# On fixed decomposition of non-deranged permutations

**Ibrahim M. and M.S.Magami**

Department of Mathematics, Usmanu Danfodiyo, University Sokoto, P.M.B. 2346 Sokoto, Nigeria

DOI: 10.29322/IJSRP.9.12.2019.p9620
http://dx.doi.org/10.29322/IJSRP.9.12.2019.p9620

***Abstract-*** Permutation statistic is one of the interesting field in Mathematics. In this paper we compute and redefined some statistics with respect to $\Gamma_1$ non deranged permutations, we also showed that the statistics *dez* (cardinality of the set *Dez)* is an Eulerian statistics on $\Gamma_1$ non deranged permutation due to its equidistributed with des (cardinality of the descent set ) and the statistics *maz* (sum of the descent of *ZDer*) is equidistributed with *Maj* (sum of the descent set).

***Index Terms-*** Descent Numbers; Descent set; Descent set of ZDer; non-deranged permutations

## I. Introduction

Permutation statistics were first introduced by [3] and then extensively studied by [10] in the last decades much progress has made, both in the discovery and the study of new statistics, and in extending these to other type of permutations such as words and restricted permutation. The concept of derangements in permutation groups (that is permutations without a fix element) has proportion in the underlying symmetric group $S_n$. Garba and Ibrahim [4] used the concept to develop a scheme for prime numbers $P \leq 5$ and $\Omega \subseteq N$ which generate the cycles of permutations (derangements) using $\omega_i = ((1)(1+i)_{mp}(1+2i)_{mp}...(1+(p-1)i)_{mp})$ to determine the arrangements. It is difficult for a set of derangements to be a permutation group because of the absence of the natural identity element (a non derangement),The construction of the generated set of permutations from the work of [4] as a permutation group was done by [11].They achieved this by embedding an identity element into the generated set of permutation(strictly derangements) with the natural permutation composition as the binary operation (the group was denoted as $G_p$)

With no doubt, patterns in permutations have been well studied for over a century. As seem to be the case, these patterns were studied on permutations arbitrary. The symmetric group $S_n$ is the set of all permutations of a set $\Gamma$ of cardinality n. There are several types of other smaller permutation groups (subgroup of $S_n$) of set $\Gamma$ ,a notable one among them is the alternating group $A_n$. On the other hand, [6] studied the representation of $\Gamma_1$-non deranged permutation group $G_P^{\Gamma_1}$ via group character, hence established that the character of every $\omega_i \in G_p^{\Gamma_1}$ is never zero. Also the non standard Young tableaux of $\Gamma_1$-non deranged permutation group $G_P^{\Gamma_1}$ has been studied by [5], they established that the Young tableaux of this permutation group is non standard. [1] studied pattern popularity in $\Gamma_1$-non deranged permutations they establish algebraically that pattern $\tau_1$ is the most popular and pattern $\tau_3, \tau_4$ *and* $\tau_5$ are equipopular in $G_P^{\Gamma_1}$ they further provided efficient algorithms and some results on popularity of patterns of length-3 in $G_P^{\Gamma_1}$.[2] studied Fuzzy on $\Gamma_1$-non deranged permutation group $G_P^{\Gamma_1}$ and discover that it is a one sided fuzzy ideal ( only right fuzzy but not left ) also the $\alpha-$level cut of $f$ coincides with $G_P^{\Gamma_1}$ if $\alpha = \dfrac{1}{p}$ [7] studied ascent on $\Gamma_1$-non deranged permutation group $G_P^{\Gamma_1}$ and discover that the union of ascent of all $\Gamma_1$-non derangement is equal to identity also observed that the difference between $Asc(\omega_i)$ and $Asc(\omega_{p-1})$ is one. [8] provide very useful theoretical properties of $\Gamma_1$-non deranged permutation s in relation to excedance and shown that the excedance set of all $\omega_i$ in $G_P^{\Gamma_1}$ such that $\omega_i \neq e$ is $\frac{1}{2}(p-1)$. More recently [9] established that the intersection of descent set of all $\Gamma_1$-non derangement is empty, also observed that the descent number is strictly less than ascent number by $p-1$. Hence we will in this paper show that the statistics *dez* (cardinality of the set *Dez)* is an Eulerian statistics on $\Gamma_1$ non deranged permutations and the statistics *maz* (sum of the descent of *ZDer*) is equidistributed with *Maj* (sum of the descent set).

## II. PRELIMINARIES

**Definition 2.1** [2]
Let $\Gamma$ be a non empty set of prime cardinality greater or equal to 5 such that $\Gamma \subset$ A bijection $\omega$ on $\Gamma$ of the form

$$\omega_i = \begin{pmatrix} 1 & 2 & 3 & . & . & . & p \\ 1 & (1+i)_{mp} & (1+2i)_{mp} & & & & (1+(p-1)i)_{mp} \end{pmatrix}$$

is called a $\Gamma_1$-non deranged permutation. We denoted $G_p$ to be the set of all $\Gamma_1$-non deranged permutations.

**Definition 2.2** [2]
The pair $G_p$ and the natural permutation composition forms a group which is denoted as $G_P^{\Gamma_1}$. This is a special permutation group which fixes the first element of $\Gamma$.

**Definition 2.3 [9]**
An descent of permutation $f = \begin{pmatrix} 1 & 2 & 3 & . & . & . & n \\ f(1) & f(2) & f(3) & . & . & . & f(n) \end{pmatrix}$ is any positive $i > n$ (where $i$ and $n$ are positive integers) where the current value is greater than the next, that is $i$ is an descent of a permutation $f(i) > f(i+1)$. The descent set of $f$, denoted as $Des(f)$, is given by $Des(f) = \{i : f(i) > f(i+1)\}$ the descent number of $f$, denoted as $des(f)$, is defined as the number of descent and is given by $des(f) = |Des(f)|$.

**Definition 2.4**
$ZDer(f)$ is the permutation derived from *f* by replacing each fixed point $f(i)$ by *0* and each other value $f(j)$ with $i \neq j$

**Definition 2.5**
$Der(f)$ is the non zero permutation of $ZDer(f)$

**Definition 2.6**
The major index of $f$ denoted by $maj(f)$ is the sum of the descent set of the permutation $f$ that is $maj(f) = \sum_{i \in Des(f)} i$

## III. MAIN RESULTS

**Proposition 3.1.**
Suppose that $G_P^{\Gamma_1}$ is $\Gamma_1$-non deranged permutations. The

$$Dez(\omega_i) = Des(\omega_i)$$

**Proof.**
Since $\omega_i = \left(1(1+i)_{mp}(1+2i)_{mp}...(1+(p-1)i)_{mp}\right)$ and

$$red(\omega_i) = \begin{pmatrix} 2 & 3 & \dots & p \\ 1 & 2 & \dots & (p-1) \end{pmatrix}$$

suppose $\omega_i = a_1 a_2 ... a_p$ and $red(\omega_i) = b_2 b_3 ... b_p$. Since the permutation $ZDer(\omega_i)$ is the permutation derived from $\omega_i$ by replacing each fixed point $a_i$ by 0 and each other value $a_j$ with $i \neq j$, therefore since 1 is the only fixed point in $\omega_i$ then $a_i$ in $ZDer(\omega_i)$ is zero. Therefore

$$ZDer(\omega_i) = \left(0 \;\; i_{mp} \;\; (2i)_{mp} \dots ((p-1)i)_{mp}\right)$$

Which is formed by subtracting 1 from each value in $\omega_i$, $\omega_i$, hence if $i < j$ in $\omega_i$ then $i < j$ in $ZDer(\omega_i)$ and if $a_i > a_j$ in $\omega_i$ then $a_i > a_j$ in $ZDer(\omega_i)$ the result follows

□

**Corollary 3.2**
Let $G_P^{\Gamma_1}$ be a $\Gamma_1$-non derangement permutations, then the descent set of Zder is equidistributed with descent set

$$\sum_{i=1}^{p-1} dez(\omega_i) = \sum_{i=1}^{p-1} des(\omega_i).$$

**Proof.**
From proposition 3.1 the $Dez(\omega_i) = Des(\omega_i)$ and since des($\omega_i$) and dez($\omega_i$) are the cardinality of Dez($\omega_i$) and Des($\omega_i$) respectively. Then dez($\omega_i$) = des($\omega_i$)
.
Hence,

$$\sum_{i=1}^{p-1} dez(\omega_i) = \sum_{i=1}^{p-1} des(\omega_i).$$

**Remark 3.3**
We can conclude that since both the set and the cardinality of DES and DEZ are equal, then they have the same properties.

**Proposition 3.4**
Suppose that $G_P^{\Gamma_1}$ is $\Gamma_1$-non deranged permutations. Then the sum of descent set of Zder is equidistributed with sum of descent set

$$\sum_{i=1}^{p-1} maz(\omega_i) = \sum_{i=1}^{p-1} maj(\omega_i)$$

**Proof.**

The $maz(\omega_i) = \sum_{i \in DEZ(\omega_i)} i.$ But it has been shown from proposition 3.1that $Dez(\omega_i) = Des(\omega_i)$

Hence

$$maz(\omega_i) = \sum_{i \in DES(\omega_i)} i.$$

by replacing $Dez(\omega_i)$ by $Des(\omega_i)$ and

$$maj(\omega_i) = \sum_{i \in DES(\omega_i)} i.$$

Therefore,

$$\sum_{i=1}^{p-1} maz(\omega_i) = \sum_{i=1}^{p-1} maj(\omega_i)$$

□

**Remark 3.5**

It is easy to notice that $Maz$ and $Maj$ of $\omega_i$ are equal since the descent set and the descent set of $ZDer$ which are $Des$ and $Dez$ are equal. And $Maz$ is the major derived from $Dez$

**Proposition 3.6**

Let $G_P^{\Gamma_1}$ be a $\Gamma_1$-non derangement permutations, then the

$$maf(\omega_i) = maj \circ Der(\omega_i)$$

***Proof.***

The $Maf(\pi) = \sum_{i \in Fix(\pi)} i - \sum_{i=1}^{Fix(\pi)} i + maj \circ Der(\pi)$

Since for every $\omega_i \in G_P^{\Gamma_1}$ only 1is fix then,

$$\sum_{i \in Fix(\pi)} i - \sum_{i=1}^{Fix(\pi)} i = 0.$$

Hence,

$$maf(\omega_i) = maj \circ Der(\omega_i)$$

□

**Proposition 3.7**

Suppose that $G_P^{\Gamma_1}$ is $\Gamma_1$-non deranged permutations. Then

$$maf(\omega_i) = dez(\omega_i)\left(\frac{p-1}{2}\right)$$

**Proof**.
Since

$$ZDer(\omega_i) = \left(0 \quad i_{mp} \quad (2i)_{mp} \quad \ldots \quad ((p-1)i)_{mp}\right)$$

And

$$Der(\omega_i) = \left(i_{mp} \quad (2i)_{mp} \quad \ldots \quad ((p-1)i)_{mp}\right)$$

but

$$maj \circ ZDer(\omega_i) = dez(\omega_i)\left(\frac{p-1}{2}\right).$$

Then since $Der(\omega_i)$ is formed by eliminating the first value (i.e 0) in $ZDer(\omega_i)$.

Therefore,

$$maj \circ Der(\omega_i) = dez(\omega_i)\left(\frac{p+1}{2}\right) - 1$$
$$= dez(\omega_i)\left(\frac{p+1}{2} - 1\right)$$
$$= dez(\omega_i)\left(\frac{p-1}{2}\right)$$

but

$$maf(\omega_i) = maj \circ Der(\omega_i)$$

Hence,

$$maf(\omega_i) = dez(\omega_i)\left(\frac{p-1}{2}\right)$$

□

**Lemma 3.8**

Let $G_P^{\Gamma_1}$ be a $\Gamma_1$-non derangement permutations, then the

$$Dez(\omega_i) \cap Dez(\omega_{P-i}) = \phi.$$

**Proof**.

Suppose $\omega_i = a_1 a_2 \ldots a_{p-1} a_p$. Then since $Dez(\omega_i) = Des(\omega_i)$ so

$$Dez = \{i : i < j \,\&\, a_i > a_j\}$$

Also, $\omega_{p-i} = a_1 a_p a_{p-1} \ldots a_2$ ,so its descent set $Des = \{i : i < j \,\&\, a_i < a_j\}$. Therefore,

$$Dez(\omega_i) \cap Dez(\omega_{P-i}) = \phi$$

□

**Proposition 3.9**

let $\omega_i \in G_P^{\Gamma_1}$ .Then the

$$Dez(\omega_i) \cup Dez(\omega_{p-i}) = Dez(\omega_{p-1})$$.

**Proof.**

Let $\omega_i = a_1 a_2 \ldots a_{p-1} a_p$ . Then the

$$Dez(\omega_i) = \{i : i < j \,\&\, a_i > a_j\}$$.

Then since $\omega_{p-i} = a_1 a_p a_{p-1} \ldots a_2$ ,its $Dez(\omega_i) = \{i : i < j \,\&\, a_i > a_j\}$ where $i \neq 1$.

Therefore,

$$Dez(\omega_i) \cup Dez(\omega_{p-i}) = \{i : i < j \,\&\, a_i > a_j \text{ or } a_i < a_j, i \neq 1\}$$

Since $\omega_i$ is a permutation then

$$Dez(\omega_i) \cup Dez(\omega_{p-1}) = \{i : i < j, i \neq 1\}$$

Since $\omega_{p-1} = 1p(p-1)\ldots 2$ then

$$Dez(\omega_{p-1}) = \{i : i < j, i \neq 1\}$$.

Hence,

$$Dez(\omega_{p-1}) = Dez(\omega_i) \cup Dez(\omega_{p-i})$$

□

**Corollary 3.10**

Let $\omega_i \in G_P^{\Gamma_1}$ . Then the

$$dez(\omega_i) + dez(\omega_{p-i}) = dez(\omega_{p-1})$$

**Proof**.

It follows from proposition 3.9

**Corollary 3.11**

Let $G_P{}^{\Gamma_1}$ be a $\Gamma_1$-non derangement permutations, then the

$$Dez(\omega_{p-1}) = \bigcup_{i=1}^{p-2} Dez(\omega_i)$$.

**Proof.**

From proposition 3.9 $Dez(\omega_i) \cup Dez(\omega_{p-i}) = Dez(\omega_{P-1})$ ,so we want to show that for any $G_p^{\Gamma_1}$ there exist $\omega_i$ and $\omega_{p-1}$ where $i \neq 1$. Since $p \geq 5$ and any $G_p^{\Gamma_1}$ consist of non –deranged permutations $\{\omega_1, \omega_2, \ldots, \omega_{p-1}\}$, therefore it has atleast $\omega_2$ and $\omega_{p-2}$ . The result follows.

## IV. CONCLUSION

This paper has provided very useful theoretical properties of the statistics descent set and descent set of ZDer and the paper also shown that the statistics *dez* (cardinality of the set *Dez)* is an Eulerian statistics on $\Gamma_1$ non deranged permutations and the statistics *maz* (sum of the descent of *ZDer*) is equidistributed with *Maj* (sum of the descent set).

## REFERENCES


[1] K.O.Aremu., A.H Ibrahim., S.Buoro and F.A.Akinola, Pattern Popularity in -non deranged permutations: An Algebraic and Algorithmic Approach.Annals. Computer Science Series15(2),(2017) 115-122.

[2] K.O.Aremu,O.Ejima and M.S.Abdullahi,On the Fuzzy -non deranged permutation group .Asian Journal of Mathematics and Computer Research 18(4) (2017),152-157.

[3] L.Euler Institutiones Calculi differntialis in "opera omnia series prime " Volx, ,(1913),Teubner,Leipzig.

[4] A.I.Garba and A.A. Ibrahim, A New Method of Constructing a Variety of Finite Group Based on Some Succession Scheme. International Journal of Physical Sciences 2(3) (2010), 23-26.

[5] A.I Garba,O.Ejima, K.O.Aremu and U.Hamisu,Non standard Young tableaux of -non deranged permutation group .Global Journal of Mathematical Analysis5(1) (2017), 21-23.

[6] A.A.Ibrahim,O. Ejima and K.O. Aremu.On the Representation of -non deranged permutation group Advance in Pure Mathematics, 6(2016),608-614.

[7] M.Ibrahim, A.A Ibrahim, A.I Garba and K.O.Aremu,Ascent on -non deranged permutation group International journal of science for global sustainability, 4(2) (2017), 27-32.

[8] M. Ibrahim and Garba A.I, Exedance on -non deranged permutations proceedings of Annual National Conference of Mathematical Association of Nigeria (MAN) , (2018),197-201.

[9] M. Ibrahim and A.I Garba, Descent on -non deranged permutation group Journal of Mathematical Association of Nigeria ABACUS , 46(1) (2019),12-18.

[10] P.A.MacMahon,.Combinatory Analysis Vol. 1 and 2 (1915)Cambridge University Press(reprinted by Chesea,New York,1955)

[11] A.Usman, and A.A.Ibrahim, A New Generating Function for Aunu Patterns: Application in Integer Group Modulo n. Nigerian Journal of Basic and Apllied Sciences , 9(1) (2011),1-4.


## AUTHORS


**First Author** – Ibrahim M, Department of Mathematics, Usmanu Danfodiyo, University Sokoto, P.M.B. 2346 Sokoto, Nigeria

**Second Author** – M.S.Magami, Department of Mathematics, Usmanu Danfodiyo, University Sokoto, P.M.B. 2346 Sokoto, Nigeria

Corresponding Author's Email:
muhammad.ibrahim@udusok.edu.ng