EXISTENCE THEORY OF RANDOM DIFFERENTIAL EQUATIONS

D. S. Palimkar
Department of Mathematics,
Vasantrao Naik College, Nanded
PIN-431603 (M.S.) INDIA
E-mail:-dspalimkar@rediffmail.com

Abstract
In this paper, the existence result of nonlinear first order ordinary random differential equations is proved using random fixed point theorem.

Index Terms: Random differential equation, multi-valued function, initial value problem
2000Mathematics Subject Classifications: 60H25,47H40, 47N20.

I. STATEMENT OF THE PROBLEM

Let \( R \) denote the real line and \( R_+ \), the set of nonnegative real numbers. Let \( C(R_+, R) \) denote the class of real-valued functions defined and continuous on \( R_+ \). Given a measurable space \((\Omega, A)\) and a measurable function \( x : \Omega \to C(R_+, R) \), consider the initial value problem of nonlinear first order ordinary random differential equations (RDE)

\[
\begin{align*}
    x'(t, \omega) - k(t, \omega)x(t, \omega) &= f(t, x(t, \omega), \omega), \quad a.e. \ t \in R_+ \\
    x(0, \omega) &= q(\omega)
\end{align*}
\]

for all \( \omega \in \Omega \), where \( k : R_+ \times \Omega \to R_+ \), \( q : \Omega \to R \) and \( f : R_+ \times R \times \Omega \to R \). By a random solution of the RDE (1.1) I mean a measurable function \( x : \Omega \to AC(R_+, R) \) that satisfies the equations in (1.1), where \( AC(R_+, R) \) is the space of absolutely continuous real-valued functions defined on \( R_+ \).

The initial value problems of ordinary differential equations have been studied in the literature on bounded as well as unbounded intervals of the real line for different aspects of the solution. See for example, Burton and Furumochi [2], Dhage [3]. Similarly, the initial value problem of random differential equations have also been discussed in the literature for existence theorems on bounded intervals, however, the study of such random equations has not been made on unbounded intervals of the real line for any aspects of the random solutions. Some results along this lines appear in Itoh [5], Bharucha-Reid [1] and Dhage [4].

Therefore, nonlinear random differential equations on unbounded intervals need to pay attention to the existence of the random solutions. The present paper proposes to discuss the existence results for random differential equations (1.1) on the right half \( R_+ \) of the real line \( R \). The classical fixed point theory, in particular, random version of Schauder’s fixed point theorem will be employed to prove the main result of this paper. I claim that the results of this paper is new to the literature on random differential equations.

II. AUXILIARY RESULTS

The following theorem is often used in the study of nonlinear discontinuous random differential equations. I also need this result in the subsequent part of this paper.

**Theorem 2.1 (Itoh[5])**. Let \( X \) be a non-empty, closed convex bounded subset of the separable Banach space \( E \) and let \( Q : \Omega \times X \to X \) be a compact and continuous random operator. Then the random equation \( Q(\omega)x = x \) has a random solution, that is there is a measurable function \( \xi : \Omega \to X \) such that \( Q(\omega)\xi(\omega) = \xi(\omega) \) for all \( \omega \in \Omega \).

**Theorem 2.2 (Carathéodory)**. Let \( Q : \Omega \times E \to E \) be a mapping such that \( Q(\cdot, x) \) is measurable for all \( x \in E \) and \( Q(\omega, \cdot) \) is continuous for all \( \omega \in \Omega \). Then the map \( (\omega, x) \to Q(\omega, x) \) is jointly measurable.
III. EXISTENCE RESULTS

I need the following definition in the sequel

**Definition 4.1.** A function \( f : \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R} \) is called **random Carathéodory** if

i) the map \( \omega \to f(t, x, \omega) \) is measurable for all \( t \in \mathbb{R} \) and \( x \in \mathbb{R} \) and (ii) the map \( (t, x) \to f(t, x, \omega) \) is jointly continuous for all \( \omega \in \Omega \). Furthermore, a random Carathéodory function \( f : \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R} \) is called **random \( L^1 \)-Carathéodory**, if there exists a function \( h \in L^1(\mathbb{R}_+, \mathbb{R}) \) such that

\[
|f(t, x, \omega)| \leq h(t) \quad a.e. \ t \in \mathbb{R}_+
\]

for all \( \omega \in \Omega \) and \( x \in \mathbb{R} \). The function \( h \) is the growth function of \( f \) on \( \mathbb{R} \times \mathbb{R} \times \Omega \).

I consider the following set of hypotheses follows:

\( (H_0) \) The function \( k : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) is measurable and bounded. Moreover, for each \( \omega \in \Omega \),

\[
\lim_{t \to \infty} K(t, \omega) = \infty
\]

where

\[
K(t, \omega) = \int_0^t k(s, \omega) ds.
\]

\( (H_1) \) The function \( q : \Omega \to \mathbb{R} \) is measurable and bounded. Moreover,

\[
\text{ess sup}_{\omega \in \Omega} |q(\omega)| = c_1
\]

for some real number \( c_1 > 0 \).

\( (H_2) \) The function \( f \) is random \( L^1 \)-Carathéodory with growth function \( h \) on \( \mathbb{R}_+ \). Moreover,

\[
\lim_{t \to \infty} e^{-K(t, \omega)} \int_0^t e^{K(s, \omega)} h(s) ds = 0
\]

for all \( \omega \in \Omega \).

**Remark 4.2.** If the hypothesis \( (H_2) \) holds, then the function \( w : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+ \) defined by

\[
w(t, \omega) = e^{-K(t, \omega)} \int_0^t e^{K(s, \omega)} h(s) ds
\]

is continuous and the number

\[
w = \sup_{t \geq 0} w(t, \omega) = \sup_{t \geq 0} e^{-K(t, \omega)} \int_0^t e^{K(s, \omega)} h(s) ds
\]

exists for all \( \omega \in \Omega \). Hypothesis \( (H_2) \) has been considered in a number of papers in the literature. See for example, Dhage [3], Burton and Furumochi [2] and the references therein.

IV. MAIN RESULT

**Theorem 4.3.** Assume that the hypotheses \( (H_0) \) through \( (H_2) \) hold. Then the RDE (1.1) admits a random solution.

**Proof.** Now RDE (1.1) is equivalent to the random equation

\[
x(t, \omega) = q(\omega) e^{K(t, \omega)} + e^{K(t, \omega)} \int_0^t e^{-K(s, \omega)} f(s, x(s, \omega), \omega) ds
\]

for all \( t \in \mathbb{R}_+ \) and \( \omega \in \Omega \).
Set $E = BC \left( R_+, R \right)$. For a given function $x : \Omega \rightarrow E$, define a mapping $Q$ on $\Omega \times E$ by

$$Q(\omega)x(t, \omega) = q(\omega)e^{K(t, \omega)} + e^{K(t, \omega)} \int_{0}^{t} e^{-K(s, \omega)} f(s, x(s, \omega), \omega)ds$$

(4.2)

for all $t \in R_+$ and $\omega \in \Omega$. For the sake of convenience, I write $Q(\omega)x(t, \omega) = Q(\omega)x(t)$ omitting the double appearance of $\omega$ merge it into $Q(\omega)$.

Clearly, $Q$ defines a mapping $Q : \Omega \times E \rightarrow E$. To see this, let $x \in E$ be arbitrary. Then for each $\omega \in \Omega$, the continuity of map $t \rightarrow Q(\omega)x(t)$ follows from the fact that exponential $e^{K(t, \omega)}$ and the indefinite integral

$$\int_{0}^{t} f(s, x(s, \omega), \omega)ds$$

are continuous functions of $t$ on $R_+$. Next, I show that the function $Q(\omega)x : R_+ \rightarrow R$ is bounded for each $\omega \in \Omega$. Now by hypotheses $(H_1)$ and $(H_2)$,

$$|Q(\omega)x(t)| \leq |q(\omega)|e^{K(t, \omega)} + e^{K(t, \omega)} \int_{0}^{t} e^{-K(s, \omega)} |f(s, x(s, \omega), \omega)|ds$$

(4.3)

$$\leq c_1e^{K(t, \omega)} + e^{K(t, \omega)} \int_{0}^{t} e^{-K(s, \omega)} h(s)ds$$

$$\leq c_1 + W$$

for all $\omega \in \Omega$. As a result, $Q : \Omega \times E \rightarrow E$.

Define a closed ball $B_r(0)$ in the Banach space $E$ centered at origin of radius $r = c_1 + W$ from (4.3),

$$|Q(\omega)x| \leq c_1 + W$$

for all $\omega \in \Omega$ and $x \in E$. Hence $Q : \Omega \times E \rightarrow B_r(0)$, and in particular, $Q$ defines a map $Q : B_r(0) \rightarrow B_r(0)$. Now I show that $Q$ satisfies all the conditions of Theorem 2.1 with $X = B_r(0)$.

Firstly, I show that $Q$ is a random operator on $\Omega \times B_r(0)$ in to $B_r(0)$. By hypothesis $(H_2)$, the map $\omega \rightarrow f(t, x, \omega)$ is measurable by the Carathéodory theorem. Since a continuous function is measurable, the map $t \rightarrow e^{K(t, \omega)}$ is measurable and so the product $t \rightarrow e^{K(t, \omega)} f(t, x(t, \omega), \omega)$ is measurable in $\omega$ for all $t \in R_+$ and $x \in R$. Since the integral is a limit of the finite sum of measurable functions, I have that the function

$$\omega \rightarrow \int_{0}^{t} e^{-K(s, \omega)} f(s, x(s, \omega), \omega)ds$$

is measurable for all $t \in R_+$. Consequently, the map $\omega \rightarrow Q(\omega)x$ is measurable for all $x \in E$ and that $Q$ is a random operator on $\Omega \times B_r(0)$.

Secondly, I show that the random operator $Q(\omega)$ is continuous on $B_r(0)$. By hypothesis $(H_2)$

$$\lim_{t \rightarrow \infty} w(t, \omega) = \lim_{t \rightarrow \infty} e^{K(t, \omega)} \int_{0}^{t} e^{-K(s, \omega)} h(s)ds = 0$$

there is a real number $T > 0$ such that $w(t) < \frac{c}{4}$ for all $t \geq T$.

I show that the continuity of the random operator $Q(\omega)$ in the following two cases:

**Case I.** Let $t \in [0, T]$ and let $\{x_n\}$ be a sequence of points in $B_r(0)$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$. Then, by the dominated convergence theorem,
\[
\lim_{n \to \infty} Q(\omega)x_n(t) = \lim_{n \to \infty} \left( q(\omega)e^{K(t,\omega)} + e^{K(t,\omega)} \int_0^t e^{-K(s,\omega)} f(s, x_n(s, \omega), \omega) ds \right)
\]
\[
= q(\omega)e^{K(t,\omega)} + \lim_{n \to \infty} e^{K(t,\omega)} \int_0^t e^{-K(s,\omega)} f(s, x_n(s, \omega), \omega) ds
\]
\[
= q(\omega)e^{K(t,\omega)} + e^{K(t,\omega)} \int_0^t \left( e^{-K(s,\omega)} f(s, x(s, \omega), \omega) \right) ds
\]
\[
= q(\omega)e^{K(t,\omega)} + \int_0^t \left( e^{-K(s,\omega)} f(s, x(s, \omega), \omega) \right) ds
\]
for all \( t \in [0, T] \) and \( \omega \in \Omega \).

**Case II.** Suppose that \( t \geq T \). Then,
\[
|Q(\omega)x_n(t) - Q(\omega)x(t)| = \left| q(\omega)e^{K(t,\omega)} + e^{K(t,\omega)} \int_0^t e^{-K(s,\omega)} f(s, x_n(s, \omega), \omega) ds - e^{K(t,\omega)} \int_0^t e^{-K(s,\omega)} f(s, x(s, \omega), \omega) ds \right|
\]
\[
\leq \left| e^{K(t,\omega)} \int_0^t e^{-K(s,\omega)} f(s, x_n(s, \omega), \omega) ds \right| + \left| e^{K(t,\omega)} \int_0^t e^{-K(s,\omega)} f(s, x(s, \omega), \omega) ds \right|
\]
\[
\leq e^{K(t,\omega)} \int_0^t e^{-K(s,\omega)} f(s, x_n(s, \omega), \omega) ds + e^{K(t,\omega)} \int_0^t e^{-K(s,\omega)} f(s, x(s, \omega), \omega) ds
\]
\[
\leq 2 \omega(t) < \varepsilon
\]
for all \( t \geq T \) and \( \omega \in \Omega \). Since \( \varepsilon \) is arbitrary, one has \( \lim_{n \to \infty} Q(\omega)x_n(t) = Q(\omega)x(t) \) all \( t \geq T \) and \( \omega \in \Omega \).

Now combining the Case I with Case II, I conclude that \( Q(\omega) \) is a pointwise continuous random operator on \( \tilde{B}_r(0) \) into itself. Further, it is shown below that the family of functions \( \{Q(\omega)x_n\} \) is an equi-continuous set in \( E \) for a fixed \( \omega \in \Omega \). Hence, the above convergence is uniform on \( R_+ \) and consequently, \( Q(\omega) \) is a continuous random operator on \( \tilde{B}_r(0) \) into itself.

Next, I show that \( Q(\omega) \) is a compact random operator on \( \tilde{B}_r(0) \). Let \( \omega \in \Omega \) be fixed and consider a sequence \( \{Q(\omega)x_n\} \) of points in \( \tilde{B}_r(0) \). To finish, it is enough to show that the sequence \( \{Q(\omega)x_n\} \) has a Cauchy subsequence for each \( \omega \in \Omega \). Clearly \( \{Q(\omega)x_n\} \) is a uniformly bounded subset of \( \tilde{B}_r(0) \).

I show that it is an equi-continuous sequence of functions on \( R_+ \).

Let \( \varepsilon > 0 \) be given. Since \( \lim_{t \to \infty} w(t, \omega) = 0 \), there exists a real number \( T > 0 \) such for \( t < \frac{\varepsilon}{4} \) for \( t \geq T \). I consider the following three cases:

**Case I.** Let \( t_1, t_2 \in [0, T] \). Then,
\[
|Q(\omega)x_n(t_1) - Q(\omega)x_n(t_2)| \\
\leq \left| q(\omega)e^{K(t_1,\omega)} + e^{K(t_1,\omega)} \int_0^{t_1} e^{-K(s,\omega)} f(s, x_n(s, \omega), \omega) ds - q(\omega)e^{K(t_2,\omega)} - e^{K(t_2,\omega)} \int_0^{t_2} e^{-K(s,\omega)} f(s, x(s, \omega), \omega) ds \right|
\]
\[
\leq \left| q(\omega)e^{K(t_1,\omega)} - q(\omega)e^{K(t_2,\omega)} \right| + \left| e^{K(t_1,\omega)} \int_0^{t_1} e^{-K(s,\omega)} f(s, x_n(s, \omega), \omega) ds - e^{K(t_2,\omega)} \int_0^{t_2} e^{-K(s,\omega)} f(s, x(s, \omega), \omega) ds \right|
\]
\[
\leq e^{K(t_1,\omega)} \int_0^{t_1} e^{-K(s,\omega)} f(s, x_n(s, \omega), \omega) ds - e^{K(t_2,\omega)} \int_0^{t_2} e^{-K(s,\omega)} f(s, x(s, \omega), \omega) ds
\]
\[
\begin{align*}
&\leq |q_\omega| |e^{K(t_1, \omega)} - e^{K(t_2, \omega)}| + \\
&\left| e^{K(t_1, \omega)} - e^{K(t_2, \omega)} \right| \int_0^{t_1} e^{-K(s, \omega)} f(s, x_s(s, \omega), \omega) \, ds - e^{K(t_2, \omega)} \int_0^{t_2} e^{-K(s, \omega)} f(s, x_s(s, \omega), \omega) \, ds + \\
&\left| e^{K(t_2, \omega)} - e^{K(t_1, \omega)} \right| \int_0^{t_2} e^{-K(s, \omega)} f(s, x_s(s, \omega), \omega) \, ds - e^{K(t_1, \omega)} \int_0^{t_1} e^{-K(s, \omega)} f(s, x_s(s, \omega), \omega) \, ds \\
&\leq |q_\omega| |e^{K(t_1, \omega)} - e^{K(t_2, \omega)}| + \left| e^{K(t_1, \omega)} - e^{K(t_2, \omega)} \right| |f(s, x_s(s, \omega), \omega)| ds + |p(t_1) - p(t_2)| \\
&\leq [c_1 + e^k \|h\|_\omega] |e^{K(t_1, \omega)} - e^{K(t_2, \omega)}| + |p(t_1) - p(t_2)|
\end{align*}
\]

for all \( n \in \mathbb{N} \), where \( k \) is the bound for \( K(t, \omega) \) and \( p(t) = \int_0^{t} e^{-K(s, \omega)} h(s) \, ds \). Since the functions \( e^{K(t, \omega)} \) and \( p(t) \) are continuous on \([0, T]\), they are uniformly continuous there. Hence,

\[ |Q_\omega x_n(t_1) - Q_\omega x_n(t_2)| \to 0 \text{ as } t_1 \to t_2 \text{ uniformly for all } t_1, t_2 \in [0, T] \text{ and for all } n \in \mathbb{N}.
\]

**Case II.** If \( t_1, t_2 \in [T, \infty] \), then

\[
\begin{align*}
&\left| Q_\omega x_n(t_1) - Q_\omega x_n(t_2) \right| \\
&\leq e^{K(t_1, \omega)} \int_0^{t_1} e^{-K(s, \omega)} f(s, x_s(s, \omega), \omega) \, ds - e^{K(t_2, \omega)} \int_0^{t_2} e^{-K(s, \omega)} f(s, x_s(s, \omega), \omega) \, ds \\
&\leq e^{K(t_1, \omega)} \int_0^{t_1} e^{-K(s, \omega)} |f(s, x_s(s, \omega), \omega)| \, ds + e^{K(t_2, \omega)} \int_0^{t_2} e^{-K(s, \omega)} |f(s, x_s(s, \omega), \omega)| \, ds \\
&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon.
\end{align*}
\]

Since \( \varepsilon \) is arbitrary, one has

\[ |Q_\omega x_n(t_1) - Q_\omega x_n(t_2)| \to 0 \text{ as } t_1 \to t_2 \text{ uniformly for all } n \in \mathbb{N}.
\]

**Case III.** If \( t_1 < T < t_2 \), then

\[
\begin{align*}
&\left| Q_\omega x_n(t_1) - Q_\omega x_n(t_2) \right| \\
&\leq \left| Q_\omega x_n(t_1) - Q_\omega x_n(T) \right| + \left| Q_\omega x_n(T) - Q_\omega x_n(t_2) \right| \\
&\text{As } t_1 \to t_2, t_1 \to T \text{ and } t_2 \to T, \text{ and so} \\
&\left| Q_\omega x_n(t_1) - Q_\omega x_n(T) \right| \to 0 \\
\text{and} \\
&\left| Q_\omega x_n(T) - Q_\omega x_n(T) \right| \to 0 \\
&\text{as } t_1 \to t_2 \text{ uniformly for all } n \in \mathbb{N}.
\end{align*}
\]

Hence

\[ |Q_\omega x_n(t_1) - Q_\omega x_n(t_2)| \to 0 \text{ as } t_1 \to t_2 \text{ uniformly for all } t_1 < T < t_2 \text{ and for all } n \in \mathbb{N}. \text{ Thus, in all three cases}
\]

\[
\left| Q_\omega x_n(t_1) - Q_\omega x_n(t_2) \right| \to 0 \text{ as } t_1 \to t_2 \text{ uniformly for all } t_1, t_2 \in \mathbb{R} \text{ and for all } n \in \mathbb{N}.
\]

www.ijsrp.org
This shows that \( \{Q(\omega)x_n\} \) is a equicontinuous sequence in \( X \). Now an application of Arzelá-Ascoli theorem yields that \( \{Q(\omega)x_n\} \) has a uniformly convergent subsequence on the compact subset \( [0, T] \) of \( R \).

Without loss of generality, call the subsequence to be the sequence itself. I show that \( \{Q(\omega)x_n\} \) is Cauchy in \( X \). Now shows that \( |Q(\omega)x_n(t) - Q(\omega)x(t)| \to 0 \text{ as } n \to \infty \) for all \( t \in [0, T] \). Then for given \( \varepsilon > 0 \) there exist \( n_0 \in \mathbb{N} \) such that

\[
\sup_{0 \leq t \leq T} e^{k(p, \omega)} \int_0^t e^{-k(1, \omega)} \left| f(s, x_m(s, \omega), \omega) - f(s, x_n(s, \omega), \omega) \right| ds < \frac{\varepsilon}{2}
\]

for all \( m, n \geq n_0 \). Therefore, if \( m, n \geq n_0 \), then we have

\[
|Q(\omega)x_m - Q(\omega)x_n| = \\
\sup_{0 \leq t \leq T} e^{k(p, \omega)} \int_0^t e^{-k(1, \omega)} \left| f(s, x_m(s, \omega), \omega) - f(s, x_n(s, \omega), \omega) \right| ds \\
\leq \sup_{0 \leq t \leq T} e^{k(p, \omega)} \int_0^t e^{-k(1, \omega)} \left| f(s, x_m(s, \omega), \omega) - f(s, x_n(s, \omega), \omega) \right| ds + \\
\sup_{0 \leq t \leq T} e^{k(p, \omega)} \int_0^t e^{-k(1, \omega)} \left| f(s, x_n(s, \omega), \omega) \right| ds + e^{k(p, \omega)} \int_0^t e^{-k(1, \omega)} \left| f(s, x_n(s, \omega), \omega) \right| ds
\]

\[
< \varepsilon
\]

This shows that \( \{Q(\omega)x_n\} \subset Q(\omega)(\bar{B}, (0)) \subset (\bar{B}, (0)) \) is Cauchy. Since \( X \) is complete, \( \{Q(\omega)x_n\} \) converges to a point in \( X \). As \( Q(\omega)(\bar{B}, (0)) \) is closed, the sequence \( \{Q(\omega)x_n\} \) converges to a point in \( Q(\omega)(\bar{B}, (0)) \). Hence \( Q(\omega)(\bar{B}, (0)) \) is relatively compact for each \( \omega \in \Omega \) and consequently \( Q \) is a continuous and compact random operator on \( \Omega \times (\bar{B}, (0)) \). Now an application of Theorem 2.1 to the operator \( Q(\omega) \) on \( (\bar{B}, (0)) \) yields that \( Q \) has a fixed point in \( (\bar{B}, (0)) \) which further implies that the RDE (1.1) random solution on \( R_+ \).

ACKNOWLEDGMENT

This research article is some output result of Minor Research Project funded by UGC, New Delhi.

REFERENCES