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A Comparative Study on Concomitant of Order Statistics and Record Statistics for Weighted Inverse Gaussian Distribution

Kishore K. Das*, Bhanita Das and Bhupen K. Baruah*****

*Department of Statistics, Gauhati University, Guwahati-781014, India
***Department of Chemistry, Gauhati University, Guwahati-781014, India

****Corresponding Author**
Bhanita Das
Department of Statistics
Gauhati University, Guwahati-781014
Assam, India
Phone: +9108822233531(M); Fax: +91- 0361-2700288
E-mail: bhanitadas83@gmail.com

Abstract- A study on bivariate and trivariate weighted inverse Gaussian distribution (WIGD) using a specific concept of pseudo distribution has been attempted in this paper. Product moment of bivariate pseudo WIGD has been obtained. Distribution of r-th concomitant of order statistics for the resulting distribution has been obtained with some ordering properties. An extension to bivariate concomitant for trivariate pseudo WIGD also has been discussed. Concomitant of second order upper record statistics and joint distribution of second and fourth order concomitant of upper record statistics for bivariate pseudo WIGD have been studied along with their properties. Numerical values of survival functions have been computed for comparison of concomitant of order statistics and record statistics. This study reveals that for both concomitant of order statistics and concomitant of second order upper record statistics for WIGD survival probability decreases as time increases for fixed values of the parameters and it increases for increasing value of shape parameter holding time and scale parameter as fixed.

Index Terms- WIGD, Bivariate pseudo distribution, Concomitant, Order statistics, Record statistics.

I. INTRODUCTION

Order statistics and record statistics are used extensively in statistical models and inference and both describes random variables arranged in order of magnitude. Usually, the ordered values of independent and identically distributed (IID) samples arranged in ascending order of magnitude are known as order statistics (Aleem, 1998). The simplest and most important functions of order statistics is the sample cumulative distribution function (cdf) $F_n(x)$. Suppose $X_1, X_2, \cdots\cdots X_n$ are n jointly distributed random variables then arranging the $X_i's$ in increasing order of magnitude, $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$ are said to be smallest, second smallest and largest order statistics.

According to David, (1981) if $X_1, X_2, \cdots\cdots X_n$ is a random sample from a continuous population with probability density function (pdf) $f(x)$ and cumulative distribution function (cdf) $F(x)$ and $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$ are the order statistics obtained by arranging the random sample in increasing order of magnitude, then the probability density function (pdf) of i-th order statistics $X_{i:n,}$ $1 \leq i \leq n$ is given by

$$f_{i:n}(x) = \frac{n!}{(n-i)!\,(i-1)!}\{F(x)\}^{i-1}\{1-F(x)\}^{n-i}f(x), \quad -\infty < x < \infty \qquad (1.1)$$

Study of bivariate order statistics motivated us to go for concomitant of order statistics. To develop order statistics, for bivariate and multivariate distributions, concomitant variables play a significant role and are used in situations where knowledge of initial values of a characteristic is important. The term concomitant of order statistics was first induced and applied extensively by David, (1973). According to Hanif, (2007) in collecting any data for an observation several characteristics are often recorded, some of them are considered as primary and others can be observed from the primary data automatically. The later one is called concomitant or going along with variables, explanatory variables or covariables. Suppose we have a random sample of size n from an absolutely continuous bivariate population (X, Y). For $1 \leq r, s \leq n$, let $X_{r:n}$ and $Y_{s:n}$ be the order statistics of X and Y sample values respectively, then the Y value associated with $X_{r:n}$ is called concomitant of the r-th order statistics or an induced order statistics and is denoted by $Y_{[r:n]}$ (Bhattacharya, 1974). The distribution of r-th concomitant is given by David and Nagaraja, (2003) as

$$g_{[r]}(y) = \int_{-\infty}^{\infty} f(y|x)\, f_{r:n}(x)dx, \quad x > 0, y > 0 \qquad (1.2)$$

where $f_{r:n}(x)$ is the r-th order statistics and $f(y|x)$ is the conditional distribution. Shahbaz and Shahbaz, (2011) have extended the distribution of concomitant of order statistics to bivariate concomitant of order statistics when a random sample is available from trivariate distribution function $F(x, y_{1,} y_2)$. According to Shahbaz and Shahbaz, (2011) the distribution of bivariate concomitant is given as

$$f_{r:n}(y_r) = \int_{-\infty}^{\infty} f(y_{1,} y_2|x)\ f_{r:n}(x)dx,\ \ (x, y_1, y_2) > 0 \qquad (1.3)$$

where $f(y_{1,} y_2|x)$ is the conditional distribution.

Literature review reported that concomitant of order statistics were studied for several bivariate pseudo distributions under assumption of identically and independently distributed (IID) or independent samples (Shahbaz *et al.*, 2009b; Shahbaz and Ahamad, 2009a and Shahbaz and Shahbaz, 2010). The pseudo distributions are new induction to statistical probability distribution theory and the idea was developed by Filus and Filus, (2006) and concept is used by various authors to derive several bivariate distributions from the existing univariate distribution to measure the failure rate of reliability. These distributions have found applications in situations where standard probability distributions are not suitable. The general setup for the density function of bivariate pseudo distributions is given by Filus and Filus, (2006) as

$$f(x, y) = f(x; q)f\big(y; f(x)\big), x > 0, y > 0 \qquad (1.4)$$

where $f(x)$ is some function of random variable x.

Concomitant of record statistics is also a new branch of statistics with several applications to measure the failure rate of reliability (Chandler, 1952 and Ahsanullah, 2010). Concomitant of record statistics can arise in several applications. Suppose individuals are to be selected on the basis of measurement of an attribute A (say income) whose high value is desirable. Suppose B (say expenditure) is an associated attribute which is known to be correlated with A. While B is easy to measure, A is not. So the individuals are first measured on the basis of their B values and only those having B value bigger (or smaller) than all previous observations on B qualify to be measured for their A values and sequence of A values thus measured are concomitant of records. Many such examples arise in the real field where concomitant of records are useful.

In many practical situations it has been found that the only available data are records. Record values are observations that exceed or fall below current extreme value on a sequence of random variables. Considering a sequence of pair wise r.v.'s $(X_{i,} Y_i), i \geq 1$, when the experimenter is interested in studying just the sequence of records of the first component, the second component associated with a record value of the first one is termed as the concomitant of that record value (Amini and Ahmadi, 2005). Record values were first discussed by Chandler, (1952). Feller, (1966) has given some examples of record values in the context of gambling problems. Nevzorov, (1988) mentioned several applications of record values. Various distributional properties of upper record values have been found in Ahsanullah, (1995) and Nagaraja, (1988).

For a random variable X distribution of k-th upper record, $X_{U(K)}$ has been given by Ahsanullah, (1995) as

$$f_{k:n}(x_k) = \frac{1}{\Gamma(k)} f(x_k)[R(x_k)]^{(k-1)}, \qquad (1.5)$$

Where $R(x) = -\log[1 - F(x)]$ and $F(x)$ is the distribution function of x.

The joint distribution of k-th and p-th upper records, $X_{U(K)}$ and $X_{U(p)}$ has been obtained by Ahsanullah, (1995) using the following expression

$$f\big(x_{k,} x_p\big) = \frac{1}{\Gamma(k)\Gamma(p-k)} r(x_k) f\big(x_p\big)[R(x_k)]^{(k-1)}\big[R(x_k) - R\big(x_p\big)\big]^{(p-k-1)}, \qquad (1.6)$$

where $r(x_k) = \frac{d}{dx_k} R(x_k)$.

Ahsanullah, (1995) obtained the distribution of k-th concomitant of upper record statistics as

$$f(y_k) = \int_{-\infty}^{\infty} f(y|x)\, f_{k:n}(x_k) \qquad (1.7)$$

where $f_{k:n}(x_k)$ is the distribution of k-th upper record statistics given in equation (1.5). Ahsanullah, (1995) has also given joint distribution of concomitant of k-th and p-th upper record values as

$$f\big(y_{[k],} y_{[p]}\big) = \iint_{-\infty}^{\infty} f(y_k|x_k)\, f\big(y_p|x_p\big) f\big(x_{k,} x_p\big) dx_k dx, \qquad (1.8)$$

where $f\big(x_{k,} x_p\big)$ is given in equation (1.6).

According to David and Galambos, (1974); Balasubramanian and Beg, (1998) and Beg and Ahsanullah, (2008) concomitants of order statistics attracted a considerable amount of attention in the literature but concomitants of record statistics received comparatively little attention. Therefore, in this paper an attempt has been made to study concomitant of order statistics and concomitant of record statistics for WIGD which has its applicability in drinking water quality data (Das *et al.*, (2011)) using the concept of bivariate pseudo distribution.

In the estimation of water quality parameters a large number physico-chemical parameters are found to be interrelated with each other showing positive or negative correlation among them. So bivariate distribution theory is applicable in such cases. In water quality data it is reported that iron (Fe) and arsenic (As) are positively correlated with each other, whereas, Fe and fluoride (F) are negatively correlated. From the estimated value of Fe we can observe the probable concentration of As as well as F in water. Similarly, microbial concentration in water also can be estimated from DO data. For these type of situations the concept of concomitant arises naturally.

II. RESULTS AND FINDINGS

Bivariate Pseudo WIGD

According to Das *et al.*, (2011) the pdf of WIGD is given by

$$f_{X_w}(x) = \frac{\left(\frac{\lambda}{2\mu^2}\right)^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)} x^{\frac{1}{2}} \exp\left(-\frac{\lambda x}{2\mu^2}\right), x, (\mu, \lambda) > 0 \qquad (2.1)$$

Suppose that the random variable (r.v.) Y also has WIGD with parameter $\varphi(x)$, where $\varphi(x)$ is some function of X having pdf

$$f_{X_w}(y) = \frac{(\varphi(x))^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)} y^{\frac{1}{2}} \exp(-\varphi(x)y), y > 0, \varphi(x) > 0 \qquad (2.2)$$

Defining equation (2.1) and (2.2) as a compound distribution of two r.v.'s the bivariate pseudo WIGD has the density

$$f_w(x,y) = \frac{\left(\frac{\lambda}{2\mu^2}\right)^{\frac{3}{2}} (\varphi(x))^{\frac{3}{2}}}{\left(\Gamma\left(\frac{3}{2}\right)\right)^2} (xy)^{\frac{1}{2}} \exp\left[-\left\{\frac{\lambda x}{2\mu^2} + \varphi(x)y\right\}\right], x > 0, y > 0, \varphi(x) > 0 \qquad (2.3)$$

Considering $\varphi(x) = x$ in equation (2.3) the distribution becomes

$$f_w(x,y) = \frac{\left(\frac{\lambda}{2\mu^2}\right)^{\frac{3}{2}} x^2}{\left(\Gamma\left(\frac{3}{2}\right)\right)^2} y^{\frac{1}{2}} \exp\left[-\left\{\frac{\lambda x}{2\mu^2} + xy\right\}\right], y, x > 0 \qquad (2.4)$$

The product moment of equation (2.4) is

$$\acute{\mu}_{p,q} = \frac{\Gamma\left(q + \frac{3}{2}\right) \Gamma(p - q + \frac{3}{2}) \left(\frac{\lambda}{2\mu^2}\right)^{q-p}}{\left(\Gamma\left(\frac{3}{2}\right)\right)^2}$$

Concomitant of Order Statistics for Bivariate Pseudo WIGD

To obtain the distribution of r-th concomitant we first find out the distribution of r-th order statistics for marginal distribution of X given in equation (2.1) as

$$f_{r:n_w}(x) = \frac{n!}{(r-1)!\,(n-r)!} k^r \left\{\sum_{j=0}^{\infty} \frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!} \frac{x^{\frac{3}{2}+j}}{\frac{3}{2}+j}\right\}^{r-1} \left\{1 - \sum_{j=0}^{\infty} \frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!} \frac{x^{\frac{3}{2}+j}}{\frac{3}{2}+j}\right\}^{n-r} \sum_{j=0}^{\infty} \frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!} x^{j+\frac{1}{2}}, x > 0, (\mu, \lambda) > 0 \qquad (3.1)$$

Also the conditional distribution of Y given X for equation (2.4) is

$$f_w(y|x) = \frac{1}{\Gamma\left(\frac{3}{2}\right)} x^{\frac{3}{2}} y^{\frac{1}{2}} \exp[-xy], x > 0, y > 0 \qquad (3.2)$$

Using equations (3.1) and (3.2) in equation (1.2), the distribution of r-th concomitant of order statistics for bivariate pseudo WIGD is obtained as

$$f_{[r:n]}(y) = \frac{n!\,k_1}{(r-1)!\,(n-r)!} k^r \int_0^{\infty} \left\{\sum_{j=0}^{\infty} \frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!} \frac{x^{\frac{3}{2}+j}}{\frac{3}{2}+j}\right\}^{r-1} \left\{1 - \sum_{j=0}^{\infty} \frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!} \frac{x^{\frac{3}{2}+j}}{\frac{3}{2}+j}\right\}^{n-r} \sum_{j=0}^{\infty} \frac{\left(-\frac{1}{2\alpha^2}\right)^j}{j!} x^{2+j} y^{\frac{1}{2}} \exp[-xy], x > 0, y > 0 \qquad (3.3)$$

Where $k_1 = \frac{1}{\Gamma\left(\frac{3}{2}\right)}$

The expression for p-th moment of the r-th concomitant for bivariate pseudo WIGD is obtained as

$$\acute{\mu}_p = \frac{n!\,k_1}{(r-1)!\,(n-r)!} k^r \iint_0^{\infty} y^{p+\frac{1}{2}} \left\{\sum_{j=0}^{\infty} \frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!} \frac{x^{\frac{3}{2}+j}}{\frac{3}{2}+j}\right\}^{r-1} \left\{1 - \sum_{j=0}^{\infty} \frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!} \frac{x^{\frac{3}{2}+j}}{\frac{3}{2}+j}\right\}^{n-r} \sum_{j=0}^{\infty} \frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!} x^{2+j} y^{\frac{1}{2}} \exp(-xy)\,dxdy, x > 0, y > 0 \qquad (3.4)$$

Survival function for r-th concomitant of order statistics for bivariate pseudo WIGD is

$$S(t) = 1 - \frac{n!\,k_1}{(r-1)!\,(n-r)!} k^r \int_0^t \int_0^{\infty} \left\{\sum_{j=0}^{\infty} \frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!} \frac{x^{\frac{3}{2}+j}}{\frac{3}{2}+j}\right\}^{r-1} \left\{1 - \sum_{j=0}^{\infty} \frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!} \frac{x^{\frac{3}{2}+j}}{\frac{3}{2}+j}\right\}^{n-r} \sum_{j=0}^{\infty} \frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!} x^{2+j} y^{\frac{1}{2}} \exp(-xy)\,dxdy, x, y > 0 \qquad (3.5)$$

The conditional hazard rate of Y given $X = x$ is

$$\lambda(y|x) = \frac{f(y|x)}{1-F(y|x)} = \frac{\frac{1}{\Gamma\left(\frac{3}{2}\right)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} x^{j+\frac{3}{2}} y^{j+\frac{1}{2}}}{1 - \frac{1}{\Gamma\left(\frac{3}{2}\right)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{(xy)^{j+\frac{3}{2}}}{j+\frac{3}{2}}}$$

Which is decreasing function in x and y.
Again the conditional survival function

$$P(Y > y|X = x) = \frac{1}{f(x)} \int_x^{\infty} f(xy)dy = 1 - \frac{1}{\Gamma\left(\frac{3}{2}\right)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{(xy)^{j+\frac{3}{2}}}{j+\frac{3}{2}}$$

is decreasing function in x, therefore according to properties given by Tahmasebi and Behboodian, (**2010**) it follows that $Y_{[r]}$ has increasing failure rate (IFR) distribution and $Y_{[r]} \leq Y_{[k]}$(entropy ordering) for $1 \leq r \leq k \leq n$ and y is stochastically decreasing and is denoted by SD$(Y|V)$.

Extention to Bivariate Concomitant from Trivariate Pseudo Distributions

The trivariate pseudo WIGD has been defined as a compound distribution of three random variables. Suppose that the random variable Y has WIGD with pdf given in equation (2.1) and the r.v. X_1 and X_2 also has WIGD with parameter $\varphi_1(x)$ and $\varphi_2(x, y_1)$ respectively with pdf

$$f\left(y_1; \varphi_1(x)\right) = \frac{\left(\varphi_1(x)\right)^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)} y_1^{\frac{1}{2}} \exp(-\varphi_1(x)y), y > 0, \varphi_1(x) > 0 \qquad (4.1)$$

And

$$f\left(y_2; \varphi_2(x, y_1)\right) = \frac{\left(\varphi_2(x, y_1)\right)^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)} y_2^{\frac{1}{2}} \exp(-\varphi_2(x, y_1), y_2), \varphi_2(x, y_1), y_2 > 0, \qquad (4.2)$$

Then the compound distribution of equations (2.1), (4.1) and (4.2) gives trivariate pseudo WIGD as

$$f(x, y_1, y_2) = \frac{\left(\varphi_1(x)\right)^{\frac{3}{2}} \left(\varphi_2(x, y_1)\right)^{\frac{3}{2}} \left(\frac{\lambda}{2\mu^2}\right)^{\frac{3}{2}}}{\left[\Gamma\left(\frac{3}{2}\right)\right]^3} (xy_1y_2)^{\frac{1}{2}} \exp\left[-\left(\frac{\lambda x}{2\mu^2} + (-\varphi_1(x)y_1 + \varphi_2(x, y_1)y_2)\right)\right] \qquad (4.3)$$

Using $\varphi_1(x) = x$ and $\varphi_2(x, y_1) = xy_1$, the density given in (4.3) becomes

$$f(x, y_1, y_2) = \frac{(x)^{\frac{3}{2}}(xy_1)^{\frac{3}{2}} \left(\frac{\lambda}{2\mu^2}\right)^{\frac{3}{2}}}{\left[\Gamma\left(\frac{3}{2}\right)\right]^3} (xy_1y_2)^{\frac{1}{2}} \exp\left[-\mathrm{x}\left(\frac{\lambda}{2\mu^2} + y_1(1 + y_2)\right)\right] \qquad (4.4)$$

The product moment of trivariate WIGD is obtained as

$$\mu'_{p,q,s} = \frac{\lambda^{q-p} \Gamma\left(p - q + \frac{3}{2}\right) \Gamma\left(s + \frac{3}{2}\right) \Gamma\left(q - s + \frac{3}{2}\right)}{(2\mu^2)^{q-p} \left[\Gamma\left(\frac{3}{2}\right)\right]^3}$$

The conditional distribution of x given y_1 and y_2 can be obtained by using

$$f(y_1, y_2) = \frac{\left(\frac{\lambda}{2\mu^2}\right)^{\frac{3}{2}}}{\left[\Gamma\left(\frac{3}{2}\right)\right]^3} \frac{\Gamma\left(\frac{9}{2}\right)}{\left[\frac{\lambda}{2\mu^2} + y_1(1 + y_2)\right]^{\frac{9}{2}}}$$

Therefore,

$$f(x|y_1, y_2) = \frac{1}{\Gamma\frac{9}{2}} \left[\frac{\lambda}{2\mu^2} + y_1(1 + y_2)\right]^{\frac{9}{2}} x^{\frac{7}{2}} \exp\left[-\mathrm{x}\left(\frac{\lambda}{2\mu^2} + y_1(1 + y_2)\right)\right]$$

For trivariate pseudo WBSD the p-th order moment is

$$E(x^p|y_1, y_2) = \frac{\Gamma\left(p + \frac{9}{2}\right)}{\left\{\frac{\lambda}{2\mu^2} + y_1(1 + y_2)\right\}^p \Gamma\frac{9}{2}}$$

Considering $p = 1$, the conditional mean for trivariate pseudo WIGD is

$$E(x|y_1, y_2) = \frac{9}{2\left[\frac{\lambda}{2\mu^2} + y_1(1 + y_2)\right]}$$

And $p = 2$, gives

$$E(x^2|y_1, y_2) = \frac{99}{4\left[\frac{\lambda}{2\mu^2} + y_1(1 + y_2)\right]^2}$$

Variance of trivariate pseudo WIGD is

$$Var\ (x|y_1, y_2) = \frac{8}{4\left[\frac{\lambda}{2\mu^2} + y_1(1 + y_2)\right]^2}$$

Again we have,

$$f(y_1, y_2|x) = \frac{1}{\left(\Gamma\frac{3}{2}\right)^2} x^3 y_1{}^2 y_2{}^{\frac{1}{2}} \exp\left[-\mathrm{x}\left(\frac{\lambda}{2\mu^2} + y_1(1 + y_2)\right)\right] \qquad (4.5)$$

Using equations (3.1) and (4.5) in equation (1.3) the pdf of bivariate concomitant from trivariate pseudo WIGD is obtained as

$$f_{[r:n]}(y) = \frac{1}{\left(\Gamma\frac{3}{2}\right)^2}\frac{n!\,{k_1}^r}{(r-1)!\,(n-r)!}\int_0^\infty \left\{\sum_{j=0}^{\infty}\frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!}\frac{x^{\frac{3}{2}+j}}{\frac{3}{2}+j}\right\}^{r-1}\left\{1 - \sum_{j=0}^{\infty}\frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!}\frac{x^{\frac{3}{2}+j}}{\frac{3}{2}+j}\right\}^{n-r} \sum_{j=0}^{\infty}\frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!}x^{j+\frac{7}{2}}\,{y_1}^2\,{y_2}^{\frac{1}{2}}\exp\left[-x\left(\frac{\lambda}{2\mu^2}+y_1(1+y_2)\right)\right]dx,\; x>0, y>0 \quad (4.6)$$

Distribution of Concomitant of Record Statistics for WIGD

Distribution of k-th concomitant of upper record statistics for bivariate pseudo WIGD has been obtained as a particular case of $k=2$ and only for identically and independently distributed cases. The distribution of k-th concomitant of record statistics can be obtained by using equation (1.7). The distribution given in equation (1.7) requires the distribution of k-th record statistics given in equation (1.5) for r.v. X and conditional distribution. If r.v. X follows WIGD, then using the cdf of WIGD and equation (1.5) the distribution of k-th record statistics for WIGD is obtained by using

$$R(x) = -\log[1-F(x)] = -\log\left[1 - c\sum_{j=0}^{\infty}\frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!}\frac{x^{\frac{3}{2}+j}}{\frac{3}{2}+j}\right] = \sum_{m=1}^{\infty}\frac{c^m}{m}\left[\sum_{j=0}^{\infty}\frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!}\frac{x^{\frac{3}{2}+j}}{\frac{3}{2}+j}\right]^m, x>0, \mu, \lambda>0$$

Where $c = \frac{\left(\frac{\lambda}{2\mu^2}\right)^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)}$. Using equation (1.5) the distribution of second order upper record statistics $X_{U(2)}$ for bivariate pseudo WIGD is

$$f_{2:n}(x_2) = \sum_{m=1}^{\infty}\frac{c^{m+1}}{m\Gamma(2)}\sum_{j=0}^{\infty}\left[\frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!}\right]^m\frac{x^{\left(\frac{3}{2}+j\right)m+\frac{1}{2}}}{\left(\frac{3}{2}+j\right)^m}\exp\left(-\frac{\lambda x}{2\mu^2}\right), x>0, \mu, \lambda>0$$

Using the distribution of second order upper record statistics $X_{U(2)}$ for WIGD and conditional distribution of WIGD distribution of second order concomitant of upper record statistics for bivariate pseudo WIGD is obtained as

$$f(y_{[2]}) = \sum_{m=1}^{\infty}\sum_{j=0}^{\infty}\frac{c^{m+1}}{m\Gamma(2)\Gamma\left(\frac{3}{2}\right)}\left[\frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!}\right]^m \frac{y^{\frac{1}{2}}}{\left(\frac{3}{2}+j\right)^m}\frac{\Gamma\left[\left(\frac{3}{2}+j\right)m+3\right]}{\left(y+\frac{\lambda}{2\mu^2}\right)^{\left(j+\frac{3}{2}\right)m+3}} \quad (5.1)$$

The r-th moment about origin for concomitant of upper record statistics for bivariate pseudo WIGD is obtained as

$$\acute{\mu}_r = \sum_{m=1}^{\infty}\sum_{j=0}^{\infty}\left[\frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!}\right]^m\frac{\left(\frac{\lambda}{2\mu^2}\right)^{r-mj}}{\left(\frac{3}{2}+j\right)^m m\Gamma(2)\left(\Gamma\left(\frac{3}{2}\right)\right)^{m+2}}\Gamma\left(r+\frac{3}{2}\right)\Gamma\left[\left(j+\frac{3}{2}\right)m+\frac{3}{2}-r\right] \quad (5.2)$$

The survival function of the second order concomitant of upper record statistics for bivariate pseudo WIGD is

$$S(y_{[2]}) = 1 - \sum_{m=1}^{\infty}\sum_{j=0}^{\infty}\left[\frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!}\right]^m\frac{\Gamma\left[\left(\frac{3}{2}+j\right)m+3\right]}{\left(\frac{3}{2}+j\right)^m}\frac{\left(\frac{\lambda}{2\mu^2}\right)^{-mj}}{m\Gamma(2)\left(\Gamma\left(\frac{3}{2}\right)\right)^{m+2}}\beta_{\frac{2\mu^2 y}{\lambda}}(1.5,(1.5+j)m+1.5) \quad (5.3)$$

Joint Distribution of Second and Fourth Order Concomitant of Upper Record Statistics for Bivariate Pseudo WIGD

The joint distribution of two concomitant of record statistics can be obtained using equation (1.8).But expression (1.8) requires the joint distribution of two record statistics given in equation (1.6). Therefore first we derive the joint distribution of two record statistics.

The joint distribution for random variables $X_1 = X_{u(2)}$ and $X_2 = X_{u(4)}$ for WIGD is as follows

$$f(x_1, x_2) = \sum_{m=1}^{\infty}\sum_{j=0}^{\infty}\left[\frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!}\right]^m\frac{c^{2(1-m)}}{m^{-2}}\frac{1}{\left(\frac{3}{2}+j\right)^{-2m}}\,x_1^{\left(\frac{3}{2}+j\right)m+\left(j+\frac{1}{2}\right)}\left\{{x_2}^{\left(\frac{3}{2}+j\right)m} - {x_1}^{\left(\frac{3}{2}+j\right)m}\right\}\frac{{x_2}^{\left(j+\frac{1}{2}\right)}}{1-c\sum_{j=0}^{\infty}\frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!}\frac{x^{\left(\frac{3}{2}+j\right)}}{\left(\frac{3}{2}+j\right)}} \quad (6.1)$$

Now, using the conditional distribution of Y given X of bivariate pseudo WIGD and joint distribution of $Y_1 = X_{u(2)}$ and $Y_2 = X_{u(4)}$ in equation (6.1), the joint distribution of second and fourth concomitant is obtained as

$f(y_{[1]}, y_{[2]})$

$$= y_1^{\frac{1}{2}} y_2^{\frac{1}{2}} \sum_{m=1}^{\infty} \sum_{j=0}^{\infty} \left[\frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!} \right]^{2(1-m)} \frac{c^{2(1-m)}}{m^{-2}} \frac{1}{\left(\frac{3}{2}+j\right)^{-2m}}$$

$$\int_0^{\infty} x_2^{\left(j+\frac{1}{2}\right)} \exp(-x_2 y_2)\, I(x_2)\, dx_2 \qquad (6.2)$$

Where

$$I(x_2) = \int_0^{x_2} x_1^{\left(\frac{3}{2}+j\right)m+(j+2)} \left\{ x_2^{\left(\frac{3}{2}+j\right)m} - x_1^{\left(\frac{3}{2}+j\right)m} \right\} \exp(-x_2 y_2) \frac{1}{1 - c\sum_{j=0}^{\infty} \frac{\left(-\frac{\lambda}{2\mu^2}\right)^j}{j!} \frac{x_2^{\left(\frac{3}{2}+j\right)}}{\left(\frac{3}{2}+j\right)}}$$

Numerical Values of Survival Functions for Concomitant of Order Statistics and Concomitant of Record Statistics for Bivariate Pseudo WIGD

To study the survivability of concomitant of order statistics and second order upper record statistics numerical values of the survival functions for bivariate pseudo WIGD have been obtained for different values of the parameter λ and y using equations (3.5) and (5.3). From **Table 1** and **Table 2** it is clear that for both concomitant of order statistics and concomitant of second order upper record statistics for WIGD survival probability decreases as y increases for fixed values of the parameters λ and μ and it increases for increasing value of λ holding y and μ as fixed.

Table 2: Numerical values for survival functions second order concomitant of upper record statistics for WIGD

λ	5.0	5.1	5.2	5.3	5.4	5.5
y	$\mu = 1$ $k = 2$ $m = 1$ $j = 0$					
.001	0.97918	0.97979	0.98037	0.98092	0.98145	0.98195
.002	0.94121	0.94293	0.94456	0.94612	0.94761	0.94903
.003	0.89217	0.89532	0.89832	0.90117	0.90390	0.90650
.004	0.83425	0.83908	0.84368	0.84807	0.85225	0.85625
.005	0.76872	0.77546	0.78187	0.78799	0.79382	0.79939
.006	0.69645	0.70528	0.71369	0.72171	0.72936	0.73667
.007	0.61809	0.62918	0.63975	0.64983	0.65945	0.66863
.008	0.53412	0.54764	0.56053	0.57281	0.58453	0.59572
.009	0.44496	0.46105	0.47639	0.49101	0.50496	0.51828
0.01	0.35094	0.36974	0.38766	0.40474	0.42104	0.43661

Table 1: Numerical values for survival functions of r-th concomitant of order statistics for WIGD

λ	2.0	3.0	4.0	5.0	6.0	7.0
y	$\mu = 1$ $r = 1$ $n = 1$					
0.01	0.99569	0.99765	0.99847	0.99890	0.99916	0.99946
0.02	0.98784	0.99337	0.99569	0.99691	0.99765	0.99847
0.03	0.97774	0.98784	0.99209	0.99433	0.99569	0.99719
0.04	0.96583	0.98132	0.98784	0.99129	0.99337	0.99569
0.05	0.95239	0.97395	0.98304	0.98784	0.99074	0.99398
0.06	0.93762	0.96583	0.97774	0.98404	0.98784	0.99209
0.07	0.92163	0.95703	0.97199	0.97991	0.98470	0.99004
0.08	0.90456	0.94761	0.96583	0.97549	0.98132	0.98784
0.09	0.88648	0.93762	0.95929	0.97079	0.97774	0.98550
0.10	0.86747	0.92709	0.95239	0.96583	0.97395	0.98304

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AUTHORS

First Author – Kishore K. Das, M. Sc., Ph.D
Department of Statistics, Gauhati University,
Guwahati, Assam, India

Second Author – Bhanita Das, M. Sc., Ph.D
Department of Statistics, Gauhati University,
Guwahati, Assam, India

Third Author – Bhupen K. Baruah, M. Sc., Ph.D
Department of Chemistry, Gauhati University,
Guwahati, Assam, India

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