((\lambda)^{(5)}+(a_{28}')^{(5)}+(p_{28})^{(5)}\right)(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)\right]$$

$$\left(\left((\lambda)^{(5)}+(b_{28}')^{(5)}-(r_{28})^{(5)}\right)s_{(29),(29)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(29)}T_{29}^{*}\right)$$

$$+\left(\left((\lambda)^{(5)}+(a_{29}')^{(5)}+(p_{29})^{(5)}\right)(q_{28})^{(5)}G_{28}^{*}+(a_{28})^{(5)}(q_{29})^{(5)}G_{29}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}')^{(5)}-(r_{28})^{(5)}\right)s_{(29),(28)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(28)}T_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}')^{(5)}+(a_{29}')^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$\left(\left((\lambda)^{(5)}\right)^{2}+\left((b_{28}')^{(5)}+(b_{29}')^{(5)}-(r_{28})^{(5)}+(r_{29})^{(5)}\right)(\lambda)^{(5)}\right)$$

$$+\left(\left((\lambda)^{(5)}\right)^{2}+\left((a_{28}')^{(5)}+(a_{29}')^{(5)}+(p_{28})^{(5)}+(p_{29})^{(5)}\right)(\lambda)^{(5)}\right)(q_{30})^{(5)}G_{30}$$

$$+\left((\lambda)^{(5)}+(a_{28}')^{(5)}+(p_{28})^{(5)}\right)\left((a_{30})^{(5)}(q_{29})^{(5)}G_{29}^{*}+(a_{29})^{(5)}(a_{30})^{(5)}(q_{28})^{(5)}G_{28}^{*}\right)$$

$$\left(\left((\lambda)^{(5)}+(b_{28}')^{(5)}-(r_{28})^{(5)}\right)s_{(29),(30)}T_{29}^{*}+(b_{29})^{(5)}s_{(28),(30)}T_{28}^{*}\right)\}=0$$

+

$$\left((\lambda)^{(6)}+(b_{34}')^{(6)}-(r_{34})^{(6)}\right)\{\left((\lambda)^{(6)}+(a_{34}')^{(6)}+(p_{34})^{(6)}\right)$$

$$\left[\left(\left((\lambda)^{(6)}+(a'_{32})^{(6)}+(p_{32})^{(6)}\right)(q_{33})^{(6)}G^*_{33}+(a_{33})^{(6)}(q_{32})^{(6)}G^*_{32}\right)\right]$$

$$\left(\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(33)}T^*_{33}+(b_{33})^{(6)}s_{(32),(33)}T^*_{33}\right)$$

$$+\left(\left((\lambda)^{(6)}+(a'_{33})^{(6)}+(p_{33})^{(6)}\right)(q_{32})^{(6)}G^*_{32}+(a_{32})^{(6)}(q_{33})^{(6)}G^*_{33}\right)$$

$$\left(\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(32)}T^*_{33}+(b_{33})^{(6)}s_{(32),(32)}T^*_{32}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^2+\left((a'_{32})^{(6)}+(a'_{33})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$\left(\left((\lambda)^{(6)}\right)^2+\left((b'_{32})^{(6)}+(b'_{33})^{(6)}-(r_{32})^{(6)}+(r_{33})^{(6)}\right)(\lambda)^{(6)}\right)$$

$$+\left(\left((\lambda)^{(6)}\right)^2+\left((a'_{32})^{(6)}+(a'_{33})^{(6)}+(p_{32})^{(6)}+(p_{33})^{(6)}\right)(\lambda)^{(6)}\right)(q_{34})^{(6)}G_{34}$$

$$+\left((\lambda)^{(6)}+(a'_{32})^{(6)}+(p_{32})^{(6)}\right)\left((a_{34})^{(6)}(q_{33})^{(6)}G^*_{33}+(a_{33})^{(6)}(a_{34})^{(6)}(q_{32})^{(6)}G^*_{32}\right)$$

$$\left(\left((\lambda)^{(6)}+(b'_{32})^{(6)}-(r_{32})^{(6)}\right)s_{(33),(34)}T^*_{33}+(b_{33})^{(6)}s_{(32),(34)}T^*_{32}\right)\}=0$$

$+$

$$\left((\lambda)^{(7)}+(b'_{38})^{(7)}-(r_{38})^{(7)}\right)\{\left((\lambda)^{(7)}+(a'_{38})^{(7)}+(p_{38})^{(7)}\right)$$

$$\left[\left(\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(q_{36})^{(7)}G^*_{36}\right)\right]$$

$$\left(\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(37)}T^*_{37}+(b_{37})^{(7)}s_{(36),(37)}T^*_{37}\right)$$

$$+\left(\left((\lambda)^{(7)}+(a'_{37})^{(7)}+(p_{37})^{(7)}\right)(q_{36})^{(7)}G^*_{36}+(a_{36})^{(7)}(q_{37})^{(7)}G^*_{37}\right)$$

$$\left(\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(36)}T^*_{37}+(b_{37})^{(7)}s_{(36),(36)}T^*_{36}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$\left(\left((\lambda)^{(7)}\right)^2+\left((b'_{36})^{(7)}+(b'_{37})^{(7)}-(r_{36})^{(7)}+(r_{37})^{(7)}\right)(\lambda)^{(7)}\right)$$

$$+\left(\left((\lambda)^{(7)}\right)^2+\left((a'_{36})^{(7)}+(a'_{37})^{(7)}+(p_{36})^{(7)}+(p_{37})^{(7)}\right)(\lambda)^{(7)}\right)(q_{38})^{(7)}G_{38}$$

$$+\left((\lambda)^{(7)}+(a'_{36})^{(7)}+(p_{36})^{(7)}\right)\left((a_{38})^{(7)}(q_{37})^{(7)}G^*_{37}+(a_{37})^{(7)}(a_{38})^{(7)}(q_{36})^{(7)}G^*_{36}\right)$$

$$\left(\left((\lambda)^{(7)}+(b'_{36})^{(7)}-(r_{36})^{(7)}\right)s_{(37),(38)}T^*_{37}+(b_{37})^{(7)}s_{(36),(38)}T^*_{36}\right)\}=0$$

$+$

$$\left((\lambda)^{(8)}+(b'_{42})^{(8)}-(r_{42})^{(8)}\right)\{\left((\lambda)^{(8)}+(a'_{42})^{(8)}+(p_{42})^{(8)}\right)$$

$$\left[\left(\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(q_{40})^{(8)}G^*_{40}\right)\right]$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(41)}T^*_{41}+(b_{41})^{(8)}s_{(40),(41)}T^*_{41}\right)$$

$$+\left(\left((\lambda)^{(8)}+(a'_{41})^{(8)}+(p_{41})^{(8)}\right)(q_{40})^{(8)}G^*_{40}+(a_{40})^{(8)}(q_{41})^{(8)}G^*_{41}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(40)}T^*_{41}+(b_{41})^{(8)}s_{(40),(40)}T^*_{40}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$\left(\left((\lambda)^{(8)}\right)^2+\left((b'_{40})^{(8)}+(b'_{41})^{(8)}-(r_{40})^{(8)}+(r_{41})^{(8)}\right)(\lambda)^{(8)}\right)$$

$$+\left(\left((\lambda)^{(8)}\right)^2+\left((a'_{40})^{(8)}+(a'_{41})^{(8)}+(p_{40})^{(8)}+(p_{41})^{(8)}\right)(\lambda)^{(8)}\right)(q_{42})^{(8)}G_{42}$$

$$+\left((\lambda)^{(8)}+(a'_{40})^{(8)}+(p_{40})^{(8)}\right)\left((a_{42})^{(8)}(q_{41})^{(8)}G^*_{41}+(a_{41})^{(8)}(a_{42})^{(8)}(q_{40})^{(8)}G^*_{40}\right)$$

$$\left(\left((\lambda)^{(8)}+(b'_{40})^{(8)}-(r_{40})^{(8)}\right)s_{(41),(42)}T^*_{41}+(b_{41})^{(8)}s_{(40),(42)}T^*_{40}\right)\}=0$$

$+$

$$\left((\lambda)^{(9)}+(b'_{46})^{(9)}-(r_{46})^{(9)}\right)\{\left((\lambda)^{(9)}+(a'_{46})^{(9)}+(p_{46})^{(9)}\right)$$

$$\left[\left(\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(q_{44})^{(9)}G^*_{44}\right)\right]$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(45)}T^*_{45}+(b_{45})^{(9)}s_{(44),(45)}T^*_{45}\right)$$

$$+\left(\left((\lambda)^{(9)}+(a'_{45})^{(9)}+(p_{45})^{(9)}\right)(q_{44})^{(9)}G^*_{44}+(a_{44})^{(9)}(q_{45})^{(9)}G^*_{45}\right)$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(44)}T^*_{45}+(b_{45})^{(9)}s_{(44),(44)}T^*_{44}\right)$$

$$\left(\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)$$

$$\left(\left((\lambda)^{(9)}\right)^2+\left((b'_{44})^{(9)}+(b'_{45})^{(9)}-(r_{44})^{(9)}+(r_{45})^{(9)}\right)(\lambda)^{(9)}\right)$$

$$+\left(\left((\lambda)^{(9)}\right)^2+\left((a'_{44})^{(9)}+(a'_{45})^{(9)}+(p_{44})^{(9)}+(p_{45})^{(9)}\right)(\lambda)^{(9)}\right)(q_{46})^{(9)}G_{46}$$

$$+\left((\lambda)^{(9)}+(a'_{44})^{(9)}+(p_{44})^{(9)}\right)\left((a_{46})^{(9)}(q_{45})^{(9)}G^*_{45}+(a_{45})^{(9)}(a_{46})^{(9)}(q_{44})^{(9)}G^*_{44}\right)$$

$$\left(\left((\lambda)^{(9)}+(b'_{44})^{(9)}-(r_{44})^{(9)}\right)s_{(45),(46)}T^*_{45}+(b_{45})^{(9)}s_{(44),(46)}T^*_{44}\right)\}=0$$

**And as one sees, all the coefficients are positive. It follows that all the roots have negative real part, and this proves the theorem.**

## REFERENCES


[1] Wald 1984, pp. 299–300

[2] Schutz, Bernard F. (2003). Gravity from the ground up. Cambridge University Press. p. 110. ISBN 0-521-45506-5

[3] Davies, P. C. W. (1978). "Thermodynamics of Black Holes". Reports on Progress in Physics 41 (8): 1313–1355. Bibcode 1978RPPh...41.1313D. doi:10.1088/0034-4885/41/8/004.

[4] Michell, J. (1784). "On the Means of Discovering the Distance, Magnitude, &c. of the Fixed Stars, in Consequence of the Diminution of the Velocity of Their Light, in Case Such a Diminution Should is Found to Take Place in any of them, and Such Other Data Should be Procured from Observations, as would be Farther Necessary for That Purpose". Philosophical Transactions of the Royal Society 74 (0): 35–57. Bibcode1784RSPT...74...35M . Doi:10.1098/rstl.1784.0008. JSTOR 106576.

[5] Gillispie, C. C. (2000). Pierre-Simon Laplace, 1749–1827: a life in exact science. Princeton paperbacks. Princeton University Press. p. 175. ISBN 0-691-05027-9.

[6] Israel, W. (1989). "Dark stars: the evolution of an idea". In Hawking, S. W.; Israel, W... 300 Years of Gravitation. Cambridge University Press. ISBN 978-0-521-37976-2.

[7] Thorne 1994, pp. 123–124

[8] Schwarzschild, K. (1916). "Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie". Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften 7: 189–196. And Schwarzschild, K. (1916). "Über das Gravitationsfeld eines Kugel aus inkompressibler Flüssigkeit nach der Einsteinschen Theorie". Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften 18: 424–434. English Translation

[9] Droste, J. (1917). "On the field of a single centre in Einstein's theory of gravitation, and the motion of a particle in that field". Proceedings Royal Academy Amsterdam (KNAW) 19 (1): 197–215.

[10] Kox, A.J. (1992). "General Relativity in the Netherlands: 1915-1920". In Eisenstaedt, J.; Kox, A.J... Studies in the history of general relativity. Birkhäuser. p. 41. ISBN 978-0-8176-3479-7.

[11] Hooft, G. (2009). Introduction to the Theory of Black Holes. Institute for Theoretical Physics / Spinoza Institute. pp. 47–48.

[12] Venkataraman, G. (1992). Chandrasekhar and his limit. Universities Press. p. 89.ISBN 81-7371-035-X.

[13] Detweiler, S. (1981). "Resource letter BH-1: Black holes". American Journal of Physics 49 (5): 394–400. Bibcode 1981AmJPh...49...394D. doi:10.1119/1.12686.

[14] Harpaz, A. (1994). Stellar evolution. A K Peters. p. 105. ISBN 1-56881-012-1.

[15] Oppenheimer, J. R.; Volkoff, G. M. (1939). "On Massive Neutron Cores". Physical Review 55 (4): 374–381. Bibcode 1939PhRv...55...374O.doi:10.1103/PhysRev.55.374

[16] Ruffini, R.; Wheeler, J. A. (1971). "Introducing the black hole". Physics Today (1): 30–41.

[17] Finkelstein, D. (1958). "Past-Future Asymmetry of the Gravitational Field of a Point Particle". Physical Review 110 (4): 965–967. Bibcode 1958PhRv..110..965F. Doi:10.1103/Phys Rev. 110.965.

[18] Kruskal, M. (1960). "Maximal Extension of Schwarzschild Metric". Physical Review 119(5): 1743. Bibcode 1960PhRv..119.1743K. doi:10.1103/PhysRev.119.1743

[19] Hewish, A. et al. (1968), "Observation of a Rapidly Pulsating Radio Source", Nature217 (5130): 709–713, Bibcode 1968Natur.217...709H, doi: 10.1038/217709a0

[20] Pilkington, J. D. H. et al. (1968), "Observations of some further Pulsed Radio Sources", Nature 218 (5137): 126–129, Bibcode 1968Natur.218...126P, doi: 10.1038/218126a0

[21] Hewish, A. (1970). "Pulsars". Annual Review of Astronomy and Astrophysics 8 (1): 265–296. Bibcode 1970ARA&A...8...265H.doi:10.1146/annurev.aa.08.090170.001405

[22] Newman, E. T. et al. (1965), "Metric of a Rotating, Charged Mass", Journal of Mathematical Physics 6 (6): 918, Bibcode 1965JMP.....6...918N,doi:10.1063/1.1704351

[23] Israel, W. (1967). "Event Horizons in Static Vacuum Space-Times". Physical Review164 (5): 1776. Bibcode 1967PhRv...164.1776I. doi:10.1103/PhysRev.164.1776

[24] Carter, B. (1971). "Axisymmetric Black Hole Has Only Two Degrees of Freedom". Physical 26 (6): 331. Bibcode 1971PhRvL..26..331C.doi:10.1103/PhysRevLett.26.331.

[25] Carter, B. (1977). "The vacuum black hole uniqueness theorem and it's conceivable

generalizations". Proceedings of the 1st Marcel Grossmann meeting on general relativity. pp. 243–254.
[26] Robinson, D. (1975). "Uniqueness of the Kerr Black Hole". Physical Review Letters 34(14): 905. Bibcode 1975PhRvL...34...905R. doi:10.1103/PhysRevLett.34.905.
[27] Heusler, M. (1998). "Stationary Black Holes: Uniqueness and Beyond". Living Reviews in Relativity 1 (6). Retrieved 2011-02-08.
[28] Penrose, R. (1965). "Gravitational Collapse and Space-Time Singularities". Physical 14 (3): 57. Bibcode 1965PhRvL..14...57P.doi:10.1103/PhysRevLett.14.57.
[29] Ford, L. H. (2003). "The Classical Singularity Theorems and Their Quantum Loopholes". International Journal of Theoretical Physics 42 (6): 1219.Doi:10.1023/ A:1025754515197.
[30] Bardeen, J. M.; Carter, B.; Hawking, S. W. (1973). "The four laws of black hole mechanics". Communications in Mathematical Physics 31 (2): 161–170. Bibcode1973C MaPh...31...161B. Doi: 10.1007/BF01645742. MR MR0334798.Zbl 1125.83309.
[31] Hawking, S. W. (1974). "Black hole explosions?" Nature 248 (5443): 30–31.Bibcode 1974Natur.248...30H. Doi: 10.1038/248030a0.
[32] Quinion, M. (26 April 2008). "Black Hole". World Wide Words. Retrieved 2008-06-17.
[33] Carroll 2004, p. 253
[34] Thorne, K. S.; Price, R. H. (1986). Black holes: the membrane paradigm. Yale University Press. ISBN 978-0-300-03770-8.
[35] Anderson, Warren G. (1996). "The Black Hole Information Loss Problem". Usenet Physics FAQ. Retrieved 2009-03-24.
[36] Preskill, J. (1994-10-21). "Black holes and information: A crisis in quantum physics". Caltech Theory Seminar.
[37] Seeds, Michael A.; Backman, Dana E. (2007), Perspectives on Astronomy, Cengage Learning, p. 167, ISBN 0-495-11352-2
[38] Shapiro, S. L.; Teukolsky, S. A. (1983). Black holes, white dwarfs, and neutron stars: the physics of compact objects. John Wiley and Sons. p. 357. ISBN 0-471-87316-0.
[39] Wald, R. M. (1997). "Gravitational Collapse and Cosmic Censorship". ArXiv:gr-qc/9710068 [gr-qc].
[40] Berger, B. K. (2002). "Numerical Approaches to Spacetime Singularities". Living Reviews in Relativity 5. Retrieved 2007-08-04.
[41] McClintock, J. E.; Shafee, R.; Narayan, R.; Remillard, R. A.; Davis, S. W.; Li, L.-X. (2006). "The Spin of the Near-Extreme Kerr Black Hole GRS 1915+105". Astrophysical Journal 652 (1): 518–539. ArXiv: astro-ph/0606076. Bibcode 2006ApJ...652...518M.Doi:10.1086/508457.
[42] Wald 1984, pp. 124–125
[43] Thorne, Misner & Wheeler 1973, p. 848
[44] Wheeler 2007, p. 179
[45] Carroll 2004, Ch. 5.4 and 7.3
[46] Carroll 2004, p. 217
[47] Carroll 2004, p. 218
[48] "Inside a black hole". Knowing the universe and its secrets. Retrieved 2009-03-26.
[49] Carroll 2004, p. 222
[50] Emparan, R.; Reall, H. S. (2008). "Black Holes in Higher Dimensions". Living Reviews in Relativity 11 (6). ArXiv: 0801.3471. Bibcode 2008LRR....11....6E. Retrieved 2011-02-10.
[51] Obers, N. A. (2009). Papantonopoulos, Eleftherios. ed. "Black Holes in Higher-Dimensional Gravity". Lecture Notes in Physics 769: 211–258. arXiv:0802.0519.doi:10.1007/978-3-540-88460-6.
[52] hawking & Ellis 1973, Ch. 9.3
[53] Carroll 2004, p. 205
[54] Carroll 2004, pp. 264–265
[55] Carroll 2004, p. 252
[56] Lewis, G. F.; Kwan, J. (2007). "No Way Back: Maximizing Survival Time below the Schwarzschild Event Horizon". Publications of the Astronomical Society of Australia 24(2): 46–52. ArXiv: 0705.1029. Bibcode 2007PASA...24...46L.Doi:10.1071/AS07012
[57] Wheeler 2007, p. 182

[58] Carroll 2004, pp. 257–259 and 265–266
[59] Droz, S.; Israel, W.; Morsink, S. M. (1996). "Black holes: the inside story". Physics World 9 (1): 34–37. Bibcode 1996PhyW....9...34D.
[60] Carroll 2004, p. 266
[61] Poisson, E.; Israel, W. (1990). "Internal structure of black holes". Physical Review D41 (6): 1796. Bibcode 1990PhRvD...41.1796P. doi:10.1103/PhysRevD.41.1796.
[62] Wald 1984, p. 212
[63] Hamade, R. (1996). "Black Holes and Quantum Gravity". Cambridge Relativity and Cosmology. University of Cambridge. Retrieved 2009-03-26.
[64] Palmer, D... "Ask an Astrophysicist: Quantum Gravity and Black Holes". NASA. Retrieved 2009-03-26.
[65] a b Nitta, Daisuke; Chiba, Takeshi; Sugiyama, Naoshi (September 2011), "Shadows of colliding black holes", Physical Review D 84 (6), arXiv: 1106.242, Bibcode2011PhRvD.. 84f3008N , doi:10.1103/PhysRevD.84.063008
[66] Nemiroff, R. J. (1993). "Visual distortions near a neutron star and black hole". American Journal of Physics 61 (7): 619. ArXiv: astro-ph/9312003. Bibcode1993AmJPh.. .61...619N. doi:10.1 119/1.17224.
[67] Carroll 2004, Ch. 6.6
[68] Carroll 2004, Ch. 6.7
[69] Einstein, A. (1939). "On A Stationary System with Spherical Symmetry Consisting of Many Gravitating Masses". Annals of Mathematics 40 (4): 922–936.Doi:10.2307/1968902.
[70] Kerr, R. P. (2009). "The Kerr and Kerr-Schild metrics". In Wiltshire, D. L.; Visser, M.; Scott, S. M... The Kerr Spacetime. Cambridge University Press. ArXiv: 0706.1109.ISBN 978-0-521-88512-6.
[71] Hawking, S. W.; Penrose, R. (January 1970). "The Singularities of Gravitational Collapse and Cosmology". Proceedings of the Royal Society A 314 (1519): 529–548.Bibcode 1970RSPSA.314..529H. doi:10.1098/rspa.1970.0021. JSTOR 2416467.
[72] Carroll 2004, Section 5.8
[73] Rees, M. J.; Volonteri, M. (2007). "Massive black holes: formation and evolution". In Karas, V.; Matt, G... Black Holes from Stars to Galaxies—Across the Range of Masses. Cambridge University Press. pp. 51–58. ArXiv: astro-ph/0701512. ISBN 978-0-521-86347-6.
[74] Penrose, R. (2002). "Gravitational Collapse: The Role of General Relativity". General Relativity and Gravitation 34 (7): 1141. Bibcode 2002GReGr...34.1141P.doi:10.1023/A:1016578408204.
[75] Carr, B. J. (2005). "Primordial Black Holes: Do They Exist and Are They Useful?". In Suzuki, H.; Yokoyama, J.; Suto, Y. et al... Inflating Horizon of Particle Astrophysics and Cosmology. Universal Academy Press. ArXiv: astro-ph/0511743. ISBN 4-946443-94-0.
[76] Giddings, S. B.; Thomas, S. (2002). "High energy colliders as black hole factories: The end of short distance physics". Physical Review D 65 (5): 056010. ArXiv:hep-ph/0106219. Bibcode 2002PhRvD...65e6010G.doi:10.1103/PhysRevD.65.056010.
[77] Harada, T. (2006). "Is there a black hole minimum mass?" Physical Review D 74 (8): 084004. ArXiv: gr-qc/0609055. Bibcode 2006PhRvD.. 74h4004H.doi:10.1103/ PhysRevD. 74.084004.
[78] Arkani–Hamed, N.; Dimopoulos, S.; Dvali, G. (1998). "The hierarchy problem and new dimensions at a millimeter". Physics Letters B 429 (3–4): 263. ArXiv: hep-ph/9803315.Bibcode 1998PhLB...429...263A. doi: 10.1016/S0370-2693(98)00466-3.
[79] LHC Safety Assessment Group. "Review of the Safety of LHC Collisions". CERN.
[80] Cavaglià, M. (2010). "Particle accelerators as black hole factories?". Einstein-Online (Max Planck Institute for Gravitational Physics (Albert Einstein Institute)) 4: 1010.
[81] Vesperini, E.; McMillan, S. L. W.; D'Ercole, A. et al. (2010). "Intermediate-Mass Black Holes in Early Globular Clusters". The Astrophysical Journal Letters 713 (1): L41–L44.arXiv:1003.3470. Bibcode 2010ApJ...713L...41V. doi:10.1088/2041-8205/713/1/L41.
[82] Zwart, S. F. P.; Baumgardt, H.; Hut, P. et al. (2004). "Formation of massive black holes through runaway collisions in dense young star clusters". Nature 428 (6984): 724.arXiv: astro-ph/0402622. Bibcode 2004Natur.428...724P.doi:10.1038/nature02448. PMID 15085124.
[83] O'leary, R. M.; Rasio, F. A.; Fregeau, J. M. et al. (2006). "Binary Mergers and Growth of Black Holes in Dense Star Clusters". The Astrophysical Journal 637 (2): 937.ArXiv: astro-

ph/0508224. Bibcode 2006ApJ...637...937O. Doi: 10.1086/498446.
[84] Page, D. N. (2005). "Hawking radiation and black hole thermodynamics". New Journal of Physics 7: 203. ArXiv: hep-th/0409024. Bibcode 2005NJPh....7...203P.Doi:10.1088/1367-2630/7/1/203.
[85] Carroll 2004, Ch. 9.6
[86] "Evaporating black holes?" Einstein online. Max Planck Institute for Gravitational Physics. 2010. Retrieved 2010-12-12.
[87] Giddings, S. B.; Mangano, M. L. (2008). "Astrophysical implications of hypothetical stable TeV-scale black holes". Physical Review D 78 (3): 035009. ArXiv: 0806.3381.Bibcode 2008PhRvD.. 78c5009G. doi:10.1103/PhysRevD.78.035009.
[88] Peskin, M. E. (2008). "The end of the world at the Large Hadron Collider?" Physics 1: 14. Bibcode 2008PhyOJ...1...14P. Doi:10.1103/Physics.1.14.
[89] Fichtel, C. E.; Bertsch, D. L.; Dingus, B. L. et al. (1994). "Search of the energetic gamma-ray experiment telescope (EGRET) data for high-energy gamma-ray microsecond bursts". Astrophysical Journal 434 (2): 557–559. Bibcode1994ApJ...434...557F. Doi: 10.1086/ 174758.
[90] Naeye, R... "Testing Fundamental Physics". NASA. Retrieved 2008-09-16.
[91] "Event Horizon Telescope". MIT Haystack Observatory. Retrieved 6 April 2012.
[92] McClintock, J. E.; Remillard, R. A. (2006). "Black Hole Binaries". In Lewin, W.; van der Klis, M... Compact Stellar X-ray Sources. Cambridge University Press. ArXiv: astro-ph/0306213. ISBN 0-521-82659-4. Section 4.1.5.
[93] Celotti, A.; Miller, J. C.; Sciama, D. W. (1999). "Astrophysical evidence for the existence of black holes". Classical and Quantum Gravity 16 (12A): A3–A21.ArXiv:astro-ph/9912186. Doi:10.1088/0264-9381/16/12A/301.
[94] Winter, L. M.; Mushotzky, R. F.; Reynolds, C. S. (2006). "XMM‐Newton Archival Study of the Ultra luminous X‐Ray Population in Nearby Galaxies". The Astrophysical Journal 649(2): 730. ArXiv: astro-ph/0512480. Bibcode 2006ApJ...649...730W.Doi:10.1086/506579.
[95] Bolton, C. T. (1972). "Identification of Cygnus X-1 with HDE 226868". Nature 235(5336): 271–273. Bibcode 1972Natur.235..271B. doi: 10.1038/235271b0.
[96] Webster, B. L.; Murdin, P. (1972). "Cygnus X-1—a Spectroscopic Binary with a Heavy Companion?" Nature 235 (5332): 37–38. Bibcode 1972Natur.235...37W.Doi:10.1038/235037a0.
[97] Rolston, B. (10 November 1997). "The First Black Hole". The bulletin. University of Toronto. Archived from the original on 2008-05-02. Retrieved 2008-03-11.
[98] Shipman, H. L. (1 January 1975). "The implausible history of triple star models for Cygnus X-1 Evidence for a black hole". Astrophysical Letters 16 (1): 9–12. Bibcode1975ApL....16....9S. Doi: 10.1016/S0304-8853(99)00384-4.
[99] Narayan, R.; McClintock, J. (2008). "Advection-dominated accretion and the black hole event horizon". New Astronomy Reviews 51 (10–12): 733. ArXiv: 0803.0322. Bibcode2008 NewAR..51... 733N. doi:10.1016/j.newar.2008.03.002.
[100] "NASA scientists identify smallest known black hole" (Press release). Goddard Space Flight Center. 2008-04-01. Retrieved 2009-03-14.
[101] Krolik, J. H. (1999). Active Galactic Nuclei. Princeton University Press. Ch. 1.2.ISBN 0-691-01151-6.
[102] Sparke, L. S.; Gallagher, J. S. (2000). Galaxies in the Universe: An Introduction. Cambridge University Press. Ch. 9.1. ISBN [[Special: Book Sources/0-521-59704-4|0-521-59704-4]].
[103] Kormendy, J.; Richstone, D. (1995). "Inward Bound—The Search For Supermassive Black Holes In Galactic Nuclei". Annual Reviews of Astronomy and Astrophysics 33 (1): 581–624. Bibcode 1995ARA&A...33...581K.doi:10.1146/annurev.aa.33.090195.003053
[104] King, A. (2003). "Black Holes, Galaxy Formation, and the MBH-σ Relation". The Astrophysical Journal Letters 596 (1): 27–29. ArXiv: astro-ph/0308342. Bibcode2003ApJ...596L...27K. doi: 10.1086/379143.
[105] Ferrarese, L.; Merritt, D. (2000). "A Fundamental Relation between Supermassive Black Holes and their Host Galaxies". The Astrophysical Journal Letters 539 (1): 9–12.arXiv: astro-ph/0006053. Bibcode 2000ApJ...539L...9F. Doi: 10.1086/312838.
[106] A Black Hole's Dinner is Fast Approaching". ESO Press Release. Retrieved 6 February 2012.
[107] Gillessen, S.; Eisenhauer, F.; Trippe, S. et al. (2009). "Monitoring Stellar Orbits around the

Massive Black Hole in the Galactic Center". The Astrophysical Journal 692(2): 1075. ArXiv: 0810.4674. Bibcode 2009ApJ...692.1075G. Doi:10.1088/0004-637X/692/2/1075.

[108] Ghez, A. M.; Klein, B. L.; Morris, M. et al. (1998). "High Proper-Motion Stars in the Vicinity of Sagittarius A*: Evidence for a Supermassive Black Hole at the Center of Our Galaxy". The Astrophysical Journal 509 (2): 678. ArXiv: astro-ph/9807210. Bibcode1998ApJ... 509...678G. Doi: 10.1086/306528.

[109] Bozza, V. (2010). "Gravitational Lensing by Black Holes". General Relativity and Gravitation (42): 2269–2300. ArXiv: 0911.2187. Bibcode 2010GReGr..42.2269B. doi:10.1007/ s10714 -010-0988-2.

[110] Barack, L.; Cutler, C. (2004). "LISA captures sources: Approximate waveforms, signal-to-noise ratios, and parameter estimation accuracy". Physical Review D (69): 082005.ArXiv:gr-qc/0310125. Bibcode 2004PhRvD..69h2005B.doi:10.1103/PhysRevD.69.082005.

[111] Kovacs, Z.; Cheng, K. S.; Harko, T. (2009). "Can stellar mass black holes be quark stars?". Monthly Notices of the Royal Astronomical Society 400 (3): 1632–1642.arXiv:0908.2672. Bibcode 2009MNRAS.400.1632K. doi:10.1111/j.1365-2966.2009.15571.x.

[112] Kusenko, A. (2006). "Properties and signatures of supersymmetric Q-balls".arXiv:hep-ph/0612159 [hep-ph].

[113] Hansson, J.; Sandin, F. (2005). "Preon stars: a new class of cosmic compact objects". Physics Letters B 616 (1–2): 1. ArXiv:astro-ph/0410417. Bibcode2005PhLB. .616....1H. doi:10.1016/ j.physletb.2005.04.034.

[114] Kiefer, C. (2006). "Quantum gravity: general introduction and recent developments".Annalen der Physik 15 (1–2): 129. ArXiv:gr-qc/0508120. Bibcode2006AnP...518...129K. Doi:10.1002/ andp.200510175.

[115] Skenderis, K.; Taylor, M. (2008). "The fuzzball proposal for black holes". Physics Reports 467 (4–5): 117. ArXiv:0804.0552. Bibcode 2008PhR...467..117S.doi:10.1016/j.physrep.2008.08.001.

[116] Hawking, S. W. (1971). "Gravitational Radiation from Colliding Black Holes". Physical Review Letters 26 (21): 1344–1346. Bibcode 1971PhRvL..26.1344H.doi:10.1103/PhysRevLett.26.1344.

[117] Wald, R. M. (2001). "The Thermodynamics of Black Holes". Living Reviews in Relativity 4 (6). ArXiv:gr-qc/9912119. Bibcode 1999gr.qc....12119W. Retrieved 2011-02-10.

[118] Hooft, G. (2001). "The Holographic Principle". In Zichichi, A... Basics and highlights in fundamental physics. Subnuclear series. 37. World Scientific. ArXiv: hep-th/0003004.ISBN 978-981-02-4536-8.

[119] Strominger, A.; Vafa, C. (1996). "Microscopic origin of the Bekenstein-Hawking entropy". Physics Letters B 379 (1–4): 99. ArXiv: hep-th/9601029. Bibcode1996PhLB...379 ...99S. Doi: 10.1016/0370-2693(96)00345-0.

[120] Carlip, S. (2009). "Black Hole Thermodynamics and Statistical Mechanics". Lecture Notes in Physics 769: 89. ArXiv: 0807.4520. Doi:10.1007/978-3-540-88460-6_3.

[121] Hawking, S. W... "Does God Play Dice?". Www.hawking.org.uk. Retrieved 2009-03-14.

[122] Giddings, S. B. (1995). "The black hole information paradox". Particles, Strings and Cosmology. Johns Hopkins Workshop on Current Problems in Particle Theory 19 and the PASCOS Interdisciplinary Symposium 5. ArXiv: hep-th/9508151.

[123] Mathur, S. D. (2011). "The information paradox: conflicts and resolutions". XXV International Symposium on Lepton Photon Interactions at High Energies.arXiv:1201.2079.