On total regularity of the join of two interval valued fuzzy graphs

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Abstract- In this paper, we define the total degree (TD) of a vertex in the join of two interval valued fuzzy graphs (IVFGs) and investigate their totally regular property (TRP). In general, the join of two totally regular IVFGs need not be a totally regular interval valued fuzzy graph (TRIVFG). We obtain some necessary and sufficient conditions for the join of two TRIVFGs to be totally regular.

Index Terms- Interval valued fuzzy graph, total regularity, join

I. INTRODUCTION

Fuzzy graph theory is finding an increasing number of applications in modeling real time systems where the level of information inherent in the system varies with different levels of precision. In the applied field, the success of the use of fuzzy set theory depends on the choice of the membership function that we make. However, there are applications in which experts do not have precise knowledge of the function that should be taken. In these cases, it is appropriate to represent the membership degree of each element of the fuzzy set by means of an interval. From these considerations arises the extension of fuzzy sets called the theory of interval valued fuzzy sets. That is, fuzzy sets such that the membership degree of each element of the fuzzy set is given by a closed subinterval of the interval [0,1]. Replacing the membership functions of vertices and edges in fuzzy graphs by interval valued fuzzy sets such that they satisfy some particular conditions, *interval valued fuzzy graphs (IVFGs)* were defined. Thus IVFGs provide a better description of vagueness and uncertainty within the specific interval than the traditional fuzzy graph. The basic concepts of fuzzy sets, interval valued fuzzy sets and fuzzy graph theory can be found in [8], [19] and [18] to [23].

In 2009, Hongmei and Lianhua[6] introduced the definition of IVFG. Muhammad Akram and Wieslaw A. Dudek[1], defined the operations of cartesian product, composition, union and join on IVFGs and investigated some properties. They also introduced the notion of interval valued fuzzy complete graphs and obtained some properties of self complementary and self weak complementary interval valued fuzzy complete graphs. Muhammad Akram [2] also introduced interval valued fuzzy line graphs. A. A. Talebi and H. Rashmanlou [20] studied on isomorphism of IVFGs. H. Rashmanlou and Young Bae Jun[13] defined the three new operations, direct product, semi strong product and strong product of IVFGs and discussed their properties on complete IVFGs. Pradip Debnath[11] introduced domination in IVFGs. H. Rashmanlou and Madhumangal Pal defined irregular IVFG[9], balanced IVFG[14] and antipodal IVFG[15] and studied their properties. They also studied on the properties of highly irregular IVFG[17] and defined isometry on IVFG[16]. Muhammad Akram, Noura Omair Alshehri and W.A Dudek [3] introduced certain types of IVFG such as balanced IVFGs, neighbourly irregular IVFGs, neighbourly total irregular IVFGs, highly irregular IVFGs, highly total irregular IVFGs. Again Muhammad Akram, M. Murtaza Yousaf and W A Dudek [4] studied the properties of self centered IVFGs. Madhumangal Pal, Sovan Samanta and H. Rashmanlou [10] defined the degree and total degree of an edge in the cartesian product and composition of two IVFG and obtained some results. Basheer Ahammed Mohideen [5] studied on strong and regular IVFGs. S. Ravi Narayanan and N. R. Santhi Maheswari [18] introduced strongly edge irregular and strongly edge totally irregular IVFG and made a comparative study between the two. A. A. Talebi, H. Rashmanlou and Reza Ameri [21] discussed product IVFGs. K. Radha and M. Vijaya [12] discussed the totally regular property of the join of two fuzzy graphs.

In this paper, we introduce and analyse the notion of TRIVFGs. Also we obtain some necessary and sufficient conditions for total regularity of IVFGs.

II. BASIC CONCEPTS

Graph theoretic terms and results used in this work are either standard or are explained as and when they first appear.

2.1. Definition (Fuzzy graph) [8]. Let V be a non empty set. A ***fuzzy graph*** is a pair of functions $G:(\sigma,\mu)$ where σ is a fuzzy subset of V and μ is a symmetric fuzzy relation on σ. That is, $\sigma: V \to [0,1]$ and $\mu: V \times V \to [0,1]$ such that $\mu(u,v) \le \sigma(u) \wedge \sigma(v)$ for all u, v in V.

2.2. Definition (Interval number) [1]. An ***interval number*** D is an interval $[a^-, a^+]$ with $0 \leq a^- \leq a^+ \leq 1$.

2.3. Remark. (i) The interval number $[a, a]$ is identified with the number $a \in [0,1]$.
(ii) $D[0,1]$ denotes the set of all interval numbers.

2.4. Definition (Some operations on D[0,1]) [1]. For interval numbers $D_1 = [a_1^-, b_1^+]$ and $D_2 = [a_2^-, b_2^+]$

- $rmin(D_1, D_2) = [min\{a_1^-, a_2^-\}, min\{b_1^+, b_2^+\}]$
- $rmax(D_1, D_2) = [max\{a_1^-, a_2^-\}, max\{b_1^+, b_2^+\}]$
- $D_1 + D_2 = [a_1^- + a_2^- - a_1^- . a_2^- , b_1^+ + b_2^+ - b_1^+ . b_2^+]$
- $D_1 \leq D_2 \Leftrightarrow a_1^- \leq a_2^- \; and \; b_1^+ \leq b_2^+$
- $D_1 = D_2 \Leftrightarrow a_1^- = a_2^- \; and \; b_1^+ = b_2^+$
- $D_1 < D_2 \Leftrightarrow D_1 \leq D_2 \; and \; D_1 \neq D_2$
- $kD = k[a_1^-, b_1^+] = [ka_1^-, kb_1^+] \; where \; 0 \leq k \leq 1.$

Then $(D[0,1], \leq, \vee, \wedge)$ is a complete lattice with $[0,0]$ as the least element and $[1,1]$ as the greatest.

2.5. Definition (Interval valued fuzzy set (IVFS)) [1]. The ***interval valued fuzzy set*** A in V is defined by $A = \{(x, [\mu_A^-(x), \mu_A^+(x)]): x \in V\}$ where $\mu_A^-(x)$ and $\mu_A^+(x)$ are fuzzy subsets of V such that $\mu_A^-(x) \leq \mu_A^+(x)$ for all $x \in V$. We shall sometimes denote the IVFS A by $[\mu_A^-(x), \mu_A^+(x)]$.

For any two IVFSs $A = [\mu_A^-(x), \mu_A^+(x)]$ and $B = [\mu_B^-(x), \mu_B^+(x)]$ in V, we define

- $A \cup B = \left\{\left(x, max\left(\mu_A^-(x), \mu_B^-(x)\right), max\left(\mu_A^+(x), \mu_B^+(x)\right)\right): x \in V\right\}$
- $A \cap B = \left\{\left(x, min\left(\mu_A^-(x), \mu_B^-(x)\right), min\left(\mu_A^+(x), \mu_B^+(x)\right)\right): x \in V\right\}$

2.6. Definition (Interval valued fuzzy relation (IVFR)) [1]. If $G^* = (V, E)$ is a graph, then by an ***interval valued fuzzy relation*** B on the set E we mean an IVFS such that $\mu_B^-(xy) \leq min\left(\mu_A^-(x), \mu_A^-(y)\right) \; and \; \mu_B^+(xy) \leq min\left(\mu_A^+(x), \mu_A^+(y)\right) \; for \; all \; xy \in E$.

2.7. Definition (Interval valued fuzzy graph (IVFG)) [1]. By an ***interval valued fuzzy graph*** of a graph $G^* = (V, E)$, we mean a pair $G = (A, B)$, where $A = [\mu_A^-(x), \mu_A^+(x)]$ is an IVFS on V and $B = [\mu_B^-(x), \mu_B^+(x)]$ is an IVFR on E. In what follows, G = (A, B) denotes such an IVFG of a graph G^* = (V, E) where V is a non-empty finite set and E $\subseteq V \times V$.

2.8. Definition (Order of an IVFG) [9]. The ***order*** of G denoted by $O(G)$ is defined by $O(G) = [O^-(G), O^+(G)]$ where $O^-(G) = \sum_{u \in V} \mu_A^-(u)$ and $O^+(G) = \sum_{u \in V} \mu_A^+(u)$.

2.9. Definition (Degree of a vertex) [9]. The ***negative degree*** of a vertex $u \in V$ is defined by $d^-(u) = \sum_{uv \in E} \mu_B^-(uv)$. Similarly, ***positive degree*** of a vertex $u \in V$ is defined by $d^+(u) = \sum_{uv \in E} \mu_B^+(uv)$. Then the ***degree*** of the vertex $u \in V$ is defined as $d(u) = [d^-(u), d^+(u)]$.

2.10. Definition (Total degree (TD) of a vertex) [9]. The ***total degree*** of the vertex $u \in V$ is defined as $td(u) = [td^-(u), td^+(u)]$ where,

$$td^-(u) = \sum_{uv \in E} \mu_B^-(uv) + \mu_A^-(u) = d^-(u) + \mu_A^-(u) \text{ and}$$

$$td^+(u) = \sum_{uv \in E} \mu_B^+(uv) + \mu_A^+(u) = d^+(u) + \mu_A^+(u).$$

2.11. Definition (Totally regular IVFG (TRIVFG)[9]. If each vertex of G has the same total degree $[k_1, k_2]$ then the graph G is called a ***TRIVFG of degree*** $[\boldsymbol{k_1}, \boldsymbol{k_2}]$ or $[\boldsymbol{k_1}, \boldsymbol{k_2}]$-***TRIVFG.***

2.12. Definition (Join of two IVFGs) [1]. The ***join*** $G_1 + G_2 = (A_1 + A_2, B_1 + B_2)$ of two IVFG $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ is defined as follows:

i. $$\begin{cases} (\mu_{A_1}^- + \mu_{A_2}^-)(x) = \mu_{A_1}^-(x) \text{ if } x \in V_1 \\ \qquad\qquad\qquad\; = \mu_{A_2}^-(x) \text{ if } x \in V_2 \\ (\mu_{A_1}^+ + \mu_{A_2}^+)(x) = \mu_{A_1}^+(x) \text{ if } x \in V_1 \\ \qquad\qquad\qquad\; = \mu_{A_2}^+(x) \text{ if } x \in V_2 \end{cases}$$

ii. $$\begin{cases} (\mu_{B_1}^- + \mu_{B_2}^-)(xy) = \mu_{B_1}^-(xy) \text{ if } xy \in E_1 \\ \qquad\qquad\qquad\;\; = \mu_{B_2}^-(xy) \text{ if } xy \in E_2 \\ (\mu_{B_1}^+ + \mu_{B_2}^+)(xy) = \mu_{B_1}^+(xy) \text{ if } xy \in E_1 \\ \qquad\qquad\qquad\;\; = \mu_{B_2}^+(xy) \text{ if } xy \in E_2 \end{cases}$$

iii. $$\begin{cases} (\mu_{B_1}^- + \mu_{B_2}^-)(xy) = \min\left(\mu_{A_1}^-(x), \mu_{A_2}^-(y)\right) \\ (\mu_{B_1}^+ + \mu_{B_2}^+)(xy) = \min\left(\mu_{A_1}^+(x), \mu_{A_2}^+(y)\right) \end{cases} \text{ if } xy \in E'$$

where E' is the set of all edges joining the nodes of V_1 and V_2.
It is assumed that both V_1 and V_2 are finite and $V_1 \cap V_2 = \emptyset$.

III. TOTALLY REGULAR PROPERTY OF THE JOIN

3.1. Remark. Since $V_1 \cap V_2 = \emptyset$, we have $E_1 \cap E_2 = \emptyset$. Hence by definition 2.10,

$$td_{G_1+G_2}^-(u) = \sum_{uv \in E_1 \cup E_2} (\mu_{B_1}^- + \mu_{B_2}^-)(uv) + \sum_{uv \in E'} \min\left(\mu_{A_1}^-(u), \mu_{A_2}^-(v)\right) + (\mu_{A_1}^- + \mu_{A_2}^-)(u)$$
$$td_{G_1+G_2}^+(u) = \sum_{uv \in E_1 \cup E_2} (\mu_{B_1}^+ + \mu_{B_2}^+)(uv) + \sum_{uv \in E'} \min\left(\mu_{A_1}^+(u), \mu_{A_2}^+(v)\right) + (\mu_{A_1}^+ + \mu_{A_2}^+)(u)$$

For any $u \in V_1$, $td_{G_1+G_2}^-(u) = \sum_{uv \in E_1} (\mu_{B_1}^-)(uv) + \sum_{uv \in E'} \min\left(\mu_{A_1}^-(u), \mu_{A_2}^-(v)\right) + (\mu_{A_1}^-)(u)$

$$= d_{G_1}^-(u) + (\mu_{A_1}^-)(u) + \sum_{uv \in E'} \min\left(\mu_{A_1}^-(u), \mu_{A_2}^-(v)\right)$$
$$= td_{G_1}^-(u) + \sum_{uv \in E'} \min\left(\mu_{A_1}^-(u), \mu_{A_2}^-(v)\right)$$

Similarly, $td_{G_1+G_2}^+(u) = td_{G_1}^+(u) + \sum_{uv \in E'} \min\left(\mu_{A_1}^+(u), \mu_{A_2}^+(v)\right)$

Also, if $u \in V_2$, $td_{G_1+G_2}^-(u) = td_{G_2}^-(u) + \sum_{uv \in E'} \min\left(\mu_{A_1}^-(u), \mu_{A_2}^-(v)\right)$

$$td_{G_1+G_2}^+(u) = td_{G_2}^+(u) + \sum_{uv \in E'} \min\left(\mu_{A_1}^+(u), \mu_{A_2}^+(v)\right)$$

In what follows, the number of vertices in G_1 and G_2 are denoted as p_1 and p_2 respectively. The proof of the following lemma is straight forward and so is omitted.

3.2. Lemma. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs.

1) If $A_1 \geq A_2, then\ td_{G_1+G_2}(u) = td_{G_1}(u) + O(G_2)\ if\ u \in V_1$
$$= td_{G_2}(u) + p_1[\mu_{A_2}^-(u), \mu_{A_2}^+(u)] \text{ if } u \in V_2$$

2) If $A_2 \geq A_1$, then $td_{G_1+G_2}(u) = td_{G_1}(u) + p_2[\mu^-_{A_1}(u), \mu^+_{A_1}(u)]$ if $u \in V_1$
$= td_{G_2}(u) + O(G_1)$ if $u \in V_2$ ■

3.3. Remark. If G_1 and G_2 are TRIVFGs, then $G_1 + G_2$ need not be a TRIVFG which is clear from the following examples.

3.4. Example. The following diagram presents two TRIVFGs G_1 and G_2 and their join $G_1 + G_2$ which is not a TRIVFG.

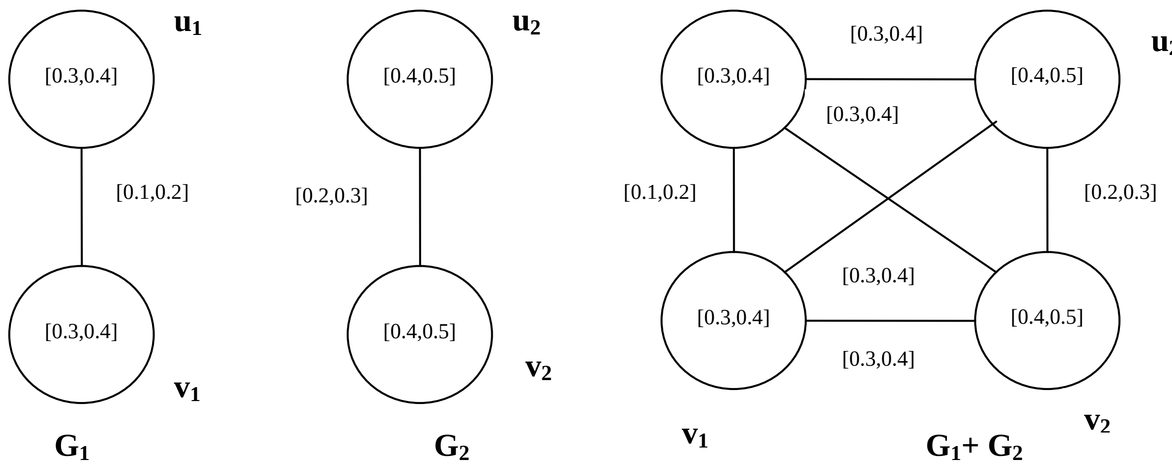

*Fig 3.1: An example to show that TRP is not carried over to joins***.**

3.5. Example. We give below another example to show that join of two TRIVFGs need not be a TRIVFG.

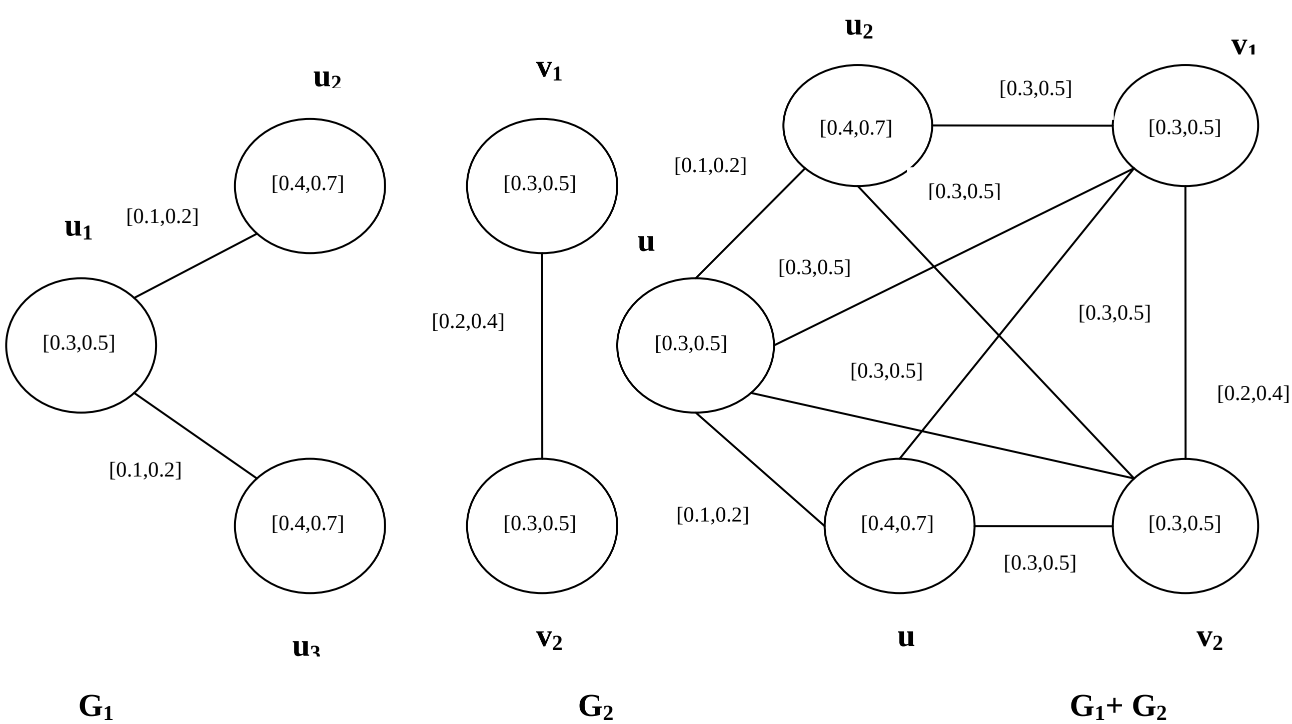

*Fig 3.2: Another example to show that TRP is not carried over to joins***.**

3.6. Example. In the following example, G_1 and G_2 are totally regular and $G_1 + G_2$ is also totally regular.

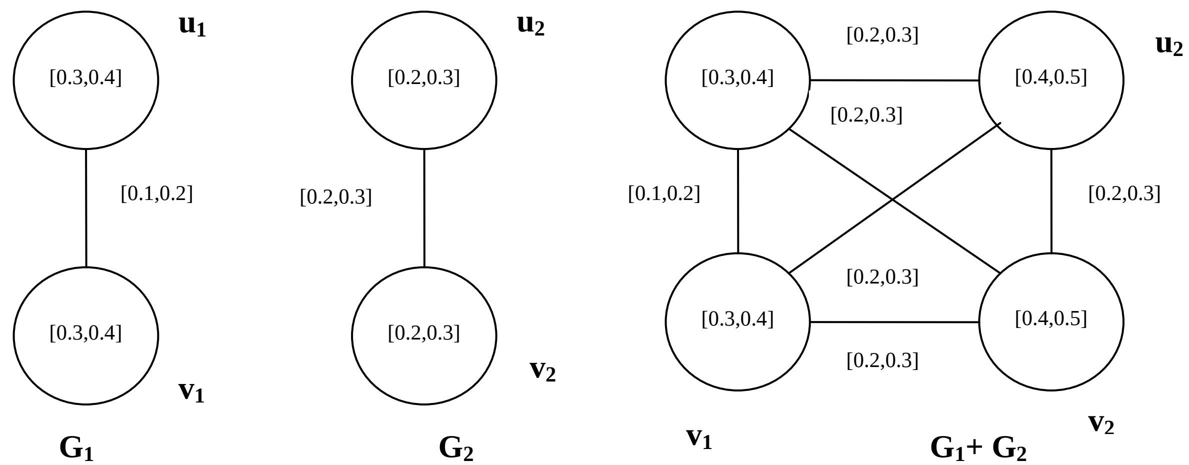

Fig 3.3: An example to show that TRP is sometimes carried over to joins.

Now, we proceed to obtain some necessary and sufficient conditions for the join of two IVFGs to be totally regular.

Theorem 3.7. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two TRIVFGs of the same degree such that $\mathrm{rmin}(A_1, A_2)$ is a constant function. Then the join $G_1 + G_2$ is a TRIVFG if and only if $p_1 = p_2$.

Proof: Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two $[k_1, k_2]$-TRIVFGs. Also let $\mathrm{rmin}(A_1, A_2) = [c_1, c_2]$ for all $u \in V_1$ and $v \in V_2$ where c_1, c_2 are constants. Then
$\mathrm{rmin}(A_1, A_2) = [\min(\mu_{A_1}^-, \mu_{A_2}^-), \min(\mu_{A_1}^+, \mu_{A_2}^+)] = [c_1, c_2]$ so that $\min(\mu_{A_1}^-, \mu_{A_2}^-) = c_1$ and $\min(\mu_{A_1}^+, \mu_{A_2}^+) = c_2$

For any $u \in V_1$, $td_{G_1+G_2}^-(u) = td_{G_1}^-(u) + \sum_{uv \in E'} \min\left(\mu_{A_1}^-(u), \mu_{A_2}^-(v)\right)$

$$= k_1 + \sum_{uv \in E'} c_1$$
$$= k_1 + c_1 p_2.$$

Similarly, $td_{G_1+G_2}^+(u) = k_2 + c_2 p_2$

Hence, $td_{G_1+G_2}(u) = [k_1 + c_1 p_2, k_2 + c_2 p_2]$

For any $v \in V_2$, $td_{G_1+G_2}(v) = [k_1 + c_1 p_1, k_2 + c_2 p_1]$

Thus $G_1 + G_2$ is a TRIVFG

$$\Leftrightarrow k_1 + c_1 p_2 = k_1 + c_1 p_1 \text{ and } k_2 + c_2 p_2 = k_2 + c_2 p_1$$
$$\Leftrightarrow c_1 p_2 = c_1 p_1 \text{ and } c_2 p_2 = c_2 p_1$$
$$\Leftrightarrow p_1 = p_2 \quad \blacksquare$$

Remark 3.8. Theorem 3.7 says that the join of two TRIVFGs of the same total degree is a TRIVFG if and only if the number of vertices of the two graphs are same. In Fig 3.2, both $G_1 and\ G_2$ are TRIVFGs of the same total degree, but $G_1 + G_2$ is not a TRIVFG. This is because the number of vertices of $G_1 and\ G_2$ are distinct. In Fig 3.3, both $G_1 and\ G_2$ are TRIVFGs with the same total degree and same number of vertices and we can see that $G_1 + G_2$ is a TRIVFG.

Corollary 3.9. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two $[k_1, k_2]$-TRIVFGs such that $A_1 \geq A_2$ $(or\ A_2 \geq A_1)$ and A_2 $(or\ A_1)$ is a constant function. Then the join $G_1 + G_2$ is a TRIVFG if and only if $p_1 = p_2$.

***Proof*:** Since $A_1 \geq A_2$, $\mu_{A_1}^{-} \geq \mu_{A_2}^{-}$ and $\mu_{A_1}^{+} \geq \mu_{A_2}^{+}$. Again since A_2 is a constant function, $\mu_{A_2}^{-}$ and $\mu_{A_2}^{+}$ are constant functions. Hence, $\mathrm{rmin}(A_1, A_2) = [\mu_{A_2}^{-}, \mu_{A_2}^{+}]$ is a constant. Similar is the case when $A_2 \geq A_1$. Hence the result follows from theorem 3.7 ■

Theorem 3.10. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs such that $p_1 = p_2$ and $\mathrm{rmin}(A_1, A_2)$ is a constant function. Then the join $G_1 + G_2$ is a TRIVFG if and only if both G_1 and G_2 are both TRIVFGs of the same degree.

Proof: Let $p_1 = p_2 = p$ and let $\mathrm{rmin}(A_1, A_2) = [c_1, c_2]$ for all $u \in V_1$ and $v \in V_2$ where c_1, c_2 are constants.

For any $u \in V_1$, $td_{G_1+G_2}^{-}(u) = td_{G_1}^{-}(u) + \sum_{uv \in E'} \min\left(\mu_{A_1}^{-}(u), \mu_{A_2}^{-}(v)\right) = td_{G_1}^{-}(u) + c_1 p$

and $td_{G_1+G_2}^{+}(u) = td_{G_1}^{+}(u) + \sum_{uv \in E'} \min\left(\mu_{A_1}^{+}(u), \mu_{A_2}^{+}(v)\right) = td_{G_1}^{+}(u) + c_2 p$.

Hence, $td_{G_1+G_2}(u) = \left[td_{G_1}^{-}(u) + c_1 p, td_{G_1}^{+}(u) + c_2 p\right]$. Similarly, for any $v \in V_2$, $td_{G_1+G_2}(v) = \left[td_{G_2}^{-}(v) + c_1 p, td_{G_2}^{+}(v) + c_2 p\right]$

Thus $G_1 + G_2$ is a TRIVFG

$\Leftrightarrow td_{G_1}^{-}(u) + c_1 p = td_{G_2}^{-}(v) + c_1 p$ and $td_{G_1}^{+}(u) + c_2 p = td_{G_2}^{+}(v) + c_2 p$

$\Leftrightarrow td_{G_1}^{-}(u) = td_{G_2}^{-}(v)$ and $td_{G_1}^{+}(u) = td_{G_2}^{+}(v)$

$\Leftrightarrow td_{G_1}(u) = td_{G_2}(v)$ where $u \in V_1$ and $v \in V_2$ are arbitrary. Hence the theorem ■

Remark 3.11. Theorems 3.7 and 3.10 fail to hold if $\mathrm{rmin}(A_1, A_2)$ is not a constant function which is clear from the following example.

Example 3.12. Here, both G_1 and G_2 are TRIVFGs with the same total degree and same number of vertices. But $\mathrm{rmin}(A_1, A_2)$ is not a constant function. We can observe from the figure that $G_1 + G_2$ is not a TRIVFG

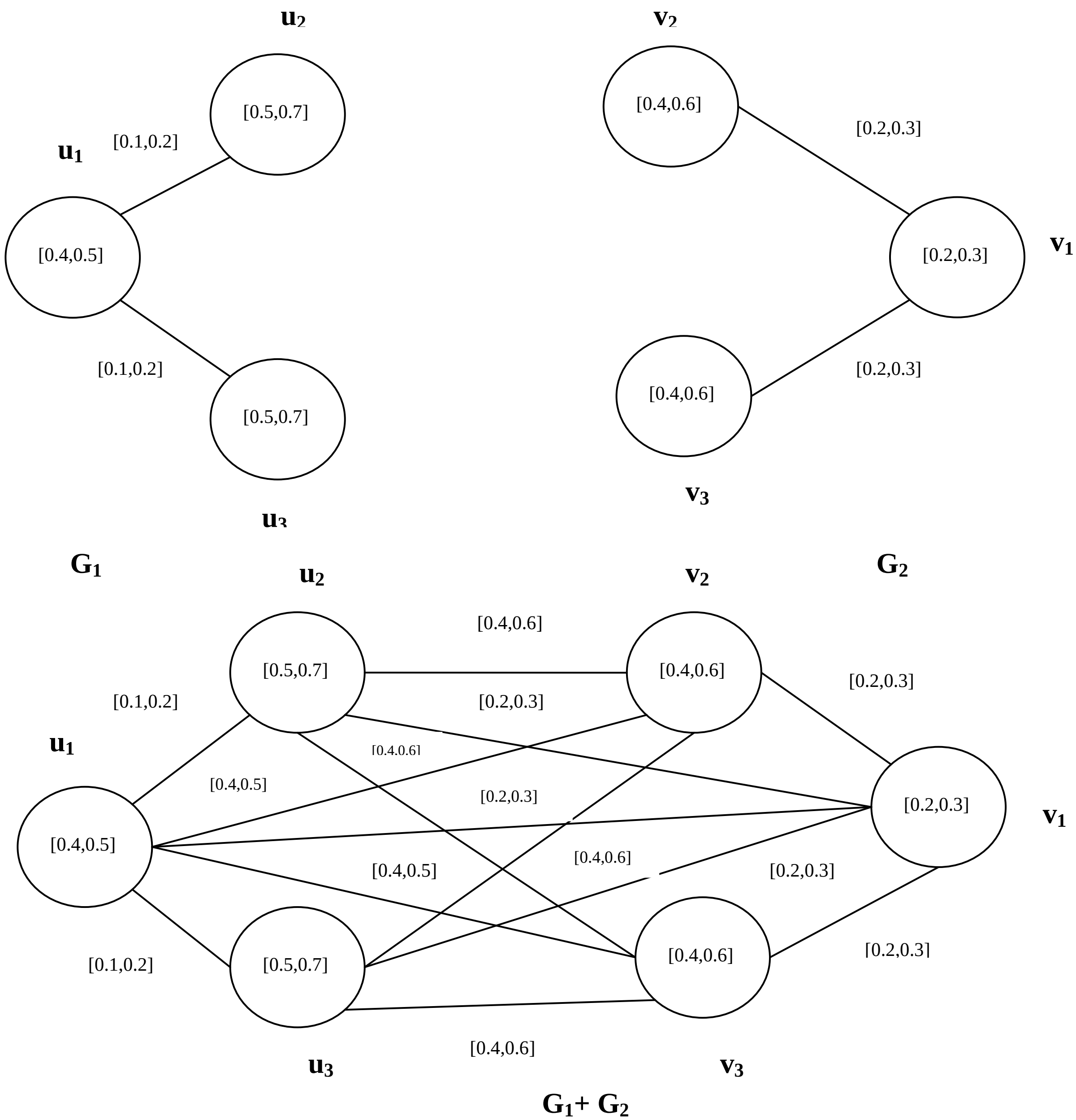

Fig 3.4 : An example to show that the condition rmin = constant is essential for TRP of G_1 + G_2

Remark 3.13. Theorem 3.10 says that the join of two TRIVFGs with the same number of vertices is a TRIVFG if and only if they have the same total degree. In Fig 3.1, both $G_1 and\ G_2$ are TRIVFGs and have the same number of vertices, but $G_1 + G_2$ is not a TRIVFG. This is because they have distinct total degrees.

Corollary 3.14. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two IVFGs such that $p_1 = p_2, A_1 \geq A_2\ (or\ A_2 \geq A_1)$ and $A_2\ (or\ A_1)$ is a constant function. Then the join $G_1 + G_2$ is a TRIVFG if and only if both $G_1 and\ G_2$ are TRIVFG of the same degree.

***Proof*:** Proof is similar to Corollary 3.9 ■

Theorem 3.15. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two TRIVFGs such that $A_1 \geq A_2\ \ (or\ A_2 \geq A_1)$. If $G_1 + G_2$ is a TRIVFG, then $A_2\ (or\ A_1)$ is a constant function.

Proof. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two TRIVFGs of total degrees $[k_1, k_2]$ and $[k_1', k_2']$ respectively. Also let $A_1 \geq A_2$.

For any $u \in V_1$, $td^-_{G_1+G_2}(u) = td^-_{G_1}(u) + \sum_{uv \in E'} \min\left(\mu^-_{A_1}(u), \mu^-_{A_2}(v)\right)$

$$= td^-_{G_1}(u) + \sum_{v \in V_2} \mu^-_{A_2}(v)$$
$$= k_1 + \sum_{v \in V_2} \mu^-_{A_2}(v)$$
$$= k_1 + O^-(G_2)$$

Similarly, $td^+_{G_1+G_2}(u) = k_2 + O^+(G_2)$

Hence, $td_{G_1+G_2}(u) = [k_1 + O^-(G_2), k_2 + O^+(G_2)]$

For any $v \in V_2$, $td^-_{G_1+G_2}(v) = td^-_{G_2}(v) + \sum_{uv \in E'} \min\left(\mu^-_{A_1}(u), \mu^-_{A_2}(v)\right)$

$$= td^-_{G_2}(v) + \sum_{u \in V_1} \mu^-_{A_2}(v)$$
$$= k_1' + p_1 \mu^-_{A_2}(v)$$

Similarly, $td^+_{G_1+G_2}(v) = k_2' + p_1 \mu^+_{A_2}(v)$

Hence, $td_{G_1+G_2}(v) = \left[k_1' + p_1 \mu^-_{A_2}(v), k_2' + p_1 \mu^+_{A_2}(v)\right]$

Thus $G_1 + G_2$ is a TRIVFG

$$\Rightarrow td_{G_1+G_2}(u) = td_{G_1+G_2}(v)$$
$$\Rightarrow [k_1 + O^-(G_2), k_2 + O^+(G_2)] = \left[k_1' + p_1 \mu^-_{A_2}(v), k_2' + p_1 \mu^+_{A_2}(v)\right]$$
$$\Rightarrow k_1 + O^-(G_2) = k_1' + p_1 \mu^-_{A_2}(v) \text{ and } k_2 + O^+(G_2) = k_2' + p_1 \mu^+_{A_2}(v)$$
$$\Rightarrow k_1 - k_1' = p_1 \mu^-_{A_2}(v) - O^-(G_2) \text{ and } k_2 - k_2' = p_1 \mu^+_{A_2}(v) - O^+(G_2)$$

For any $v, w \in V_2$, $p_1 \mu^-_{A_2}(v) - O^-(G_2) = k_1 - k_1' = p_1 \mu^-_{A_2}(w) - O^-(G_2)$ and

$$p_1 \mu^+_{A_2}(v) - O^+(G_2) = k_2 - k_2' = p_1 \mu^+_{A_2}(w) - O^+(G_2)$$
$$\Rightarrow p_1 \mu^-_{A_2}(v) = p_1 \mu^-_{A_2}(w) \text{ and } p_1 \mu^+_{A_2}(v) = p_1 \mu^+_{A_2}(w)$$
$$\Rightarrow \mu^-_{A_2}(v) = \mu^-_{A_2}(w) \text{ and } \mu^+_{A_2}(v) = \mu^+_{A_2}(w)$$

$\Rightarrow [\mu^-_{A_2}, \mu^+_{A_2}]$ is a constant function.

$\Rightarrow A_2$ is a constant function ■

The following example shows that the converse of the above theorem is not true.

Example 3.16. Here G_1 *and* G_2 are TRIVFGs such that $A_1 \geq A_2$ and A_2 is a constant function. But, $G_1 + G_2$ is not a TRIVFG.

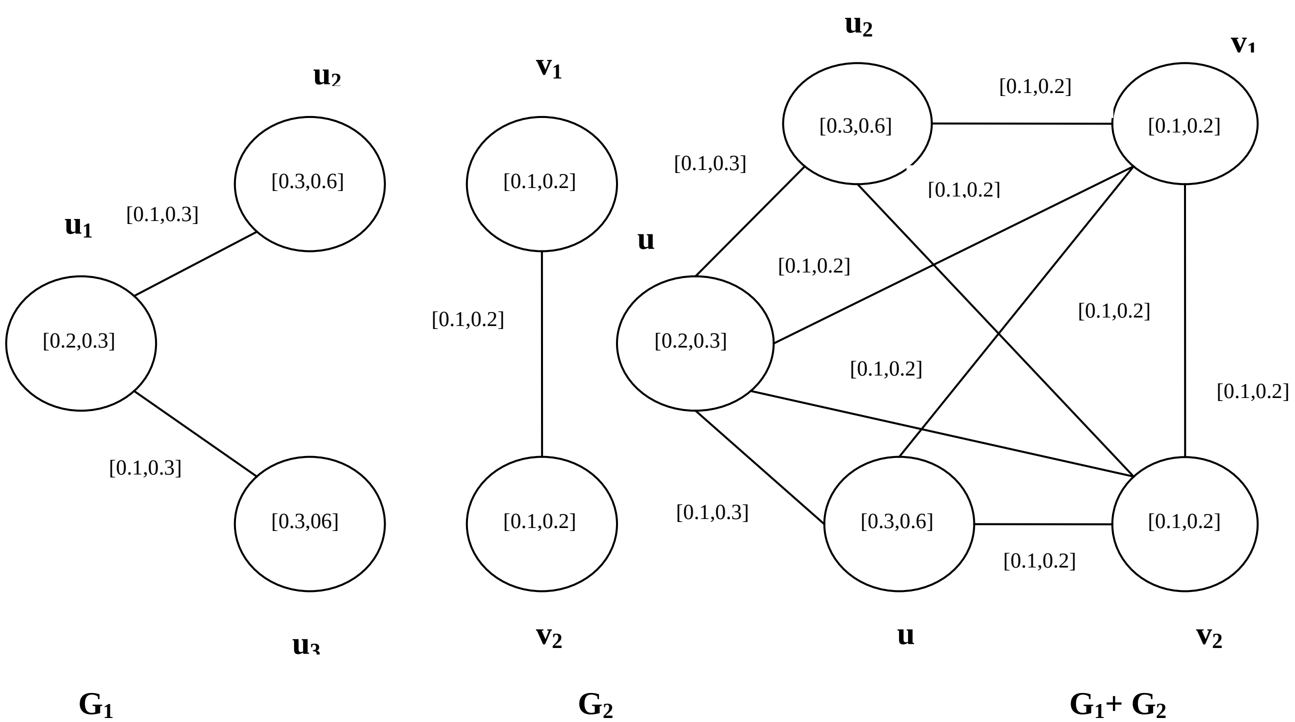

Fig 3.5 : Example to show that converse of theorem 3.15 is not true

In the next theorem, we give a necessary and sufficient condition for the join of two TRIVFGs with distinct total degrees and distinct number of vertices to be a TRIVFG.

Theorem 3.17. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two TRIVFGs of total degrees $[k_1, k_2]$ and $[k_1', k_2']$ respectively and let $\text{rmin}(A_1, A_2)$ be a constant function. Then the join $G_1 + G_2$ is a TRIVFG if and only if $[k_1 - k_1', k_2 - k_2'] = [c_1, c_2](p_1 - p_2)$ where $[c_1, c_2]$ is the constant value of $\text{rmin}(A_1, A_2)$.

Proof: For any $u \in V_1$, $td^-_{G_1+G_2}(u) = td^-_{G_1}(u) + \sum_{uv\in E'} \min\left(\mu^-_{A_1}(u), \mu^-_{A_2}(v)\right) = k_1 + c_1p_2$ and $td^+_{G_1+G_2}(u) = td^+_{G_1}(u) + \sum_{uv\in E'} \min\left(\mu^+_{A_1}(u), \mu^+_{A_2}(v)\right) = k_2 + c_2p_2$. Hence, $td_{G_1+G_2}(u) = [k_1 + c_1p_2, k_2 + c_2p_2]$. Similarly, for any $v \in V_2$, $td_{G_1+G_2}(v) = [k_1' + c_1p_1, k_2' + c_2p_1]$.

Thus $G_1 + G_2$ is a TRIVFG

$$\Leftrightarrow k_1 + c_1p_2 = k_1' + c_1p_1 \text{ and } k_2 + c_2p_2 = k_2' + c_2p_1$$
$$\Leftrightarrow k_1 - k_1' = c_1p_1 - c_1p_2 \text{ and } k_2 - k_2' = c_2p_1 - c_2p_2$$
$$\Leftrightarrow k_1 - k_1' = c_1(p_1 - p_2) \text{ and } k_2 - k_2' = c_2(p_1 - p_2)$$
$$\Leftrightarrow [k_1 - k_1', k_2 - k_2'] = [c_1, c_2](p_1 - p_2) \quad \blacksquare$$

Corollary 3.18. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two TRIVFGs such that $A_1 \geq A_2$ $(or\ A_2 \geq A_1)$ and A_2 $(or\ A_1)$ is a constant function. Then the join $G_1 + G_2$ is a TRIVFG if and only if $[k_1 - k_1', k_2 - k_2'] = [c_1, c_2](p_1 - p_2)$ where $[c_1, c_2]$ is the constant value of $\text{rmin}(A_1, A_2)$ $\blacksquare$

Example 3.19

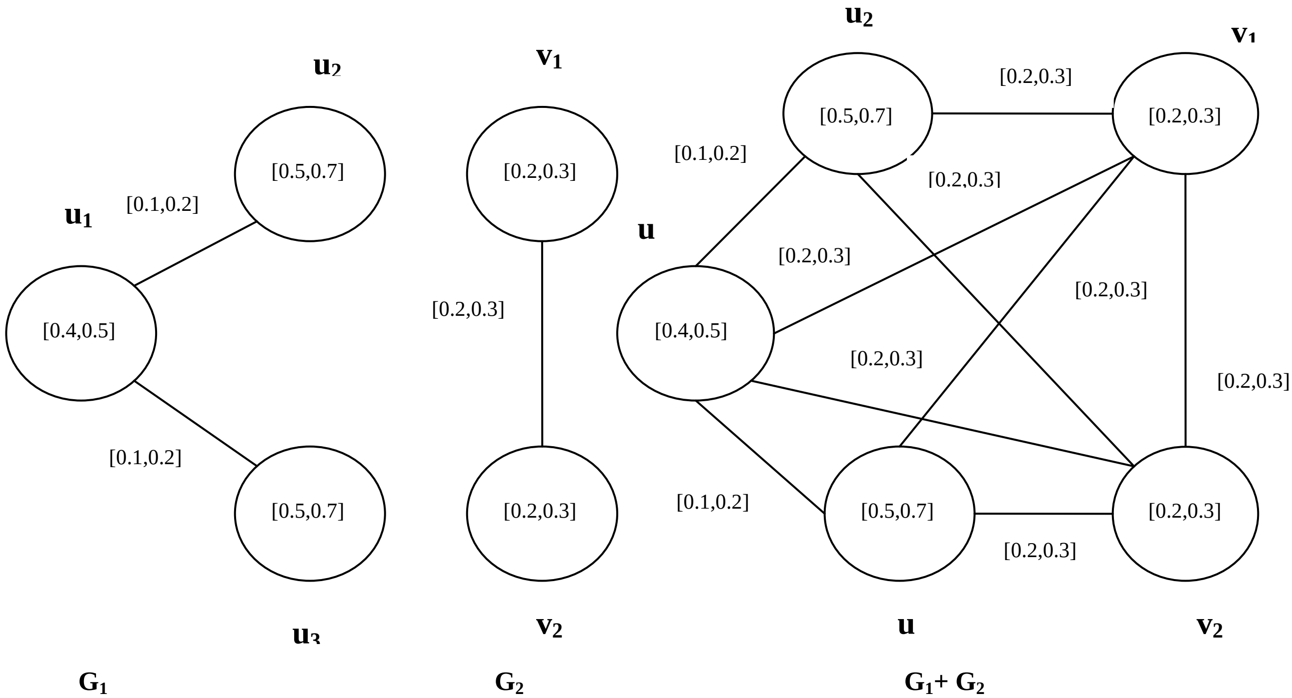

Fig 3.6 : An example to verify theorem 3.17

Here $td_{G_1} = [k_1, k_2] = [0.6, 0.9]$ and $td_{G_2} = [k_1', k_2'] = [0.4, 0.6]$. $\text{rmin}(A_1, A_2) = [c_1, c_2] = [0.2, 0.3], p_1 = 3, p_2 = 2.$
$\therefore [k_1 - k_1', k_2 - k_2'] = [0.2, 0.3] = [0.2, 0.3](3-2) = [c_1, c_2](p_1 - p_2)$. Hence $G_1 and\ G_2$ satisfy the condition given in theorem 3.17 and so $G_1 + G_2$ is a TRIVFG which is also clear from the figure.

IV. CONCLUSION

In this paper, we have introduced the notion of TRIVFGs. We have observed that the join of two TRIVFGs need not be a TRIVFG. Also, we have derived some necessary and sufficient conditions for the join of two IVFGs to be a TRIVFG.

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