Quaternions and Rotation Sequences

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Abstract- The position of a point after some rotation about the origin can simply be obtained by multiplyingits coordinates with a matrix. One reason for introducing homogeneous coordinates is tobe able to describe translation with a matrix so that multiple transformations, whether each is arotation or a translation, can be concatenated into one described by the product of their respectivematrices. However, in some applications (such as spaceship tracking), we need only be concernedwith rotations of an object, or at least independently from other transformations. In such a situation,we often need to extract the rotation axis and angle from a matrix which represents theconcatenation of multiple rotations. The homogeneous transformation matrix, however, is not wellsuited for the purpose.

***Index Terms*-** Quaternion,Quaternion triple-product, Rotation operator

I. Introduction

Up until now we have learned that a rotation in $\mathbf{R}^3$ about an axis through the origin can be representedby a 3×3 orthogonal matrix with determinant 1. However, the matrix representation seemsredundant because only four of its nine elements are independent. Also the geometric interpretationof such a matrix is not clear until we carry out several steps of calculation to extract the rotationaxis and angle. Furthermore, to compose two rotations, we need to compute the product of thetwo corresponding matrices, which requires twenty-seven multiplications and eighteen additions [1-5].

Quaternions are very efficient for analyzing situations where rotations in $\mathbf{R}^3$ are involved. Aquaternion is a 4-tuple, which is a more concise representation than a rotation matrix. Its geometricmeaning is also more obvious as the rotation axis and angle can be trivially recovered. Thequaternion algebra to be introduced will also allow us to easily compose rotations. This is becausequaternion composition takes merely sixteen multiplications and twelve additions [4-9].

HamiltonsQuaternions

Whilenumbers of theform $a+ib$, thatis, complexnumbers of rank2, weregaininggeneralacceptance, somemathematicians of thatdaysoughtothermathematicalsystemsoverthehyper-complexnumbers of, say, rank $3,4,\dots,n$. In1843 afteryears of struggling, inaneffort to createsuch a system, a suddenstroke of mathematicalinsightcameuponWilliamRowanHamilton. Historysayshehappened to beoutwalkingwithhiswifeand, reputedly, carvedthesenowfamousequationsinthestonewall of thebridge, inDublin, overwhichtheyhappened to bewalking:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Implicitintheseequations, isthat

$$\mathbf{ij} = \mathbf{i}\times\mathbf{j} = \mathbf{k} = -\mathbf{j}\times\mathbf{i} = -\mathbf{ji},$$

$$\mathbf{jk} = \mathbf{j}\times\mathbf{k} = \mathbf{i} = -\mathbf{k}\times\mathbf{j} = -\mathbf{kj},$$

$$\mathbf{ki} = \mathbf{k}\times\mathbf{i} = \mathbf{j} = -\mathbf{i}\times\mathbf{k} = -\mathbf{ik}.$$

All of quaternionalgebraproceedsfromtheseequations, e.g. theproduct of twoquaternions $\mathbf{p}$ and $\mathbf{q}$ where

$$p = p_0 + \mathbf{p} = p_0 + \mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3$$

$$q = q_0 + \mathbf{q} = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$$

canbereduced to

$$pq = p_0q_0 - \mathbf{p}\cdot\mathbf{q} + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p}\times\mathbf{q}$$

allinaccordancewithHamiltonsforegoingoriginalequations.

QuaternionAlgebra

The set of quaternions, together with the two operations of addition and multiplication, form anon-commutative [10,11]The standard orthonormal basis for $\mathbf{R}^3$ is given by three unit vectors $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, $\mathbf{k} = (0,0,1)$.

A quaternion q is defined as the sum of a scalar q_0 and avector $\mathbf{q} = (q_1, q_2, q_3)$, i.e.

$$q = q_0 + q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}.$$

Addition and Multiplication

Addition of two quaternions acts component-wise. More specifically, consider the quaternion q above and another quaternion

$$p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}.$$

Then we have

$$p + q = (p_0 + q_0) + (p_1 + q_1)\mathbf{i} + (p_2 + q_2)\mathbf{j} + (p_3 + q_3)\mathbf{k}.$$

Every quaternion q has a negative $-q$ with components $-q_i,\ i = 0,1,2,3$.

The product of two quaternions satisfies these fundamental rules introduced by Hamilton:

$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,$

$\mathbf{ij} = \mathbf{k} = -\mathbf{ji},$

$\mathbf{jk} = \mathbf{i} = -\mathbf{kj},$

$\mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$

Hence, the product of two quaternions p and q is:

$$pq = p_0q_0 - (p_1q_1 + p_2q_2p_3q_3) + p_0(q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}) + q_0(p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}) + (p_2q_3 - p_3q_2)\mathbf{i} + (p_3q_1 - p_1q_3)\mathbf{j} + (p_1q_2 - p_2q_1)\mathbf{k}.$$

Using by the inner product and cross product of two vectors in $\mathbf{R}^3$, we obtain:

$$pq = p_0q_0 - p \cdot q + \mathbf{p}_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q}.$$

In the above, $\mathbf{p} = (p_1, p_2, p_3)$ and $\mathbf{q} = (q_1, q_2, q_3)$ are the vector parts of p and q, respectively.

Complex Conjugate, Norm, and Inverse

Let $q = q_0 + \mathbf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ be a quaternion. The *complex conjugate*of q, denoted q^*, isdefined as

$$q^* = q_0 - \mathbf{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}.$$

From the definition we immediately have

$$(q^*)^* = q_0 - (-\mathbf{q}) = q,$$

$$q + q^* = 2q_0,$$

$$q^* q = (q_0 - \mathbf{q})(q_0 + \mathbf{q}) = qq^*.$$

Given two quaternions p and q, we verify that

$$(pq)^* = q^* p^*.$$

The *norm* of a quaternion q, denoted by $|q|$, is the scalar $|q| = \sqrt{q^* q}$. A quaternion is called a*unit quaternion* if its norm is 1. The norm of the product of two quaternions p and q is the productof the individual norms, for we have

$$|pq|^2 = |p|^2 |q|^2.$$

The inverse of a quaternion q is defined as

$$q^{-1} = \frac{q^*}{|q|^2}.$$

We verify that $q^{-1}q = qq^{-1} = 1$. In the case q is a unit quaternion, the inverse is itsconjugate q^*.

Power, Exponential, and Logarithm

Let we have $q = q_0 + \mathbf{q}$. That $q_0^2 + \|\mathbf{q}\|^2 = 1$ implies that there exists aunique $\theta \in [0, \pi]$ such that $\cos\theta = q_0$ and $\sin\theta = \|\mathbf{q}\|$. The quaternion can thus be rewritten in terms of θ and the unit vector $\mathbf{u} = \mathbf{q}/\|\mathbf{q}\|$:

$$q = |q|(\cos\theta + \mathbf{u}\sin\theta),$$

where $\theta = \arccos(q_0/q)$. Euler's identity for a complex number is

$$a + bi = \sqrt{a^2 + b^2}e^{i\phi},$$

where $i^2 = -1$ and $\phi = a\tan\angle(b, a)$, generalizes to the quaternion q can berewritten as

$$q = |q|e^{u\theta}.$$

This allows us to define the power of q as

$$q^\rho = |q|^\rho (e^{u\theta})^\rho = |q|^\rho (\cos(\rho\theta) + \mathbf{u}\sin(\rho\theta)), \qquad \rho \in \mathbf{R}.$$

Intuitively, the power is taken over the norm of the quaternion while a scaling is performed on its
"phase angle".

An exponential of q makes use of the Taylor expansion that treats q just as an ordinary variable:

$$e^q = \sum_{i=0}^{\infty} \frac{q^i}{i!}.$$

The sum on the right hand side has a closed form that transforms the above into

$$e^q = \exp(q_0 + \mathbf{u}\|\mathbf{q}\|) = e^{q_0}(\cos\|\mathbf{q}\| + \mathbf{u}\sin\|\mathbf{q}\|).$$

The logarithm of q is accordingly defined as

$$\ln q = \ln|q| + \mathbf{u}\arccos\left(\frac{q_0}{|q|}\right).$$

The two operations are inverses of each other as we verify that

$$e^{\ln q} = e^{\ln|q| + \mathbf{u}\arccos(q_0/|q|)} = |q|e^{\mathbf{u}\arccos(q_0/|q|)} = |q|\left(\frac{q_0}{|q|} + \mathbf{u}\frac{\|\mathbf{q}\|}{|q|}\right) = q.$$

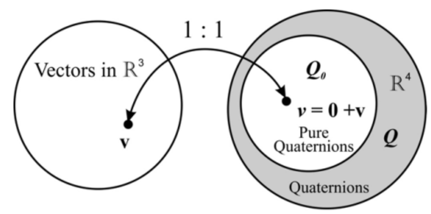

Figure 1. Correspondence: vector ↔ quaternions

Special quaternion triple-product – a rotation operator

Teorem 1. For any quaternion

$$q = q_0 + \mathbf{q} = \cos\theta + \mathbf{u}\sin\theta$$

and for any vector $\mathbf{v} \in \mathbf{R}^3$ the action of the operator
$L_q(\mathbf{v}) = q\mathbf{v}q^*$
on $\mathbf{v}$ may be interpreted geometrically as a rotation of the vector $\mathbf{v}$ through an angle 2θ about $\mathbf{q}$ as the axis of rotation [10].

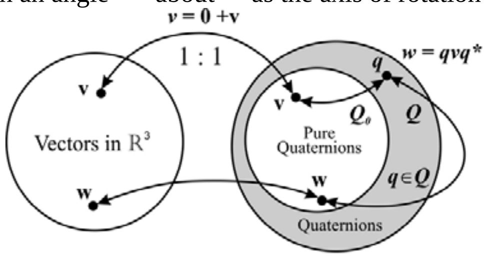

Figure 2. Quaternion operations on vectors

Rotation operator geometry

The quaternion rotation operator takes $v \to w$. That is,
$w = qvq^* = \left(q_0^2 - |\mathbf{q}|^2\right)\mathbf{v} + 2(\mathbf{q}\cdot\mathbf{v})\mathbf{q} + 2q_0(\mathbf{q}\times\mathbf{v})$.
If we write the vector $\mathbf{v}$ in the form
$\mathbf{v} = \mathbf{a} + \mathbf{n}$,
where $\mathbf{a}$ is the components of $\mathbf{v}$ along the vector part of q, and $\mathbf{n}$ is the components of $\mathbf{v}$ normal to the vector part of q. Then, it follows that
$\mathbf{a} + \mathbf{m} = q(\mathbf{a}+\mathbf{n})q^* = \left(q_0^2 - |\mathbf{q}|^2\right)(\mathbf{a}+\mathbf{n}) + 2(\mathbf{q}\cdot\mathbf{a})\mathbf{q} + 2q_0(\mathbf{q}\times\mathbf{n}) = \mathbf{w}$.

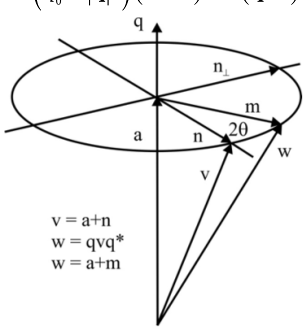

Figure 3. Rotation operator geometry

Two-rotation tracking sequence using matrices

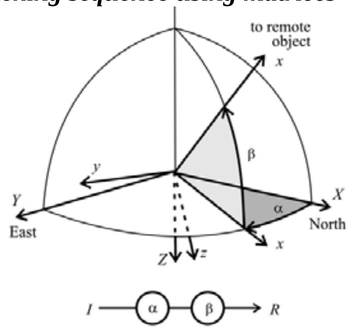

Figure 4. A rotation about Z-axis, followed by a rotation about new y-axis

$$\mathbf{R} = R^y_\beta R^z_\alpha \mathbf{I} = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{I}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta & \sin\alpha\cos\beta & -\sin\beta \\ -\sin\alpha & \cos\alpha & 0 \\ \cos\alpha\sin\beta & \sin\alpha\sin\beta & \cos\beta \end{pmatrix}\mathbf{I}$$

where $\mathbf{I}$ is identity matrix in $\mathbf{R}^3$.

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