On fixed decomposition of non-deranged permutations

Ibrahim M. and M.S. Magami

Department of Mathematics, Usmanu Danfodiyo, University Sokoto, P.M.B. 2346 Sokoto, Nigeria


Abstract- Permutation statistic is one of the interesting field in Mathematics. In this paper we compute and redefined some statistics with respect to \( \Gamma_1 \) non deranged permutations, we also showed that the statistics \( \text{dez} \) (cardinality of the set \( \text{Dec} \)) is an Eulerian statistics on \( \Gamma_1 \) non deranged permutation due to its equidistributed with \( \text{des} \) (cardinality of the descent set) and the statistics \( \text{maz} \) (sum of the descent of \( Z\text{Der} \)) is equidistributed with \( \text{Maj} \) (sum of the descent set).

Index Terms- Descent Numbers; Descent set; Descent set of \( Z\text{Der} \); non-deranged permutations

I. INTRODUCTION

Permutation statistics were first introduced by [3] and then extensively studied by [10] in the last decades much progress has made, both in the discovery and the study of new statistics, and in extending these to other type of permutations such as words and restricted permutation. The concept of derangements in permutation groups (that is permutations without a fix element) has proportion in the underlying symmetric group \( S_n \). Garba and Ibrahim [4] used the concept to develop a scheme for prime numbers, \( P \leq 5 \) and \( \Omega \subseteq N \) which generate the cycles of permutations (derangements) using \( \omega_i = ((1)(1+i)...(1+(p-1)i))_{mp} \) to determine the arrangements. It is difficult for a set of derangements to be a permutation group because of the absence of the natural identity element (a non derangement). The construction of the generated set of permutations from the work of [4] as a permutation group was done by [11]. They achieved this by embedding an identity element into the generated set of permutation (strictly derangements) with the natural permutation composition as the binary operation (the group was denoted as \( G_P \)).

With no doubt, patterns in permutations have been well studied for over a century. As seem to be the case, these patterns were studied on permutations arbitrary. The symmetric group \( S_n \) is the set of all permutations of a set \( \Gamma \) of cardinality \( n \). There are several types of other smaller permutation groups (subgroup of \( S_n \)) of set \( \Gamma \), a notable one among them is the alternating group \( A_n \). On the other hand, [6] studied the representation of \( \Gamma_1 \)-non deranged permutation group \( G_{p^{r_1}} \) via group character, hence established that the character of every \( \omega_i \in G_{p^{r_1}} \) is never zero. Also the non standard Young tableau of \( \Gamma_1 \)-non deranged permutation group \( G_{p^{r_1}} \) has been studied by [5], they established that the Young tableau of this permutation group is non standard. [1] studied pattern popularity in \( \Gamma_1 \)-non deranged permutations they establish algebraically that pattern \( \tau_{1} \) is the most popular and pattern \( \tau_{3}, \tau_{4} \) and \( \tau_{5} \) are equipopular in \( G_{p^{r_1}} \), they further provided efficient algorithms and some results on popularity of patterns of length-3 in \( G_{p^{r_1}} \).[2] studied fuzzy on \( \Gamma_1 \)-non deranged permutation group \( G_{p^{r_1}} \) and discover that it is a one sided fuzzy ideal (only right fuzzy but not left) also the \( \alpha \)-level cut of \( f \) coincides with \( G_{p^{r_1}} \) if \( \alpha = \frac{1}{p} \) [7] studied ascent on \( \Gamma_1 \)-non deranged permutation group \( G_{p^{r_1}} \) and discover that the union of ascent of all \( \Gamma_1 \)-non derangement is equal to identity also observed that the difference between \( \text{Asc}(\omega_i) \) and \( \text{Asc}(\omega_{i-1}) \) is one. [8] provide very useful theoretical properties of \( \Gamma_1 \)-non deranged permutation in relation to exceedance and shown that the exceedance set of all \( \omega_i \in G_{p^{r_1}} \) such that \( \omega_i \neq e \) is \( \frac{1}{2}(p-1) \). More recently [9] established that the intersection of descent set of all \( \Gamma_1 \)-non derangement is empty, also observed that the descent number is strictly less than ascent number by \( p-1 \). Hence we will in this paper show that the statistics \( \text{dez} \) (cardinality of the set \( \text{Dec} \)) is an Eulerian statistics on \( \Gamma_1 \) non deranged permutations and the...
statistics $maj$ (sum of the descent of $ZDer$) is equidistributed with $Maj$ (sum of the descent set).

II. PRELIMINARIES

**Definition 2.1** [2]

Let $\Gamma$ be a non empty set of prime cardinality greater or equal to 5 such that $\Gamma \subseteq A$.

A bijection $\omega$ on $\Gamma$ of the form

$$\omega = \begin{pmatrix} 1 & 2 & 3 & \ldots & p \\ f(1) & f(2) & f(3) & \ldots & f(n) \end{pmatrix}$$

is called a $\Gamma$-non deranged permutation. We denoted $G_{\Gamma}$ to be the set of all $\Gamma$-non deranged permutations.

**Definition 2.2** [2]

The pair $G_{\Gamma}$ and the natural permutation composition forms a group which is denoted as $G_{\Gamma} \times G_{\Gamma}$. This is a special permutation group which fixes the first element of $\Gamma$.

**Definition 2.3** [9]

An descent of a permutation $f = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ f(1) & f(2) & f(3) & \ldots & f(n) \end{pmatrix}$ is any positive $i > n$ (where $i$ and $n$ are positive integers) where the current value is greater than the next, that is $i$ is a descent of a permutation $f(i) > f(i+1)$. The descent set of $f$, denoted as $Des(f)$, is given by $Des(f) = \{ i : f(i) > f(i+1) \}$.

The descent of $f$, denoted as $des(f)$, is defined as the number of descent and is given by $des(f) = |Des(f)|$.

**Definition 2.4**

$ZDer(f)$ is the permutation derived from $f$ by replacing each fixed point $f(i)$ by 0 and each other value $f(j)$ with $i \neq j$.

**Definition 2.5**

$Der(f)$ is the non-zero permutation of $ZDer(f)$.

**Definition 2.6**

The major index of $f$ denoted by $maj(f)$ is the sum of the descent set of the permutation $f$ that is

$$maj(f) = \sum_{i \in des(f)} i.$$

III. MAIN RESULTS

**Proposition 3.1.**

Suppose that $G_{\Gamma}$ is $\Gamma$-non deranged permutations. Then

$$Dez(\alpha_i) = Des(\alpha_i)$$

Proof.

Since

$$\omega_i = \begin{pmatrix} 1(1+i)_{mp} & 2(1+2i)_{mp} & \ldots & (1+(p-1)i)_{mp} \\ 1 & 2 & \ldots & p \end{pmatrix}$$

is any positive $i > n$, then $a_i$ in $ZDer(\alpha_i)$ is zero. Therefore

$$ZDer(\alpha_i) = \begin{pmatrix} 0 & (2i)_{mp} & \ldots & ((p-1)i)_{mp} \\ 1 & 2 & \ldots & (p-1) \end{pmatrix}$$

Which is formed by subtracting 1 from each value in $\alpha_i$. Hence, if $i < j$ in $\alpha_i$ then $i < j$ in $ZDer(\alpha_i)$ and if $a_i > a_j$ in $\alpha_i$ then $a_i > a_j$ in $ZDer(\alpha_i)$ the result follows.

**Corollary 3.2**

Let $G_{\Gamma}$ be a $\Gamma$-non derangement permutations, then the descent set of Zder is equidistributed with descent set

$$\sum_{i=1}^{n} dez(\alpha_i) = \sum_{i=1}^{n} des(\alpha_i).$$

Proof.

From proposition 3.1 the $Dez(\alpha_i) = Des(\alpha_i)$ and since $des(\alpha_i)$ and $dez(\alpha_i)$ are the cardinality of $Dez(\alpha_i)$ and $Des(\alpha_i)$ respectively. Then $dez(\alpha_i) = des(\alpha_i)$.

Hence,

$$\sum_{i=1}^{n} dez(\alpha_i) = \sum_{i=1}^{n} des(\alpha_i).$$

**Remark 3.3**

We can conclude that since both the set and the cardinality of DES and DEZ are equal, then they have the same properties.

**Proposition 3.4**

Suppose that $G_{\Gamma}$ is $\Gamma$-non deranged permutations. Then the sum of descent set of Zder is equidistributed with sum of descent set


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\[ \sum_{i=1}^{n} maz(\omega_i) = \sum_{i=1}^{n} maj(\omega_i) \]

**Proof.**

\[ maz(\omega_i) = \sum_{i \in DEZ(\omega_i)} i. \]

The proposition 3.1 that\( DEZ(\omega_i) = Des(\omega_i) \)

Hence

\[ maz(\omega_i) = \sum_{i \in DES(\omega_i)} i. \]

by replacing \( DEZ(\omega_i) \) by \( Des(\omega_i) \) and

\[ maj(\omega_i) = \sum_{i \in DES(\omega_i)} i. \]

Therefore,

\[ \sum_{i=1}^{n} maz(\omega_i) = \sum_{i=1}^{n} maj(\omega_i) \]

\[ \square \]

**Remark 3.5**

It is easy to notice that \( Maz(\omega_i) \) and \( Maj(\omega_i) \) are equal since the descent set and the descent set of \( ZDer(\omega_i) \) which are equal. And \( Maz(\omega_i) \) is the major derived from \( Des(\omega_i) \).

**Proposition 3.6**

Let \( G_p^\Gamma_i \) be a \( \Gamma_1 \)-non derangement permutations, then the

\[ maf(\omega_i) = maj \circ Der(\omega_i) \]

**Proof.**

\[ Maf(\pi) = \sum_{i \in Fix(\pi)} i - \sum_{i=1}^{Fix(\pi)} i + maj \circ Der(\pi) \]

The

Since for every \( \omega_i \in G_p^\Gamma_i \) only 1 is fix then,

\[ \sum_{i \in Fix(\pi)} i - \sum_{i=1}^{Fix(\pi)} i = 0. \]

Hence,

\[ maf(\omega_i) = maj \circ Der(\omega_i) \]

\[ \square \]

**Proposition 3.7**

Suppose that \( G_p^\Gamma_i \) is \( \Gamma_1 \)-non deranged permutations. Then

\[ maf(\omega_i) = dez(\omega_i)\left(\frac{p-1}{2}\right) \]

**Proof.**

Since

\[ ZDer(\omega_i) = (0 \ i_{mp} \ (2i)_{mp} \ \ldots \ ((p-1)i)_{mp}) \]

And

\[ Der(\omega_i) = (i_{mp} \ (2i)_{mp} \ \ldots \ ((p-1)i)_{mp}) \]

\[ maj \circ ZDer(\omega_i) = dez(\omega_i)\left(\frac{p-1}{2}\right) \]

but

\[ \text{Therefore,} \]

\[ maf(\omega_i) = maj \circ Der(\omega_i) \]

\[ \square \]

**Lemma 3.8**

Let \( G_p^\Gamma_i \) be a \( \Gamma_1 \)-non derangement permutations, then the

\[ Dez(\omega_i) \cap Dez(\omega_{p-1}) = \phi. \]

**Proof.**

Suppose \( \omega_i = a_1a_2a_3 \ldots a_{p-1}a_p \).

Then since \( Dez(\omega_i) = Des(\omega_i) \)

\[ Des = \{i: i < j & a_i > a_j\} \]

Also, \( \omega_{p-1} = a_1a_p \ldots a_{p-1} \), so its descent set

\[ Des = \{i: i < j & a_i < a_j\} \]

Therefore,
\[ \text{Dez}(\omega) \cap \text{Dez}(\omega_{p-1}) = \phi \]

Proposition 3.9

Let \( \omega_i \in G_p^{\Gamma_1} \). Then the
\[ \text{Dez}(\omega_i) \cup \text{Dez}(\omega_{p-1}) = \text{Dez}(\omega_{p-1}) . \]

Proof.

Let \( \omega_i = a_1a_2\ldots a_{p-1}a_p \). Then the
\[ \text{Dez}(\omega_i) = \{i: i < j \& a_i > a_j\} . \]

Then since \( \omega_{p-1} = a_1a_p a_{p-1}\ldots a_2 \) its \( \text{Dez}(\omega_i) = \{i: i < j \& a_i > a_j\} \) where \( i \neq 1 \).

Therefore,
\[ \text{Dez}(\omega_i) \cup \text{Dez}(\omega_{p-1}) = \{i: i < j \& a_i > a_j \text{ or } a_i < a_j, i \neq 1\} \]

Since \( \omega_i \) is a permutation then
\[ \text{Dez}(\omega_i) \cup \text{Dez}(\omega_{p-1}) = \{i: i < j, i \neq 1\} \]

Since \( \omega_{p-1} = 1p(p-1)\ldots 2 \) then
\[ \text{Dez}(\omega_{p-1}) = \{i: i < j, i \neq 1\} . \]

Hence,
\[ \text{Dez}(\omega_{p-1}) = \text{Dez}(\omega_i) \cup \text{Dez}(\omega_{p-1}) \]

\[ \square \]

Corollary 3.10

Let \( \omega_i \in G_p^{\Gamma_1} \). Then the
\[ \text{dez}(\omega_i) + \text{dez}(\omega_{p-1}) = \text{dez}(\omega_{p-1}) \]

Proof.

It follows from proposition 3.9

Corollary 3.11

Let \( G_p^{\Gamma_1} \) be a \( \Gamma_1 \)-non derangement permutations, then the
\[ \text{Dez}(\omega_{p-1}) = \bigcup_{i=1}^{p-2} \text{Dez}(\omega_i) . \]

Proof.

From proposition 3.9 \( \text{Dez}(\omega_i) \cup \text{Dez}(\omega_{p-1}) = \text{Dez}(\omega_{p-1}) \), so we want to show that for any \( G_p^{\Gamma_1} \) there exist \( \omega_i \) and \( \omega_{p-1} \) where \( i \neq 1 \). Since \( p \geq 5 \) and any \( G_p^{\Gamma_1} \) consist of non -deranged permutations \( \{\omega_1, \omega_2, \ldots, \omega_{p-1}\} \), therefore it has at least \( \omega_2 \) and \( \omega_{p-2} \). The result follows.

IV. CONCLUSION

This paper has provided very useful theoretical properties of the statistics descent set and descent set of \( Z\text{Der} \) and the paper also shown that the statistics \( \text{dez} \) (cardinality of the set \( \text{Dez} \)) is an Eulerian statistics on \( \Gamma_1 \) non deranged permutations and the statistics \( \text{maz} \) (sum of the descent of \( Z\text{Der} \)) is equidistributed with \( \text{Maj} \) (sum of the descent set).

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AUTHORS

First Author – Ibrahim M, Department of Mathematics, Usman Danfodiyo, University Sokoto, P.M.B. 2346 Sokoto, Nigeria

Second Author – M.S.Magami, Department of Mathematics, Usman Danfodiyo, University Sokoto, P.M.B. 2346 Sokoto, Nigeria

Corresponding Author’s Email:
muhammad.ibrahim@udusok.edu.ng