Nano Regular Generalized Irresolute Maps in Nano Topological Spaces
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Abstract: In this paper we introduce and study new class of maps, namely Nano regular generalized irresolute maps, strongly Nano regular generalized continuous maps and perfectly Nano regular generalized continuous maps in Nano topological spaces. We define and analyze some of the properties of these mappings in terms of Nano rg-closed map, Nano rg-open map.

Keywords: Nano rg-closed map, Nano rg-open map, Nano rg-closure, Nano-continuity, Nano rg-continuity, Nano-irresolute, Nano rg-irresolute.

1. Introduction

2. Preliminaries
Definition 2.1: [7] A subset A of a space (U, τR(X)) is called a regular generalized closed set if rcl(A) ⊆ V whenever A ⊆ V and V is regular open.
Definition 2.2: [7] The regular generalized closure of a subset A of a space U is the intersection of all rg-closed sets containing A and is denoted by rgcl(A).
Definition 2.3: [7] The regular generalized interior of a subset A of a space U is the union of all rg-open sets contained in A and is denoted by rgInt(A).
Definition 2.4: [7] A function f: (U, τR(X)) → (V, τR(Y)) is regular generalized continuous (rg-continuous) if f⁻¹(B) is rg-closed set in U for every closed set B of V or equivalently a function f: (U, τR(X)) → (V, τR(Y)) is rg-continuous if and only if the inverse image of each open set is rg-open set.
Definition 2.5: [4] A function $f: (U, \tau_R(X)) \rightarrow (V, \tau_R'(Y))$ is said to be irresolute if the inverse image of every regular open set in $V$ is regular open in $U$.

Definition 2.6: [4] A function $f: (U, \tau_R(X)) \rightarrow (V, \tau_R'(Y))$ is said to be rg-irresolute if the inverse image of every rg-closed set in $V$ is rg-closed in $U$.

Definition 2.7: [6] A function $f: (U, \tau_R(X)) \rightarrow (V, \tau_R'(Y))$ is said to be strongly continuous if $f^{-1}(B)$ is both open and closed in $(U, \tau_R(X))$ for each subset $B$ of $(V, \tau_R'(Y))$.

Definition 2.8: [7] A function $f: (U, \tau_R(X)) \rightarrow (V, \tau_R'(Y))$ is said to be strongly rg-continuous if the inverse image of every rg-open set in $V$ is open in $U$.

Definition 2.9: [7] A function $f: (U, \tau_R(X)) \rightarrow (V, \tau_R'(Y))$ is said to be perfectly rg-continuous if the inverse image of every rg-open set in $V$ is both open and closed in $U$.

Definition 2.10: [8] Let $U$ be a non-empty finite set of objects called the Universe and $R$ be an equivalence relation on $U$ named as the indiscernibility relation. Then $U$ is divided into disjoint equivalence classes. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair $(U, R)$ is said to be the approximation space.

1. The lower approximation of $X$ with respect to $R$ is the set of all objects, which can be certainly classified as $X$ with respect to $R$ and it is denoted by $L_R(X)$.
   That is $L_R(X) = U \{ R(x): R(x) \subseteq X \}$, where $R(x)$ denotes the equivalence class determined by $x \in U$.

2. The upper approximation of $X$ with respect to $R$ is the set of all objects, which can be possibly classified as an $X$ with respect to $R$ and it is denoted by $U_R(X)$. That is $U_R(X) = U \{ R(x): R(x) \cap X \neq \emptyset \}$.

3. The boundary region of $X$ with respect to $R$ is the set of all objects, which can be classified neither as $X$ nor as not $X$ with respect to $R$ and it is denoted by $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.11: [8] If $\tau_R(X)$ is the Nano topology on $U$ with respect to $X$, then the set $\beta = \{ U_R(X), L_R(X), B_R(X) \}$ is the basis for $\tau_R(X)$.

Definition 2.12: [8] If $(U, \tau_R(X))$ is a Nano topological space with respect to $X$ where $X \subseteq U$ and if $A \subseteq U$, then

(i) The Nano interior of the set $A$ is defined as the union of all open subsets contained in $A$ and it is denoted by $NInt(A)$. That is, $NInt(A)$ is the largest Nano open subsets of $A$.

(ii) The Nano closure of $A$ is defined as the intersection of all Nano closed sets containing $A$ and is denoted by $NCl(A)$. That is, $NCl(A)$ is the smallest Nano closed set containing $A$.

Definition 2.13: [8] If $(U, \tau_R(X))$ is a Nano topological space with respect to $X$ where $X \subseteq U$ and if $A \subseteq U$, then

(i) The Nano regular interior of $A$ is defined as the union of all Nano regular open subsets of $A$ and it is denoted by $NrInt(A)$. That is, $NrInt(A)$ is the largest Nano regular open subsets of $A$.

(ii) The Nano regular closure of $A$ is defined as the intersection of all Nano regular closed sets containing $A$ and is denoted by $NrCl(A)$. That is, $NrCl(A)$ is the smallest Nano regular closed set containing $A$. 

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Definition 2.14: [8] A subset A of \((U, \tau_R(X))\) is called Nano regular-generalized closed set (briefly Nrg-closed) if \(NrgCl(A) \subseteq V\) and \(A \subseteq V\) and \(V\) is Nano regular-open in \((U, \tau_R(X))\). The subset A is called Nano rg-open in \((U, \tau_R(X))\) if \(A^c\) is Nano rg-closed.

Definition 2.15: [9] If \((U, \tau_R(X))\) is a Nano topological space with respect to \(X\) where \(X \subseteq U\) and if \(A \subseteq U\), then

(i) The Nano regular generalized interior of \(A\) is defined as the union of all Nano regular generalized open subsets of \(A\) and it is denoted by \(NrgInt(A)\). That is, \(NrgInt(A)\) is the largest Nano regular generalized open subsets of \(A\).

(ii) The Nano regular generalized closure of \(A\) is defined as the intersection of all Nano regular generalized closed sets containing \(A\) and is denoted by \(NrgCl(A)\). That is, \(NrgCl(A)\) is the smallest Nano regular generalized closed set containing \(A\).

Definition 2.16: [10] Let \((U, \tau_R(X))\) and \((V, \tau_R(Y))\) be two Nano topological spaces. Then a mapping \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))\) is Nano continuous on \(U\) if the inverse image of every Nano open set in \(V\) is Nano open in \(U\).

Definition 2.17: [10] Let \((U, \tau_R(X))\) and \((V, \tau_R(Y))\) be two Nano topological spaces. Then a mapping \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))\) is Nano rg-continuous on \(U\) if the inverse image of every Nano open set in \(V\) is Nano rg-open in \(U\).

3. Nrg - IRRESOLUTE MAPS

In this section, we introduce the concept of Nano irresolute maps, Nrg-irresolute maps in Nano topological spaces and investigate some of their properties.

Definition 3.1: A map \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))\) is Nano irresolute map if the inverse image of every Nano regular open set \(A\) in \(V\) is Nano regular open in \(U\).

Definition 3.2: A map \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))\) is said to be Nano regular generalized irresolute map (briefly Nrg-irresolute) if the inverse image of every Nrg-closed set \(A\) in \(V\) is Nrg-closed in \(U\).

Example 3.3: Let \(U = \{a, b, c, d\}\) with \(U|R = \{\{a\}, \{c\}, \{b, d\}\}\) and \(X = \{a, b\}\). Then \(\tau_R(X) = \{U, \emptyset, \{a\}, \{b, d\}\}\). Let \(V = \{x, y, z, w\}\) with \(V|R = \{\{y\}, \{w\}, \{x, z\}\}\). Then \(\tau_R'(Y) = \{V, \emptyset, \{y\}, \{y, z, w\}\}\). Define a mapping \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R'(Y))\) as, \(f(a) = x, f(b) = y, f(c) = w, f(d) = z\). Then \(f^{-1}(V) = U, f^{-1}(\emptyset) = \phi, f^{-1}((\{y, w\})) = \{a, b, c\}, f^{-1}((\{x, z\})) = \{a, c, d\}\). Thus the inverse image of every Nrg-closed set in \(V\) is Nrg-closed set in \(U\). Hence \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R'(Y))\) is Nrg-irresolute.

Theorem 3.4: Let \((U, \tau_R(X))\) and \((V, \tau_R'(Y))\) be any two Nano topological spaces, where “every Nrg-closed subset is Nano closed” and \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R'(Y))\) be a map. If \(f\) is Nrg-irresolute map then it is Nrg-continuous but not conversely.

Proof: Let \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R'(Y))\) be an Nrg-irresolute map. Then the inverse image \(f^{-1}(A)\) of every Nrg-closed set \(A\) in \(V\) is Nrg-closed set in \(U\). Since every Nano closed set is Nano regular generalized closed, the inverse image of every Nano closed set in \(V\) is Nrg-closed in \(U\). whenever the inverse image of Nrg-closed is Nrg-closed. Hence Nrg-irresolute function is Nrg-continuous.

The converse of the above theorem need not be true as seen from the following example.
Example 3.5: Consider $U = \{a, b, c, d\}$ with $U|R = \{a, \{\}, \{b, d\}\}$; $X = \{a, b, \} \subseteq U$. Define $f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))$ as the identity map. Then $f$ is Nrg-continuous but not Nrg-irresolute.

Theorem 3.6: Let $(U, \tau_R(X)), (V, \tau_R(Y))$ and $(W, \tau_R(Z))$ be Nano topological spaces. If $f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))$ and $g: (V, \tau_R(Y)) \rightarrow (W, \tau_R(Z))$ be two maps. If $f$ is Nrg-irresolute and $g$ is Nrg-continuous then $g \circ f$ is Nrg-continuous.

Proof: Let $A$ be a Nano closed set in $W$. Since the map $g: (V, \tau_R(Y)) \rightarrow (W, \tau_R(Z))$ is Nrg-continuous the inverse image $g^{-1}(A)$ is Nrg-closed set in $V$. Thus the composition $(g \circ f)^{-1}(A)$ is Nrg-closed in $U$ for every Nrg closed set $A$ in $W$. Hence the composition $(g \circ f): (U, \tau_R(X)) \rightarrow (W, \tau_R(Z))$ is Nrg-continuous.

Theorem 3.7: Let $f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))$ and $g: (V, \tau_R(Y)) \rightarrow (W, \tau_R(Z))$ be two maps. If $f$ and $g$ are both Nrg-irresolute then $(g \circ f)$ is Nrg-continuous.

Proof: It is obvious.

Theorem 3.8: Let $f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))$ and $g: (V, \tau_R(Y)) \rightarrow (W, \tau_R(Z))$ be two maps. If $f$ and $g$ are both Nrg-irresolute then $g \circ f$ is Nrg-continuous.

Proof: As the map $g: (V, \tau_R(Y)) \rightarrow (W, \tau_R(Z))$ is Nrg-irresolute, the inverse image $g^{-1}(A)$ of every Nrg-open set $A$ in $W$ is Nrg-open in $V$. Hence $g^{-1}(A)$ is Nrg-open set in $V$ and $f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))$ being Nrg-irresolute implies that $f^{-1}(g^{-1}(A))$ is Nrg-open in $U$. Thus $(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A))$ is Nrg-open in $U$ for every Nrg-open set $g^{-1}(A)$ in $V$. Hence $(g \circ f)$ is Nrg-continuous.

Theorem 3.9: Let $(U, \tau_R(X)), (V, \tau_R(Y))$ and $(W, \tau_R(Z))$ be Nano topological spaces. If $f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))$ and $g: (V, \tau_R(Y)) \rightarrow (W, \tau_R(Z))$ be two maps. If $f$ and $g$ are both Nrg-irresolute then $g \circ f$ is Nrg-irresolute.

Proof: Assume that $A$ is a Nrg-closed set in $V$. Let $X$ be any Nano regular open set in $U$ such that $f^{-1}(A) \subset X$. Then $A \subset f(X)$ where $f(X)$ is Nano regular open. Since $A$ is Nrg- closed, NCl $(A) \subset f(X)$. Which implies $f^{-1}[NCl(A)] \subset X$. By assumption $f$ is Nrg-continuous. Therefore $f^{-1}(NCl(A))$ is Nrg-closed. Hence $NCl(f^{-1}(NCl(A))) \subset f(X)$. That is NCl $(f^{-1}(A)) \subset NCl(f^{-1}(NCl(A))) \subset X$. Thus $f^{-1}(A)$ is Nrg-closed set in $U$.

Theorem 3.11: A map $f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))$ is Nrg-irresolute if and only if the inverse image of an Nrg-open set in $V$ is Nrg-open in $U$.

Proof: Let $f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))$ be a Nrg-irresolute and $A$ be a Nrg-open set in $V$. Then $A^c$ is Nrg-closed in $V$. Since $f$ is Nrg-irresolute, $f^{-1}(A^c)$ is Nrg-closed in $U$. Since $f^{-1}(A^c) = (f^{-1}(A))^c$ and so $f^{-1}(A)$ is Nrg-open in $U$.

Conversely, assume that $f^{-1}(A)$ is Nrg-open in $U$ for each Nrg-open set $A$ in $V$. Let $F$ be Nrg-open set in $V$ and by assumption $f^{-1}(F)$ is Nrg-open in $U$. Since, $f^{-1}(F) = (f^{-1}(F))^c$, we have $f^{-1}(F)$ is Nrg-closed in $U$ and so $f$ is Nrg-irresolute.

Definition 3.12: A map $f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))$ is said to be Nrg-closed map if the image of every Nano closed set in $U$ is Nrg-closed set in $V$.

Remark 3.13: Every Nano closed map is Nrg-closed but not conversely.
Proof: Let \( f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y)) \) be Nano closed map. Let \( A \) be any Nano closed set in \( U \). Then \( f(U) \) is Nano closed set but every Nano closed set is Nrg – closed set. Hence \( f \) is Nrg – closed map.

The converse of the above remark need not be true as seen from the following example,

**Example 3.14:** Consider \( U=\{a, b, c, d\} \) with \( U|R=\{\{a\},\{c\},\{b, d\}\} \) \( X=\{a, b\} \subseteq U \) \( \tau_R(X)=\{U, \phi, \{a\}, \{a, b, d\}, \{b, d\}\} \) and \( V=\{a, b, c, d\} \) with \( V|R'=\{\{a\},\{b, d\}\} \) \( Y = \{a, b\} \subseteq V \). Define \( f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y)) \) and \( g: (V, \tau_R(Y)) \rightarrow (W, \tau_R(Z)) \) as follows: 
\( f(a)=b, f(b)=c, f(c)=a, f(d)=d, g(a)=b, g(b)=a, g(c)=d, g(d)=c \). Then \( f \) and \( g \) are Nrg-closed maps. But their composition \( (g \circ f) \) is not Nrg-closed. Since for the Nano closed \( \{c\} \), \( (g \circ f) \{c\} = g \{f(c)\} = g \{a\} = b \) is not Nrg-closed.

**Definition 3.18:** A map \( f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y)) \) is said to be Nrg – open map if the image of every Nano open set in \( U \) is Nrg – open set in \( V \).

**Theorem 3.16:** A map \( f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y)) \) is Nrg-closed if and only if for each subset \( S \) of \( V \) containing \( f^{-1}(S) \) there is a Nrg-open set \( Y \) of \( V \) such that \( S \subseteq X \) and \( f^{-1}(Y) \subseteq X \).

**Proof:** Let \( B \) be a Nano closed set of \( U \). Then \( f^{-1}(V f(B)) \subseteq B \cup B \) and \( B \cup B \) is Nano open. By hypothesis there is an Nrg-open set \( Y \) of \( V \) such that \( \forall f \{B\} \subseteq Y \) and \( f^{-1}(Y) \subseteq B \cup B \). Hence it follows that \( B \subseteq f^{-1}(Y) \) and \( Y \subseteq \forall f \{B\} \subseteq V Y \). This proves that \( f(B) = \forall Y \). Since \( V Y \) is Nrg-closed then \( f(B) \) is Nrg-closed.

**Theorem 3.20:** If \( f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y)) \) is Nrg – open, then \( f^{-1}(\text{Nrg} – \text{cl}(A)) \subseteq \text{Ncl}(f^{-1}(A)) \) for each subset \( A \) of \( V \).

**Proof:** Let \( f: (U, \tau_R(X)) \rightarrow (V, \tau_R(Y)) \) be Nrg – open and \( A \) be any subset of \( V \). Then \( f^{-1}(A) \subseteq \text{Ncl}(f^{-1}(A)) \) and \( \text{Ncl}(f^{-1}(A)) \) is Nano closed set in \( U \). Then by the above theorem 3.18, there exists a Nrg – closed set \( B \) of \( V \) such that \( A \subseteq B \) and \( f^{-1}(B) \subseteq \text{Ncl}(f^{-1}(A)) \). 
Now, \( \text{Nrg} – \text{cl}(A) \subseteq \text{Nrg} - \text{cl}(B) = B \). Therefore \( f^{-1}(\text{Nrg} – \text{cl}(A)) \subseteq f^{-1}(B) \) and \( f^{-1}(\text{Nrg} – \text{cl}(A)) \subseteq f^{-1}(B) \subseteq \text{Ncl}(f^{-1}(A)) \). Thus, \( f^{-1}(\text{Nrg} – \text{cl}(A)) \subseteq \text{Ncl}(f^{-1}(A)) \) for each subset \( A \) of \( V \).
Definition 3.21: A space \((U, \tau_R(X))\) is said to be Nrg-normal if for any two disjoint Nano regular closed sets \(A\) and \(B\) there exist disjoint Nrg-open sets \(X\) and \(Y\) such that \(A \subset X; B \subset Y\).

Theorem 3.22: If \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R'(Y))\) is Nano continuous, Nrg-closed surjection from a normal space \(U\) onto a space \(V\), then \(V\) is Nrg-normal.

Proof: Let \(A\) and \(B\) be two disjoint Nano regular closed sets of \(V\). Then \(A\) and \(B\) are Nano closed sets in \(V\). Which implies that \(f^{-1}(A)\) and \(f^{-1}(B)\) are disjoint Nano closed sets of \(U\). Since \(U\) is normal, there are disjoint Nano open sets \(X, Y\) in \(U\) such that \(f^{-1}(A) \subset X\) and \(f^{-1}(B) \subset Y\). Then there are Nrg-open sets \(F, S\) in \(V\) such that \(A \subset F, B \subset S\) and \(f^{-1}(F) \subset X, f^{-1}(S) \subset Y\). Since \(X\) and \(Y\) are disjoint, that is \(f^{-1}(F) \cap f^{-1}(S) = X \cap Y = \emptyset\). Hence the space \(V\) is Nrg-normal.

Definition 3.23: Let \((U, \tau_R(X))\) and \((V, \tau_R'(Y))\) be two Nano topological spaces. Then a mapping \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R'(Y))\) is strongly Nano continuous if \(f^{-1}(A)\) is both Nano open and Nano closed in \((U, \tau_R(X))\) for each subset \(A\) of \((V, \tau_R'(Y))\).

Definition 3.24: Let \((U, \tau_R(X))\) and \((V, \tau_R'(Y))\) be two Nano topological spaces. Then a mapping \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R'(Y))\) is strongly Nano rg-continuous (briefly strongly Nrg-continuous) if \(f^{-1}(A)\) is Nano open in \((U, \tau_R(X))\) for every Nrg-open set \(A\) in \((V, \tau_R'(Y))\).

Theorem 3.25: If the map \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R'(Y))\) is strongly Nano rg-continuous, then it is Nano continuous but not conversely.

Proof: Let \(A\) be a Nano open set in \((V, \tau_R'(Y))\). As every Nano open set is Nrg-open, \(A\) is Nrg-open in \(V\). By the given hypothesis, \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R'(Y))\) is strongly Nrg-continuous and hence \(f^{-1}(A)\) is Nano open in \(U\) for every Nrg-open set \(A\) in \(V\). Hence \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R'(Y))\) is Nano continuous.

The converse of the above theorem need not be true as seen from the following example.

Example 3.26: Let \(U = \{a, b, c, d\}\) with \(U|\tau = \{\{a\}, \{c\}, \{b, d\}\}\) and \(X = \{a, b\}\). Then \(\tau_R(X) = \{U, \phi, \{a\}, \{b, d\}, \{a, b, d\}\}\). Let \(V = \{x, y, z, w\}\) with \(V|\tau' = \{\{y\}, \{w\}, \{x, z\}\}\). Then \(\tau_R'(Y) = \{V, \phi, \{y\}, \{x, y, z\}, \{x, z\}\}\). Define a mapping \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R'(Y))\) as, \(f(\text{a}) = x, f(\text{b}) = y, f(\text{c}) = w, f(\text{d}) = z\). Then \(f\) is Nano continuous. Since the inverse image of every Nano open set in \(V\) is Nano open in \(U\). But \(f\) is not strongly Nrg-continuous. Since \(f^{-1}(\{x, z, w\}) = \{a, c, d\}\) is not Nano open in \(U\) even though \(\{x, z, w\}\) is Nrg-open in \(V\). Hence \(f\) is Nano continuous is not strongly Nrg-continuous.

Theorem 3.27: If the map \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R'(Y))\) is strongly Nano continuous, then it is strongly Nrg-continuous but not conversely.

Proof: Let \(A\) be a Nrg-open set in \((V, \tau_R'(Y))\). The map \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R'(Y))\) is strongly Nano continuous and hence the inverse image of every subset of \(V\) is both Nano open and Nano closed in \((U, \tau_R(X))\). Hence \(f^{-1}(A)\) is Nano open in \((U, \tau_R(X))\) for every Nrg-open subset \(A\) of \((V, \tau_R'(Y))\). Hence \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R'(Y))\) is strongly Nrg-continuous.

Corollary 3.28: If the map \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R'(Y))\) and \(g: (V, \tau_R'(Y)) \rightarrow (W, \tau_R'(Z))\) are strongly Nrg-continuous, then their composition \((g \circ f): (U, \tau_R(X)) \rightarrow (W, \tau_R'(Z))\) is also strongly Nrg-continuous.

Theorem 3.29: Let \(f: (U, \tau_R(X)) \rightarrow (V, \tau_R'(Y))\) and \(g: (V, \tau_R'(Y)) \rightarrow (W, \tau_R'(Z))\) be any two maps. Then their composition \((g \circ f): (U, \tau_R(X)) \rightarrow (W, \tau_R'(Z))\) is

(i) Strongly Nrg-continuous if \(g\) is strongly Nrg-continuous and \(f\) is Nano continuous.

(ii) Strongly Nrg-continuous if \(g\) is strongly Nano continuous and \(f\) is Nano irresolute.
(iii) Nano continuous if \( g \) is Nrg-continuous and \( f \) is strongly Nrg-continuous.

**Proof:**
(i) Let \( A \) be a Nrg-open set in \( W \). since the map \( g: (V, \tau_R'(Y)) \to (W, \tau_R'(Z)) \) is strongly Nrg-continuous, \( g^{-1}(A) \) is Nano open set in \( V \). Since the map \( f: (U, \tau_R(X)) \to (V, \tau_R'(Y)) \) is Nano continuous, the inverse \( f^{-1}(g^{-1}(A)) \) is Nano open in \( U \). Thus the inverse image \( (g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A)) \) is Nano open in \( U \) for every Nrg-open set \( A \) in \( W \). Hence the composition \( (g \circ f): (U, \tau_R(X)) \to (W, \tau_R'(Z)) \) is strongly Nrg-continuous.

(ii) Let \( A \) be a Nrg-open set in \( W \). since the map \( g: (V, \tau_R(Y)) \to (W, \tau_R'(Z)) \) is strongly Nano-continuous, \( g^{-1}(A) \) is both Nano open and Nano closed set in \( V \). As every Nano open set is Nano regular-open, since \( g^{-1}(A) \) is Nano regular-open in \( V \). Since the map \( f: (U, \tau_R(X)) \to (V, \tau_R(Y)) \) is Nano irresolute, the inverse \( f^{-1}(g^{-1}(A)) \) is Nano-regular-open in \( U \). Thus the inverse image \( (g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A)) \) is Nano regular-open in \( U \) and \( g \) is strongly Nano continuous. Hence the map \( (g \circ f): (U, \tau_R(X)) \to (W, \tau_R'(Z)) \) is strongly Nrg-continuous.

(iii) Let \( A \) be a Nano-open set in \( W \). since the map \( g: (V, \tau_R(Y)) \to (W, \tau_R'(Z)) \) is Nrg-continuous, \( g^{-1}(A) \) is Nrg-open in \( V \). Since the map \( f: (U, \tau_R(X)) \to (V, \tau_R(Y)) \) is strongly Nrg-continuous, the inverse \( f^{-1}(g^{-1}(A)) \) is Nano-regular-open in \( U \). Thus the inverse image \( (g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A)) \) is Nano-open in \( U \) and \( g \) is strongly Nano continuous. Hence the map \( (g \circ f): (U, \tau_R(X)) \to (W, \tau_R'(Z)) \) is Nano-continuous for every Nano open set \( A \) in \( (W, \tau_R'(Z)) \).

**Theorem 3.30:** Let \( f: (U, \tau_R(X)) \to (V, \tau_R'(Y)) \) and \( g: (V, \tau_R(Y)) \to (W, \tau_R'(Z)) \) be two maps. If \( f \) is Nrg-continuous and \( g \) is strongly Nrg-continuous then \( g \circ f \) is Nrg-irresolute.

**Proof:**
Let \( A \) be a Nano open set in \( W \). since the map \( g: (V, \tau_R'(Y)) \to (W, \tau_R'(Z)) \) is Nrg-continuous the inverse image \( g^{-1}(A) \) is Nrg-closed set in \( V \). Since the map \( f: (U, \tau_R(X)) \to (V, \tau_R(Y)) \) is strongly Nrg-continuous, the inverse \( f^{-1}(g^{-1}(A)) \) of Nrg-open set in \( U \). Thus the inverse image \( (g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A)) \) is Nrg-open in \( U \) for every Nrg open set \( A \) in \( W \). Hence the map \( (g \circ f): (U, \tau_R(X)) \to (W, \tau_R'(Z)) \) is Nrg-irresolute.

**Definition 3.31:** A function \( f: (U, \tau_R(X)) \to (V, \tau_R'(Y)) \) is said to be perfectly Nrg-continuous if the inverse image of every Nrg-open set in \( V \) is both Nano-open and Nano-closed in \( U \).

**Theorem 3.32:**
Let \( f: (U, \tau_R(X)) \to (V, \tau_R'(Y)) \) and \( g: (V, \tau_R'(Y)) \to (W, \tau_R'(Z)) \) be two maps. Then their compositions \( (g \circ f): (U, \tau_R(X)) \to (W, \tau_R'(Z)) \) is

(i) If \( f \) is Nrg-continuous and \( g \) is perfectly Nrg-continuous then \( g \circ f \) is Nrg-irresolute.

(ii) If \( f \) is perfectly Nrg-continuous and \( g \) is strongly Nano-continuous then \( g \circ f \) is perfectly Nrg-continuous.

**Proof:**
(i) Let \( A \) be a Nrg-open set in \( W \). since the map \( g: (V, \tau_R'(Y)) \to (W, \tau_R'(Z)) \) is perfectly Nrg-continuous, the inverse image \( g^{-1}(A) \) is both Nano open and Nano closed in \( V \). Since the map \( f: (U, \tau_R(X)) \to (V, \tau_R(Y)) \) is Nrg-continuous, the inverse \( f^{-1}(g^{-1}(A)) \) of Nrg-open set in \( U \). Thus the inverse image \( (g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A)) \) is Nrg-open in \( U \) for every Nrg open set \( A \) in \( W \). Hence the map \( (g \circ f): (U, \tau_R(X)) \to (W, \tau_R'(Z)) \) is Nrg-irresolute.

(ii) Let \( A \) be a Nrg-open set in \( W \). since the map \( g: (V, \tau_R'(Y)) \to (W, \tau_R'(Z)) \) is strongly Nrg-continuous, the inverse image \( g^{-1}(A) \) is both Nano open and Nano closed in \( V \). As every Nano open set is Nrg-open, \( g^{-1}(A) \) is both Nrg-open and Nrg closed. Since the map \( f: (U, \tau_R(X)) \to (V, \tau_R'(Y)) \) is perfectly Nrg-continuous, the inverse \( f^{-1}(g^{-1}(A)) \) of Nano open and Nano closed set in \( U \). Thus the
inverse image \((g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A))\) is both Nano-open and Nano-closed in \(U\) for every Nrg open set \(A\) in \(W\). Hence the map 
\((g \circ f): (U, \tau_\mathbb{R}(X)) \rightarrow (W, \tau_\mathbb{R}'(Z))\) is perfectly Nrg - continuous.

References


