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A Technique for Constructing Odd-order Magic Squares Using Basic Latin Squares

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Abstract- In this paper, a technique for constructing n^2 magic squares (when n is odd) using n^2 basic Latin square is developed. Magic squares are practically important of the properties of equality in the sum of its rows, columns, diagonals. The construction is made by fixing the pivot element and arranging other elements in an orderly manner. The construction is illustrated with numerical examples. .

Index Terms- Latin square (basic), magic square (normal), pivot element, rotation, reflection

I. Introduction

The Latin squares and Greco-Latin squares are used in statistical research particularly in agricultural sciences and design of experiments whereas magic squares are used in puzzle games of cubes, pattern recognition and magic carpet constructions, magic square cipher in Cryptology etc.

Basic Latin Squares

A basic (3 x 3) Latin square can be represented with Latin letters A, B and C as

$$\begin{bmatrix} A & B & C \\ B & C & A \\ C & A & B \end{bmatrix} \quad [1]$$

$$\begin{bmatrix} B & A & C \\ C & B & A \\ A & C & B \end{bmatrix} \begin{bmatrix} C & A & B \\ A & B & C \\ B & C & A \end{bmatrix} \begin{bmatrix} B & C & A \\ A & B & C \\ C & A & B \end{bmatrix} \quad \begin{bmatrix} C & A & B \\ A & B & C \\ B & C & A \end{bmatrix} \text{etc.}$$

(Inter-changing rows and columns) are other forms of (3 x 3) Latin Squares.

In all cases Latin letters are seen once in each row and column. In a Latin square, the sums of rows and columns are equal but not the sums of diagonals.

The basic Latin Square is represented as $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Where, $\sum_i a_{ij} = \sum_j a_{ij}$ but $\sum_i d_{ij} \neq \sum_j d_{ij}$ [2]

Normal Magic Squares

On the other hand, (3 x 3) magic square (normal) with numbers 1,2,3,….,9 is represented as;

$$\begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix} \quad [3]$$

Where, the sums of the rows, columns and diagonals are equal. The above (3 x 3) magic square (normal) can be expressed as A = $\{a_{ij}\}; i, j = 1,2,3$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Satisfying $\sum_i a_{ij} = \sum_j a_{ij} = \sum_i d_{ij} = \sum_j d_{ij}; i, j = 1,2,3$ [4]

Since the elements are consecutive and not repeated and therefore normal magic square.

Magic squares (normal) may be classified an arrangement of non repeated integers ($n \geq 0$) in an array of equal rows and columns such that the sums of its rows, columns and diagonals are equal.

For a normal magic square, the following properties can be established

(a) Elements or numbers ($n \geq 0$) are consecutive

(b) Elements are not repeated

(c) Sums of the rows, columns and diagonals are equal

$$\Rightarrow \sum_i a_{ij} = \sum_j a_{ij} = \sum_i d_{ij} = \sum_j d_{ij} \text{ for all } i, j = 1,2,.....,n$$

(d) Equality property of the rows, columns and diagonals remain unaltered for rotations and reflections.

There exists different (n x n) magic square not satisfying these properties. Examples of such magic squares, not satisfying the above properties are: magic squares (special or random, prime numbers etc.)

Examples: (i) magic square (special) $\begin{bmatrix} 1 & 14 & 14 & 4 \\ 11 & 7 & 6 & 9 \\ 8 & 10 & 10 & 5 \\ 13 & 2 & 3 & 15 \end{bmatrix}$

(ii) Magic square (prime numbers) $\begin{bmatrix} 17 & 39 & 71 \\ 113 & 59 & 5 \\ 47 & 29 & 101 \end{bmatrix}$.

It satisfies; $\sum_i a_{ij} = \sum_j a_{ij} = \sum_i d_{ij} = \sum_j d_{ij}$

However, these magic squares are not normal because in

(i) the elements are repeated and non-consecutive and

(ii) the numbers (prime) are not repeated but non-consecutive

3. Symmetric properties of Basic Latin Squares

Lemma-1: A (n x n) basic Latin square (n is odd) is symmetric and non-duplicated.
Let a (3 x 3) basic Latin square be

$$\begin{bmatrix} A & B & C \\ B & C & A \\ C & A & B \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Here, $\{a_{ij}\} = \{a_{ji}\}$ for all i and $j \Rightarrow a_{13} = a_{31}$ = C, a_{23} = a_{32} = A and so on.

But $a_{11} = A \quad a_{22} = C \quad a_{33} = B$ [5]

The diagonal elements are not equal or repeated $\Rightarrow$ non-duplicated

Lemma-2: A (n x n) basic Latin square (n is even) is symmetric but duplicated
Again, let a (4 x 4) basic Latin square be

$$\begin{bmatrix} A & B & C & D \\ B & C & D & A \\ C & D & A & B \\ D & A & B & C \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Clearly, $\{a_{ij}\} = \{a_{ji}\}$ for all i and $j \Rightarrow$ Basic Latin squares (of all orders) are symmetric.

But $a_{11} = A \quad a_{22} = C \quad a_{33} = A \quad a_{44} = C$

$$\Rightarrow a_{11} = A = a_{33} \text{ and } a_{22} = C = a_{44} \qquad [6]$$

The diagonal elements are equal or repeated $\Rightarrow$ duplicated

Lemma-3: Conversely, a (n x n) square (n is odd), satisfying the symmetric and non duplication properties is a basic Latin square.
Prof: If $\{a_{ij}\} = \{a_{ji}\}$ for all i and j, then it follows that $\{a_{ij}\}$ is a Basic Latin square.

Lemma-4: In a basic Latin square (n is odd), one of the sum of diagonal is equal to the sum of rows or columns.
Prof: It follows immediately that in a basic Latin square (when n is odd),

$$\sum_i a_{ij} = \sum_j a_{ij} = \sum_i d_{ij} \text{ or } \sum_j d_{ij} \text{ holds.} \qquad [7]$$

2. METHODOLOGY

2.1 For constructing n^2 (n is odd) magic square

The technique of constructing magic square using basic Latin square principle can be expressed as follows:

Let the $(n \, x \, n)$ matrix $\{a_{ij}\}$; $i, j = 1,2,...n$ with the consecutive elements/numbers of ($a_{11} \; a_{12} \; ... a_{1n}$), ($a_{21} \; a_{22} \; ... a_{2n}$), … ($a_{n1} \; a_{n2} \; ... a_{nn}$), arranged in Basic Latin square format be; $\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n-1} & a_{1n} \\ a_{22} & a_{23} & .... & a_{2n} & a_{21} \\ ...... & & & & \\ a_{nn} & a_{n1} & ..... & a_{nn-2} & a_{nn-1} \end{bmatrix}$

giving $\sum_i a_{ij} = S$ for all i where S = $\dfrac{n(n^2+1)}{2}$ [8]

This condition will be true for all n (odd or even) due to basic Latin square property
The **pivot element** (number) in the middle cell, when n is odd can be defined as

$$a_{\frac{n+1}{2}, \frac{n+1}{2}} \qquad [9]$$

Since the pivot element is fixed, we select the row, associated with it and assign as the diagonal of the (n x n) array, fixing the pivot element in the middle and arranging the other elements in an orderly manner to get a new matrix $\{b_{ij}\}$; $i, j = 1,2,.....n$, satisfying symmetric property of Latin Square

Hence, $\sum_i b_{ij} = \sum_i d_{ij} = \sum_j d_{ij} = S$ for all i and j [10]

Again, since sum of the columns of $\{a_{ij}\}$are now the rows of $\{b_{ij}\}$.

Therefore, $\sum_i a_{ij} = \sum_i b_{ij} = S$ [11]

Hence, $\sum_i a_{ij} = \sum_j a_{ij} = \sum_i d_{ij} = \sum_j d_{ij}$ is fulfilled

$\Rightarrow \{b_{ij}\}; i, j = 1,2,...n$ is a magic square.

Hence the theorem is established as:
The (n x n) square, developed by using basic Latin square format when the pivot element is fixed and rearranging in an orderly manner represents a magic square [12]

2.2 Steps for construction of a magic square (n is odd)

The construction of magic square by using basic Latin square can be expressed in the following steps:

Step-1: First arrange the consecutive numbers ($a_{11} \; a_{12} \; ... a_{1n}$), ($a_{21} \; a_{22} \; ... a_{2n}$), … ($a_{n1} \; a_{n2} \; ... a_{nn}$) in basic Latin square form

Step-2: Determine the pivot element to be assigned in the middle cell, $a_{\frac{n}{2}+1,\ \frac{n}{2}+1}$ and select the row associated with this pivot element.

Step-3: Assign this row as diagonal elements, fixing the pivot element in the middle and arrange other elements in an orderly manner to give the desired magic square.

Check whether it satisfies the property or not,

$$\sum_i b_{ij} = \sum_j b_{ij} = \sum_i d_{ij} = \sum_j d_{ij}$$

Note:

(i) For the consecutive numbers (n ≥ 1), then pivot element (P) and sum (S) are

$$P = \frac{(n^2+1)}{2} \text{ and } S = \frac{n(n^2+1)}{2} \qquad [13]$$

(ii) If the consecutive number (n ≥ 0), then it gives the lowest magic square.

(iii) If the consecutive number starts from s+1 where s ≥ 1, the corresponding

$$P = s + \frac{(n^2+1)}{2} \text{ and } S = n\left\{s + \frac{(n^2+1)}{2}\right\} \qquad [14]$$

(iv) Maximum and minimum elements can be determined using

$$a_{\frac{n+1}{2},\ \frac{n+1}{2}} + \left(\frac{n^2-1}{2}\right) \text{ and } a_{\frac{n+1}{2},\ \frac{n+1}{2}} - \left(\frac{n^2-1}{2}\right) \qquad [15]$$

2.3 **Alternate Structures of $(n \times n)$ magic squares**

Let $\{a_{ij}\}$ be a magic square satisfying the properties (a) to (d). Equality in the sums of rows, columns and diagonals will remain unchanged for rotations and reflections

The alternate structures of a magic square can be expressed (clockwise or anticlockwise rotation)

$(k\frac{\pi}{2});\ k = \pm 1,\ \pm 2,\ \dots \pm m$) as $\{a_{ij}(k)\}$

Where $\{a_{ij}\} = \{a_{ij}(k)\}$ for all $i = 0, 4, 8, \dots\dots$ [16]

2.4 More properties

(a) Infinite number of magic squares can be generated by multiplying or adding by a number p ≥ 1 to each element of the given magic square.

Or, if $\{a_{ij}\}$ is a magic square, then p $\{a_{ij}\}$ and $\{a_{ij} + p\}$ are magic squares

(b) If the minimum element/number is 0, then $\{a_{ij}\}$ gives the lowest magic square

(c) Sum of two magic squares in the same rotation/reflection gives a magic square

(d) Sum of two magic squares in different rotation are not magic squares.

(e) Product of two magic squares is not a magic square

Magic squares in the same rotation/reflection are additive

3. NUMERICAL EXAMPLES

3.1 To construct a (3 x 3) magic square

Let the numbers be (1, 2, 3), (4, 5, 6) and (7, 8, 9)

Step-1: Latin square format gives $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 4 \\ 9 & 7 & 8 \end{bmatrix}$. [18]

It gives the column totals equal, $\sum_i a_{ij} = 15$ for all j

Here, $P = \frac{(n^2+1)}{2} = 5$ and $S = \frac{n(n^2+1)}{2} = 15$ for n = 3

Step-2: Select the row associated with the pivot element (say 5, 6, 4) and assign it as diagonal elements, fixing the pivot element in the middle (say 4, 5, 6)

Step-3: Rearrange the other elements in an orderly manner to get a new (3 x 3) array $\{b_{ij}\}\ i, j = 1, 2, 3$ as

$\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}$ which represents the 3^2 magic square

On alternate structures of 3^2 magic square

By rotation or reflex ion, alternate structures of a magic square A = $\{a_{ij}\}\ i, j = 1, 2, 3$ can be expressed in different structures (multiples of 90^0)

Let A = $\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}$ Reflection: $\begin{bmatrix} 6 & 1 & 8 \\ 7 & 5 & 3 \\ 2 & 9 & 4 \end{bmatrix}$

and Rotation ($+90^o$): $\begin{bmatrix} 4 & 3 & 8 \\ 9 & 5 & 1 \\ 2 & 7 & 6 \end{bmatrix}$

(i) $A^{(m)} = \{a_{ij}(m)\}$ with the rotation of m 90^0 clockwise or anti-clockwise, where m is real and positive or negative.

$A^{(1)} = \{a_{ij}(1)\} \Rightarrow \begin{bmatrix} 4 & 3 & 8 \\ 9 & 5 & 1 \\ 2 & 7 & 6 \end{bmatrix}$ $A^{(-1)} = \begin{bmatrix} 6 & 7 & 2 \\ 1 & 5 & 9 \\ 8 & 3 & 4 \end{bmatrix}$

$A^{(2)} = \{a_{ij}(2)\} \Rightarrow \begin{bmatrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{bmatrix}$ $A^{(-2)} = \begin{bmatrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{bmatrix}$

$A^{(3)} = \{a_{ij}(3)\} \Rightarrow \begin{bmatrix} 6 & 7 & 2 \\ 1 & 5 & 9 \\ 8 & 3 & 4 \end{bmatrix}$ $\quad A^{(-3)} = \begin{bmatrix} 4 & 3 & 8 \\ 9 & 5 & 1 \\ 2 & 7 & 6 \end{bmatrix}$

In all cases, $\sum_i a_{ij} = \sum_j a_{ij} = \sum_i d_{ij} = \sum_j d_{ij}$ are fulfilled.

3.2 To construct a 5^2 magic square

Let the consecutive numbers be (1, 2, 3, 4, 5), (6, 7, 8, 9,10), (11,12,13,14,15), (16,17,18,19, 20), (21, 22, 23, 24, 25)

Step-1: Arranging in basic Latin square format, it gives

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 7 & 8 & 9 & 10 & 6 \\ 13 & 14 & 15 & 11 & 12 \\ 19 & 20 & 16 & 17 & 18 \\ 25 & 21 & 22 & 23 & 24 \end{bmatrix}$$ Satisfying $\sum_i a_{ij}$ = S for all j,

$$P = \frac{(n^2+1)}{2} = 13 \text{ and } S = \frac{n(n^2+1)}{2} = 65$$

Step-2: Select the row associated with the pivot element (13) as (13, 14, 15, 11, 12) and assign this row as diagonal elements, fixing the pivot element (13) in the middle.

Step-3: Rearrange the other elements in an orderly manner to get a new matrix

$$\{b_{ij}\} = \begin{bmatrix} 17 & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 12 & 19 & 21 & 3 \\ 11 & 18 & 25 & 2 & 9 \end{bmatrix} \text{ where } i, j = 1, 2, 3$$

Satisfying $\sum_i a_{ij} = \sum_j a_{ij} = \sum_i d_{ij} = \sum_j d_{ij}$

3.3 To construct a 7^2 magic square

Let the consecutive numbers be (1, 2, 3, 4, 5, 6 7), (8, 9, 10, 11, 12, 13, 14), ….(43, 44, 45, 46, 47, 48, 49).

Following the steps of arranging in 7^2 basic Latin square format:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 & 13 & 14 & 8 \\ 17 & 18 & 19 & 20 & 21 & 15 & 16 \\ 25 & 26 & 27 & 28 & 22 & 23 & 24 \\ 33 & 34 & 35 & 29 & 30 & 31 & 32 \\ 41 & 42 & 36 & 37 & 38 & 39 & 40 \\ 49 & 43 & 44 & 45 & 46 & 47 & 48 \end{bmatrix}$$

Selecting the row associated with the pivot element (25) and assigning it as diagonal elements, fixing the pivot element in the middle and rearranging the other elements in an orderly manner to get a new (7 x 7) matrix

$$\begin{bmatrix} 30 & 39 & 48 & 1 & 10 & 19 & 28 \\ 38 & 47 & 7 & 9 & 18 & 27 & 29 \\ 46 & 6 & 8 & 17 & 26 & 35 & 37 \\ 5 & 14 & 16 & 25 & 34 & 36 & 45 \\ 13 & 15 & 24 & 33 & 42 & 44 & 4 \\ 21 & 23 & 32 & 41 & 43 & 3 & 12 \\ 22 & 31 & 40 & 49 & 2 & 11 & 20 \end{bmatrix}$$

It satisfies

$\sum_i a_{ij} = \sum_j a_{ij} = \sum_i d_{ij} = \sum_j d_{ij}$ where P = 25 and S = 175

3.4 Construction of 9^2 and 13^2 magic squares using basic Latin squares

Selecting the row associated with the pivot element and assigning it as diagonal elements, fixing the pivot element in the middle and rearranging the other elements in an orderly manner, one can construct the magic squares.

(a) Arranging in 9^2 basic Latin square format

$$P = \frac{(n^2+1)}{2} = 41 \text{ and } S = \frac{n(n^2+1)}{2} = 369$$

:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 10 \\ 21 & 22 & 23 & 24 & 25 & 26 & 27 & 19 & 20 \\ 31 & 32 & 33 & 34 & 35 & 36 & 28 & 29 & 30 \\ 41 & 42 & 43 & 44 & 45 & 37 & 38 & 39 & 40 \\ 51 & 52 & 53 & 54 & 46 & 47 & 48 & 49 & 50 \\ 61 & 62 & 63 & 55 & 56 & 57 & 58 & 59 & 60 \\ 71 & 72 & 64 & 65 & 66 & 67 & 68 & 69 & 70 \\ 81 & 73 & 74 & 75 & 76 & 77 & 78 & 79 & 80 \end{bmatrix}$$

$\Rightarrow$ the required 9^2 magic square is

$$\begin{bmatrix} 47 & 58 & 69 & 80 & 1 & 12 & 23 & 34 & 45 \\ 57 & 68 & 79 & 9 & 11 & 22 & 33 & 44 & 46 \\ 67 & 78 & 8 & 10 & 21 & 32 & 43 & 54 & 56 \\ 77 & 7 & 18 & 20 & 31 & 42 & 53 & 55 & 66 \\ 6 & 17 & 19 & 30 & 41 & 52 & 63 & 65 & 76 \\ 16 & 27 & 29 & 40 & 51 & 62 & 64 & 75 & 5 \\ 26 & 28 & 39 & 50 & 61 & 72 & 74 & 4 & 15 \\ 36 & 38 & 49 & 60 & 71 & 73 & 3 & 14 & 25 \\ 37 & 48 & 59 & 70 & 81 & 2 & 13 & 24 & 35 \end{bmatrix}$$

(b) Arranging in 13^2 basic Latin square format;

$$P = \frac{(n^2+1)}{2} = 85 \text{ and } S = \frac{n(n^2+1)}{2} = 1105$$

1	2	3	4	5	6	7	8	9	10	11	12	13
15	16	17	18	19	20	21	22	23	24	25	26	14
29	30	31	32	33	34	35	36	37	38	39	27	28
43	44	45	46	47	48	49	50	51	52	40	41	42
57	58	59	60	61	62	63	64	65	53	54	55	56
71	72	73	74	75	76	77	78	66	67	68	69	70
85	86	87	88	89	90	91	79	80	81	82	83	84
99	100	101	102	103	104	92	93	94	95	96	97	98
113	114	115	116	117	105	106	107	108	109	110	111	112
127	128	129	130	118	119	120	121	122	123	124	125	126
141	142	143	131	132	133	134	135	136	137	138	139	140
155	156	144	145	146	147	148	149	150	151	152	153	154
169	157	158	159	160	161	162	163	164	165	166	167	168

$\Rightarrow$ the required 13^2 magic square is

93	108	123	138	153	168	1	16	31	46	61	76	91
107	122	137	152	167	13	15	30	45	60	75	90	92
121	136	151	166	12	14	29	44	59	74	89	104	106
135	150	165	11	26	28	43	58	73	88	103	105	120
149	164	10	25	27	42	57	72	87	102	117	119	134
163	9	24	39	41	56	71	86	101	116	118	133	148
8	23	38	40	55	70	85	100	115	130	132	147	162
22	37	52	54	69	84	99	114	129	131	146	161	7
36	51	53	68	83	98	113	128	143	145	160	6	21
50	65	67	82	97	112	127	142	144	159	5	20	35
64	66	81	96	111	126	141	156	158	4	19	34	49
78	80	95	110	125	140	155	157	3	18	33	48	63
79	94	109	124	139	154	169	2	17	32	47	62	77

8. Construction of 3^2 magic squares of complex numbers

Taking two 3x3 magic squares, magic squares of complex numbers can be generated. Let the two magic squares, modified with the subtraction of 7 and 4 respectively be

$$\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{bmatrix} \text{ give } \begin{bmatrix} 1 & -6 & -1 \\ -4 & -2 & 0 \\ -3 & 2 & -5 \end{bmatrix} \text{ and } \begin{bmatrix} 4 & -5 & 2 \\ -1 & 1 & 3 \\ 0 & 5 & -2 \end{bmatrix}$$

Hence a magic square of complex numbers, can be generated as

$$\begin{bmatrix} 1+4i & -6-5i & -1+2i \\ -4-i & -2+i & 3i \\ -3 & 2+5i & -5-2i \end{bmatrix}$$ where P = -2 + i and S= -6 + 3i

Similarly, magic squares of complex numbers of any order (n is odd) can be generated.

4. CONCLUSION

The technique can be used for finding magic squares from basic Latin Squares of any order (n ≥ 1, for n is odd) easily within a shortest possible time. In this paper ,construction of odd order magic squares using basic Latin squares is shown.

However, even-order magic squares can't be constructed directly in the same process because of duplications in diagonal elements and therefore separate techniques are to be adopted.

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