Quaternions and Rotation Sequences

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Abstract- The position of a point after some rotation about the origin can simply be obtained by multiplying its coordinates with a matrix. One reason for introducing homogeneous coordinates is to be able to describe translation with a matrix so that multiple transformations, whether each is a rotation or a translation, can be concatenated into one described by the product of their respective matrices. However, in some applications (such as spaceship tracking), we need only be concerned with rotations of an object, or at least independently from other transformations. In such a situation, we often need to extract the rotation axis and angle from a matrix which represents the concatenation of multiple rotations. The homogeneous transformation matrix, however, is not well suited for the purpose.

Index Terms- Quaternion, Quaternion triple-product, Rotation operator

I. INTRODUCTION

Until now we have learned that a rotation in R^3 about an axis through the origin can be represented by a 3×3 orthogonal matrix with determinant 1. However, the matrix representation seems redundant because only four of its nine elements are independent. Also the geometric interpretation of such a matrix is not clear until we carry out several steps of calculation to extract the rotation axis and angle. Furthermore, to compose two rotations, we need to compute the product of the two corresponding matrices, which requires twenty-seven multiplications and eighteen additions [1-5].

Quaternions are very efficient for analyzing situations where rotations in R^3 are involved. A quaternion is a 4-tuple, which is a more concise representation than a rotation matrix. Its geometric meaning is also more obvious as the rotation axis and angle can be trivially recovered. The quaternion algebra to be introduced will also allow us to easily compose rotations. This is because quaternion composition takes merely sixteen multiplications and twelve additions [4-9].

Hamilton's Quaternions

While numbers of the form a + ib, that is, complex numbers of rank 2, were gaining general acceptance, some mathematicians of that day sought other mathematical systems over the hypercomplex numbers of, say, rank 3, 4, ..., n. In 1843 after years of struggling, manneffort to create such a system, a suddenstroke of mathematicalinsightcameuponWilliamRowanHamilton. History says he happened to be out walking with his wife and, reputedly, carved these now famous equations in the stone wall of the bridge, in Dublin, over which they happened to be walking:

i^2 = j^2 = k^2 = ijk = -1.

Implicit in these equations, isthat

ij = k = -j \times i = -ji,

jk = i \times k = i = -k \times j = -kj,

ki = -i \times k = k = -ik.

All of quaternion algebra proceeds from these equations, e.g. the product of two quaternions p and q where

p = p_o + p_1 i + p_2 j + p_3 k,

q = q_o + q_1 i + q_2 j + q_3 k,

can be reduced to

pq = p_o q_o - \langle i, q \rangle + \langle p, q \rangle + (q_o p_o + \langle i, j \rangle + \langle j, k \rangle + \langle k, i \rangle) k.

Addition and Multiplication

Addition of two quaternions acts component-wise. More specifically, consider the quaternion q above and another quaternion

p = p_o + p_1 i + p_2 j + p_3 k.

Then we have

p + q = (p_o + q_o) + (p_1 + q_1) i + (p_2 + q_2) j + (p_3 + q_3) k.

Every quaternion q has a negative -q with components

-q_i, i = 0, 1, 2, 3.

The product of two quaternions satisfies these fundamental rules introduced by Hamilton:
Hence, the product of two quaternions $p$ and $q$ is:

$$pq = p_1q_1 + (p_2q_3 - p_3q_2) + (p_3q_1 - p_1q_3) + (p_1q_2 - p_2q_1).$$

Using the inner product and cross product of two vectors in $\mathbb{R}^3$, we obtain:

$$pq = pq - p \cdot q + p \times q + q \times p.$$

In the above, $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ are the vector parts of $p$ and $q$, respectively.

**Complex Conjugate, Norm, and Inverse**

Let $q = q_0 + q_i + q_j + q_k$ be a quaternion. The complex conjugate of $q$, denoted $q^*$, is defined as

$$q^* = q_0 - q_i - q_j - q_k.$$

From the definition we immediately have

$$(q^*)^* = q_0 - (-q) = q,$$

$$q + q^* = 2q_0,$$

$$q^* q = (q_0 - q)(q_0 + q) = qq^*.$$

Given two quaternions $p$ and $q$, we verify that

$$(pq)^* = q^* p^*.$$

The norm of a quaternion $q$, denoted by $|q|$, is the scalar $|q| = \sqrt{q^* q}$. A quaternion is called a unit quaternion if its norm is 1. The norm of the product of two quaternions $p$ and $q$ is the product of the individual norms, for we have

$$|pq|^2 = |p|^2 |q|^2.$$

The inverse of a quaternion $q$ is defined as

$$q^{-1} = \frac{q^*}{|q|^2}.$$

We verify that $q^{-1} q = q q^{-1} = 1$. In the case $q$ is a unit quaternion, the inverse is its conjugate $q^*$.

**Power, Exponential, and Logarithm**

Let we have $q = q_0 + q_i$. That $q_0^2 + \|q\|^2 = 1$ implies that there exists a unique $\theta \in [0, \pi]$ such that $\cos \theta = q_0$ and $\sin \theta = \|q\|$. The quaternion can thus be rewritten in terms of $\theta$

and the unit vector $u = q_0/\|q\|$.

$$q = \|q\| e^{i \theta},$$

where $\theta = \arccos (q_0/\|q\|)$. Euler's identity for a complex number is

$$a + bi = \sqrt{a^2 + b^2} e^{i \phi},$$

where $i = -1$ and $\phi = a \tan \theta(b,a)$, generalizes to the quaternion $q$ can be rewritten as

$$q = \|q\| e^{i \phi}.$$

This allows us to define the power of $q$ as

$$q^p = \|q\|^p (e^{i \phi})^p = \|q\|^p (\cos(p \theta) + i \sin(p \theta)),$$  

where $p \in \mathbb{R}$.

Intuitively, the power is taken over the norm of the quaternion while a scaling is performed on its "phase angle".

An exponential of $q$ makes use of the Taylor expansion that treats $q$ just as an ordinary variable:

$$e^q = \sum_{i=0}^{\infty} \frac{q^i}{i!}.$$

The sum on the right hand side has a closed form that transforms the above into

$$e^q = \exp (q_0 + u \|q\|) = e^{q_0} (\cos \|q\| + u \sin \|q\|).$$

The logarithm of $q$ is accordingly defined as

$$\ln q = \ln \|q\| + u \arccos \left( \frac{q_0}{\|q\|} \right).$$

The two operations are inverses of each other as we verify that

$$e^{\ln q} = e^{\ln \|q\| + u \arccos (q_0/\|q\|)} = \|q\| e^{u \arccos (q_0/\|q\|)} = \|q\| (\frac{q_0}{\|q\|} + u \frac{\|q\|}{\|q\|}) = q.$$
and for any vector \( \mathbf{v} \in \mathbb{R}^3 \) the action of the operator
\[
L_q(\mathbf{v}) = q\mathbf{v}q^*
\]
on \( \mathbf{v} \) may be interpreted geometrically as a rotation of the vector \( \mathbf{v} \) through an angle \( 2\theta \) about \( \mathbf{q} \) as the axis of rotation [10].

Figure 2. Quaternion operations on vectors

**Rotation operator geometry**

The quaternion rotation operator takes \( \mathbf{v} \rightarrow w \). That is,
\[
w = q\mathbf{v}q^* = \left(q_0 - |\mathbf{q}|^2\right)\mathbf{v} + 2(\mathbf{q} \cdot \mathbf{v})\mathbf{q} + 2q_0(\mathbf{q} \times \mathbf{v}).
\]
If we write the vector \( \mathbf{v} \) in the form
\[
\mathbf{v} = \mathbf{a} + \mathbf{n},
\]
where \( \mathbf{a} \) is the components of \( \mathbf{v} \) along the vector part of \( \mathbf{q} \), and \( \mathbf{n} \) is the components of \( \mathbf{v} \) normal to the vector part of \( \mathbf{q} \). Then, it follows that
\[
\mathbf{a} + \mathbf{m} = q(\mathbf{a} + \mathbf{n})q^* = \left(q_0^2 - |\mathbf{q}|^2\right)(\mathbf{a} + \mathbf{n}) + 2(\mathbf{q} \cdot \mathbf{a})\mathbf{q} + 2q_0(\mathbf{q} \times \mathbf{n}) = w.
\]

Figure 3. Rotation operator geometry

**Two-rotation tracking sequence using matrices**

Two rotations can be tracked sequentially using matrices. The rotation about the Z-axis, followed by a rotation about the new Y-axis:

\[
\mathbf{R} = R_y' R_x',
\]

where
\[
\mathbf{R} = \begin{pmatrix}
\cos \beta & -\sin \beta & 0 \\
\sin \beta & \cos \beta & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

References: