# An energy interpretation of exact solutions of Dirac equation in terms of standing waves modes

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***Abstract-*** Solutions of Dirac equation are well known in terms of bi-spinors, but the complete determination of these spinors is always derived using the assumption that the wave function is a plane wave. We investigate in this paper the search of general solutions under the form of standing wave modes, using methods issued from electromagnetism. It is shown that such solutions exist as solutions of a system of 64 equations and 64 unknowns, and that they must be associated with a relativistic relation of energy conservation.
Complete exact solutions are presented. It is shown that these solutions highlight in a new manner the duality wave-particle, and the uncertainty principle.

***Index Terms-*** Dirac equation – spinors – standing wave modes – quantum mechanics.

## I. Introduction

Dirac equation is the equation which describes in the most accurate manner the behavior of particles in the infinite small world. In particular, it is able to take into account the spin phenomena, and to predict the existence of antimatter. These two properties are deduced from general solutions of the free Dirac equation, without any supplementary assumption. It's not possible to progress towards more explicit solutions without adding some hypothesis, and the main one which is proposed is the assumption that the matter wave behaves like a plane wave. Free solutions of the Dirac equation are always presented under this condition as it can be seen in books at the state of art of the knowledge in this domain [1][2][3][4][5][6]. Papers linked to this subject [7][8][9] don't explore the way of standing wave modes, no more than publications looking for exact solutions in electromagnetic or gravitational potentials [10][11][12][13].
Let us consider a particle at rest: it's difficult to imagine how the matter wave works if it behaves like a plane wave inside the matter. A more pertinent description should be made using a wave working on standing wave modes. In such a description, the energy matter is supposed to be captive in a certain volume, in the same way as electromagnetic energy may be captive in a cavity. We know that in such a situation, the energy is exchanged inside the cavity between to kind of energy: electric energy and magnetic energy. Later in this paper, we will not wonder on the way in which the matter energy is confined in a region of space: the answer to this question is not known to us. But we will show that if we make the assumption that the internal energy is exchanged in stationary modes, this led to a quantum physics determinist and consistent with the actual probabilistic theory.
The following of this paper is composed with two sections. The first one explain how we can find exact solutions of Dirac equation in terms of standing wave modes and give explicit formulations of these exact solutions. The second one proposes energy interpretations of these new solutions and shows how they agree with the classical theory.

## II. The Dirac system for standing waves

We have to solve the Dirac system (1). No solutions of this system exists in terms of standing waves. We have then to formalize a method to found such a kind of solutions.

$$\begin{aligned}
\eta\psi_0 &= j\frac{\partial\psi_0}{\partial(ct)} + j\frac{\partial\psi_3}{\partial x} + \frac{\partial\psi_3}{\partial y} + j\frac{\partial\psi_2}{\partial z} \\
\eta\psi_1 &= j\frac{\partial\psi_1}{\partial(ct)} + j\frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_2}{\partial y} - j\frac{\partial\psi_3}{\partial z} \qquad \text{with} \quad \eta = \frac{mc}{\hbar} \\
\eta\psi_2 &= -j\frac{\partial\psi_2}{\partial(ct)} - j\frac{\partial\psi_1}{\partial x} - \frac{\partial\psi_1}{\partial y} - j\frac{\partial\psi_0}{\partial z} \\
\eta\psi_3 &= -j\frac{\partial\psi_3}{\partial(ct)} - j\frac{\partial\psi_0}{\partial x} + \frac{\partial\psi_0}{\partial y} + j\frac{\partial\psi_1}{\partial z}
\end{aligned} \tag{1}$$

We assume in a first time, that any solution of the Dirac system in Cartesian coordinates may be developed on a whole base of standing wave modes. This base is built with sixteen terms which represent all combinations of a product of four sine and cosine functions.

We adopt for each wave function $\psi_i$, for i = 0, 1, 2, 3 the following development:

$$\begin{aligned}\psi_i = &\{a_iS_xS_yS_z + b_iS_xC_yS_z + c_iC_xS_yS_z + d_iC_xC_yS_z + e_iS_xS_yC_z + f_iS_xC_yC_z + g_iC_xS_yC_z + h_iC_xC_yC_z\}C_t \\ &+ \{i_iS_xS_yS_z + j_iS_xC_yS_z + k_iC_xS_yS_z + l_iC_xC_yS_z + m_iS_xS_yC_z + n_iS_xC_yC_z + o_iC_xS_yC_z + p_iC_xC_yC_z\}S_t\end{aligned} \quad (2)$$

In this expression, the following abbreviated notation has been used:

$S_x = \sin(k_x\, x)$ $S_y = \sin(k_y\, y)$ $S_z = \sin(k_z\, z)$ $S_t = \sin(k_t\, x_t)$ (3)

$C_x = \cos(k_x\, x)$ $C_y = \cos(k_y\, y)$ $C_z = \cos(k_z\, z)$ $C_t = \cos(k_t\, x_t)$ (4)

The wave vector is represented by its $k_x$, $k_y$, $k_z$ components, while for a homogeneous notation and consistent with relativity, the product ωt has been replaced with the expression $k_t.x_t$, which allows to highlight in (3) and (4) the product of two four-vectors:

$$\text{4-vector position: } \begin{pmatrix} x \\ y \\ z \\ x_t = ct \end{pmatrix} \qquad \text{4-vector wave: } \begin{pmatrix} k_x \\ k_y \\ k_z \\ k_t = \frac{\omega}{c} \end{pmatrix} \quad (5)$$

Coefficients $a_i$, $b_i$, $c_i$, $d_i$, $e_i$, $f_i$, $g_i$, $h_i$, $i_i$, $j_i$, $k_i$, $l_i$, $m_i$, $n_i$, $o_i$ $p_i$ , for i = 0, 1, 2, 3, are real constants that weigh each of the modes and will serve as unknowns in the search for the wave functions $\psi_0$ , $\psi_1$ , $\psi_2$ , $\psi_3$ solutions of the DIRAC system.

This leads, for each equation of the DIRAC system, to express the partial derivatives of the wave functions $\psi_0$ , $\psi_1$ , $\psi_2$ , $\psi_3$ and to formulate a homogeneous system of 16 equations for the coefficients $a_i$, $b_i$, $c_i$, $d_i$, $e_i$, $f_i$, $g_i$, $h_i$, $i_i$, $j_i$, $k_i$, $l_i$, $m_i$, $n_i$, $o_i$ $p_i$. Calculations are a bit laborious but without difficulties. They are presented in their entirety in [15]
The obtained global system is therefore a homogeneous system of 64 equations with 64 unknowns. It is presented in (6).

$$\begin{array}{ll}
-jk_ta_0 - jk_xk_3 - k_yj_3 - jk_zm_2 - \eta i_0 = 0 & -jk_ta_1 - jk_xk_2 + k_yj_2 + jk_zm_3 - \eta i_1 = 0 \\
-jk_tb_0 - jk_xl_3 + k_yi_3 - jk_zn_2 - \eta j_0 = 0 & -jk_tb_1 - jk_xl_2 - k_yi_2 + jk_zn_3 - \eta j_1 = 0 \\
-jk_tc_0 + jk_xi_3 - k_yl_3 - jk_zo_2 - \eta k_0 = 0 & -jk_tc_1 + jk_xi_2 + k_yl_2 + jk_zo_3 - \eta k_1 = 0 \\
-jk_td_0 + jk_xj_3 + k_yk_3 - jk_zp_2 - \eta l_0 = 0 & -jk_td_1 + jk_xj_2 - k_yk_2 + jk_zp_3 - \eta l_1 = 0 \\
-jk_te_0 - jk_xo_3 - k_yn_3 + jk_zi_2 - \eta m_0 = 0 & -jk_te_1 - jk_xo_2 + k_yn_2 - jk_zi_3 - \eta m_1 = 0 \\
-jk_tf_0 - jk_xp_3 + k_ym_3 + jk_zj_2 - \eta n_0 = 0 & -jk_tf_1 - jk_xp_2 - k_ym_2 - jk_zj_3 - \eta n_1 = 0 \\
-jk_tg_0 + jk_xm_3 - k_yp_3 + jk_zk_2 - \eta o_0 = 0 & -jk_tg_1 + jk_xm_2 + k_yp_2 - jk_zk_3 - \eta o_1 = 0 \\
-jk_th_0 + jk_xn_3 + k_yo_3 + jk_zl_2 - \eta p_0 = 0 & -jk_th_1 + jk_xn_2 - k_yo_2 - jk_zl_3 - \eta p_1 = 0 \\
jk_ti_0 - jk_xc_3 - k_yb_3 - jk_ze_2 - \eta a_0 = 0 & jk_ti_1 - jk_xc_2 + k_yb_2 + jk_ze_3 - \eta a_1 = 0 \\
jk_tj_0 - jk_xd_3 + k_ya_3 - jk_zf_2 - \eta b_0 = 0 & jk_tj_1 - jk_xd_2 - k_ya_2 + jk_zf_3 - \eta b_1 = 0 \\
jk_tk_0 + jk_xa_3 - k_yd_3 - jk_zg_2 - \eta c_0 = 0 & jk_tk_1 + jk_xa_2 + k_yd_2 + jk_zg_3 - \eta c_1 = 0 \\
jk_tl_0 + jk_xb_3 + k_yc_3 - jk_zh_2 - \eta d_0 = 0 & jk_tl_1 + jk_xb_2 - k_yc_2 + jk_zh_3 - \eta d_1 = 0 \\
jk_tm_0 - jk_xg_3 - k_yf_3 + jk_za_2 - \eta e_0 = 0 & jk_tm_1 - jk_xg_2 + k_yf_2 - jk_za_3 - \eta e_1 = 0 \\
jk_tn_0 - jk_xh_3 + k_ye_3 + jk_zb_2 - \eta f_0 = 0 & jk_tn_1 - jk_xh_2 - k_ye_2 - jk_zb_3 - \eta f_1 = 0 \\
jk_to_0 + jk_xe_3 - k_yh_3 + jk_zc_2 - \eta g_0 = 0 & jk_to_1 + jk_xe_2 + k_yh_2 - jk_zc_3 - \eta g_1 = 0 \\
jk_tp_0 + jk_xf_3 + k_yg_3 + jk_zd_2 - \eta h_0 = 0 & jk_tp_1 + jk_xf_2 - k_yg_2 - jk_zd_3 - \eta h_1 = 0
\end{array} \quad (6)$$

$$jk_t a_2 + jk_x k_1 + k_y j_1 + jk_z m_0 - \eta i_2 = 0$$
$$jk_t b_2 + jk_x l_1 - k_y i_1 + jk_z n_0 - \eta j_2 = 0$$
$$jk_t c_2 - jk_x i_1 + k_y l_1 + jk_z o_0 - \eta k_2 = 0$$
$$jk_t d_2 - jk_x j_1 - k_y k_1 + jk_z p_0 - \eta l_2 = 0$$
$$jk_t e_2 + jk_x o_1 + k_y n_1 - jk_z i_0 - \eta m_2 = 0$$
$$jk_t f_2 + jk_x p_1 - k_y m_1 - jk_z j_0 - \eta n_2 = 0$$
$$jk_t g_2 - jk_x m_1 + k_y p_1 - jk_z k_0 - \eta o_2 = 0$$
$$jk_t h_2 - jk_x n_1 - k_y o_1 - jk_z l_0 - \eta p_2 = 0$$
$$-jk_t i_2 + jk_x c_1 + k_y b_1 + jk_z e_0 - \eta a_2 = 0$$
$$-jk_t j_2 + jk_x d_1 - k_y a_1 + jk_z f_0 - \eta b_2 = 0$$
$$-jk_t k_2 - jk_x a_1 + k_y d_1 + jk_z g_0 - \eta c_2 = 0$$
$$-jk_t l_2 - jk_x b_1 - k_y c_1 + jk_z h_0 - \eta d_2 = 0$$
$$-jk_t m_2 + jk_x g_1 + k_y f_1 - jk_z a_0 - \eta e_2 = 0$$
$$-jk_t n_2 + jk_x h_1 - k_y e_1 - jk_z b_0 - \eta f_2 = 0$$
$$-jk_t o_2 - jk_x e_1 + k_y h_1 - jk_z c_0 - \eta g_2 = 0$$
$$-jk_t p_2 - jk_x f_1 - k_y g_1 - jk_z d_0 - \eta h_2 = 0$$

$$jk_t a_3 + jk_x k_0 - k_y j_0 - jk_z m_1 - \eta i_3 = 0$$
$$jk_t b_3 + jk_x l_0 + k_y i_0 - jk_z n_1 - \eta j_3 = 0$$
$$jk_t c_3 - jk_x i_0 - k_y l_0 - jk_z o_1 - \eta k_3 = 0$$
$$jk_t d_3 - jk_x j_0 + k_y k_0 - jk_z p_1 - \eta l_3 = 0$$
$$jk_t e_3 + jk_x o_0 - k_y n_0 + jk_z i_1 - \eta m_3 = 0$$
$$jk_t f_3 + jk_x p_0 + k_y m_0 + jk_z j_1 - \eta n_3 = 0$$
$$jk_t g_3 - jk_x m_0 - k_y p_0 + jk_z k_1 - \eta o_3 = 0$$
$$jk_t h_3 - jk_x n_0 + k_y o_0 + jk_z l_1 - \eta p_3 = 0$$
$$-jk_t i_3 + jk_x c_0 - k_y b_0 - jk_z e_1 - \eta a_3 = 0$$
$$-jk_t j_3 + jk_x d_0 + k_y a_0 - jk_z f_1 - \eta b_3 = 0$$
$$-jk_t k_3 - jk_x a_0 - k_y d_0 - jk_z g_1 - \eta c_3 = 0$$
$$-jk_t l_3 - jk_x b_0 + k_y c_0 - jk_z h_1 - \eta d_3 = 0$$
$$-jk_t m_3 + jk_x g_0 - k_y f_0 + jk_z a_1 - \eta e_3 = 0$$
$$-jk_t n_3 + jk_x h_0 + k_y e_0 + jk_z b_1 - \eta f_3 = 0$$
$$-jk_t o_3 - jk_x e_0 - k_y h_0 + jk_z c_1 - \eta g_3 = 0$$
$$-jk_t p_3 - jk_x f_0 + k_y g_0 + jk_z d_1 - \eta h_3 = 0$$

It is a homogeneous system that allows non-zero solution only if its determinant is zero. But the condition to cancel a determinant of a system of 64 equations with 64 unknowns is difficult to obtain.
This condition of cancellation may be reached in another way by searching the condition imposed by all standing wave modes to be solution of Klein-Gordon equation (7).

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial x_t^2}\right](\psi_i)=\frac{m^2c^2}{\hbar^2}(\psi_i) \tag{7}$$

By substituting any function $\psi_i$ from (2) in (7), we get:

$$\left(-k_x^2-k_y^2-k_z^2+k_t^2\right)(\psi_i)=\eta^2(\psi_i) \tag{8}$$

This condition express that the pseudo norm of the four vector wave (5) is constant and ensure the nullity of the determinant of the Dirac system for standing wave modes (6).

Under condition (8), the Dirac system for standing wave modes has a great number of solutions whose presentation should need several pages and which may be consulted in [15]. One can exhibit a particular one (9), which is representative and which will serve for discussion in the next section.

$$\begin{aligned}
\psi_0 &= 0\\
\psi_1 &= -k_t\sin(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t) + j\eta\sin(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)\\
\psi_2 &= jk_y\sin(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t) + k_x\cos(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)\\
\psi_3 &= k_z\sin(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t)
\end{aligned} \tag{9}$$

The verification that (9) is solution of the Dirac system (1) under condition (8) is straightforward and needs only elementary calculus. To help the reader in this verification, the main needed expressions are reported in appendix.

## III. AN ENERGY INTERPRETATION

From a mathematical point of view, we have any freedom to multiply all of the wave functions present in (9) by a constant, and one that seems indicated in this case is equal to $\hbar c$. By replacing $\eta = \frac{m_0 c}{\hbar}$, we obtain:

$$
\begin{aligned}
\psi_0 &= 0 \\
\psi_1 &= -(\hbar\omega)\sin(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t) + j(m_0c^2)\sin(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t) \\
\psi_2 &= j(\hbar c k_y)\sin(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t) + (\hbar c k_x)\cos(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t) \\
\psi_3 &= (\hbar c k_z)\sin(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t)
\end{aligned} \tag{10}
$$

Each wave function has now an energy dimension, and one make the observation that in this solution, each term contains an energy of different kind, considering that two pulse energies in orthogonal directions are differentiated.

- $(\hbar\omega)$ will be called wave energy
- $(m_0c^2)$ will be called mass energy
- $(\hbar c k_x),(\hbar c k_y),(\hbar c k_z)$ will be called impulse energy along direction x, y, z

These energies exchange in the same way as electromagnetic energy in a cavity [14], but with some specificities. The first spinor $(\psi_0,\psi_1)$ carries exchanges between wave energy and mass energy, and the second spinor $(\psi_2,\psi_3)$ carries exchanges between impulse energies. We propose to analyze in more details these energy exchanges in the next paragraphs.

### *A. The wave-particle duality*

If we look at a point where the mass energy is extremum, we have to put in (10) $|\sin(k_x.x)| = |\cos(k_y.y)| = |\cos(k_z.z)| = 1$. The solution (10) then takes the form:

$$
\begin{aligned}
\psi_0 &= 0 \\
\psi_1 &= -(\hbar\omega)\cos(k_t x_t) + j(m_0c^2)\sin(k_t x_t) \\
\psi_2 &= 0 \\
\psi_3 &= 0
\end{aligned} \tag{11}
$$

where it is recognized the wave energy ($\hbar\omega$) and the mass energy ($m_0c^2$). The result that teaches us the relationship (11) is that these energies are evolving in time quadrature, and that when one is extremum, the other is null. In other words, when the particle is in its total mass form, it has no wave energy, and when it occurs in its total wave form, it presents no mass energy. Energy present in the particle so alternates between mass and wave forms to the pulse ω defined by the equation of conservation of quantum energy (8) that for impulse energy equal to zero is simply written:

$$\hbar^2\omega^2 = (m_0c^2)^2 \tag{12}$$

It may be assumed that it is in this ongoing exchange of energy that lies the mystery of the wave-particle duality which appears, in the light of the relationship (11), sometimes in the form of mass, sometimes in the form of wave.

### *B. The uncertainty principle*

This principle, enunciated by HEISENBERG, during the early days of quantum mechanics, was popularized in the expression: "it is impossible to know both the position and momentum of a particle". From a physical point of view, it is whole contained in a relationship that connects the uncertainty on position Δx and the uncertainty on momentum Δpx of a particle in the quantum world:

$$\Delta x.\Delta p_x > \frac{\hbar}{2} \tag{13}$$

In view of the exact solution (10), there is a question that naturally comes to mind. This solution is perfectly deterministic: each type of energy is known, in theory, with infinite precision for a position (x, y, z) and an instant (t) given. This state indeed seems in contradiction with the HEISENBERG uncertainty principle.

To remove this contradiction, we must first admit in the form of postulate the following conclusion: an observer can obtain information from a physical system only if it exchanges energy with this system.

Let us place on a point in space (x, y, z) where mass energy of the particle is extremum. The position of this mass energy can then be determined with any precision desired, according to the rules of classical mechanics. To ensure that this condition is achieved, the coordinates x, y, z must check as previously:

$$\sin^2(k_x x) = \cos^2(k_y y) = \cos^2(k_z z) = 1 \tag{14}$$

What requires:

$$\cos(k_x x) = \sin(k_y y) = \sin(k_z z) = 0 \tag{15}$$

It appears the following remarkable result: all impulse energy present in the particle are equal to zero at this location.
In other words, if we move to a point where we can, through an exchange of energy with the energy of mass, know with precision the position of the particle, we cannot get any information on its momentum at this point because its impulse energy is zero at this place.

The reciprocal is expressed in the following way: if one moves to a place where the impulse energy following x is maximum, then mass energy and impulse energy along y and z are zero. A similar property is checked by permutation on the variables x, y, z.
These observations allow to understand how a completely deterministic theory built on exact stationary solutions to the DIRAC equation remains compatible with the HEISENBERG uncertainty principle.

## IV. CONCLUSION

This paper shows how it is possible to find exact solutions of Dirac equation in terms of standing wave modes, upon a relativistic condition of energy conservation. These new solutions can be interpreted as internal energy exchanges within the particle, and it is shown how this determinist interpretation is fully compatible with the vision of Copenhagen school. It justifies the wave particle duality in indicating in what manner the internal energy to the particle alternately goes in the form of mass energy and wave energy and it shows how the HEISENBERG uncertainty principle is interpreted by indicating how the mass energy and impulse energy are not simultaneously present in the same place.

## APPENDIX

We give in appendix elements which allow a straightforward verification of the proposed solution (9). Verification of the sums is left to the reader care.

First equation of the Dirac system (1):

$$\eta\psi_0 = j\frac{\partial\psi_0}{\partial x_t} + j\frac{\partial\psi_3}{\partial x} + \frac{\partial\psi_3}{\partial y} + j\frac{\partial\psi_2}{\partial z}$$

$$\eta\psi_0 = 0$$

$$j\frac{\partial\psi_0}{\partial x_t} = 0$$

$$j\frac{\partial\psi_3}{\partial x} = jk_z k_x \cos(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t)$$

$$\frac{\partial\psi_3}{\partial y} = -k_z k_y \sin(k_x x)\sin(k_y y)\sin(k_z z)\sin(k_t x_t)$$

$$j\frac{\partial\psi_2}{\partial z} = k_y k_z \sin(k_x x)\sin(k_y y)\sin(k_z z)\sin(k_t x_t) - jk_x k_z \cos(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t)$$

Second equation of the Dirac system (1):

$$\eta\psi_1 = j\frac{\partial\psi_1}{\partial x_t} + j\frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_2}{\partial y} - j\frac{\partial\psi_3}{\partial z}$$

$$\eta\psi_1 = -k_t\eta\sin(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t) + j\eta^2\sin(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)$$

$$j\frac{\partial\psi_1}{\partial x_t} = jk_t^2\sin(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t) - \eta k_t\sin(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t)$$

$$j\frac{\partial\psi_2}{\partial x} = -k_y k_x\cos(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t) - jk_x^2\sin(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)$$

$$-\frac{\partial\psi_2}{\partial y} = -jk_y^2\sin(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t) + k_x k_y\cos(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t)$$

$$-j\frac{\partial\psi_3}{\partial z} = -jk_z^2\sin(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)$$

The sum leads to the relation of energy conservation (8):

$$\eta^2 = k_t^2 - k_x^2 - k_y^2 - k_z^2$$

Third equation of the Dirac system (1):

$$\eta\psi_2 = -j\frac{\partial\psi_2}{\partial x_t} - j\frac{\partial\psi_1}{\partial x} - \frac{\partial\psi_1}{\partial y} - j\frac{\partial\psi_0}{\partial z}$$

$$\eta\psi_2 = j\eta k_y \sin(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t) + \eta k_x \cos(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)$$

$$-j\frac{\partial\psi_2}{\partial x_t} = k_y k_t \sin(k_x x)\sin(k_y y)\cos(k_z z)\cos(k_t x_t) - jk_x k_t \cos(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t)$$

$$-j\frac{\partial\psi_1}{\partial x} = jk_t k_x \cos(k_x x)\cos(k_y y)\cos(k_z z)\cos(k_t x_t) + \eta k_x \cos(k_x x)\cos(k_y y)\cos(k_z z)\sin(k_t x_t)$$

$$-\frac{\partial\psi_1}{\partial y} = -k_t k_y \sin(k_x x)\sin(k_y y)\cos(k_z z)\cos(k_t x_t) + j\eta k_y \sin(k_x x)\sin(k_y y)\cos(k_z z)\sin(k_t x_t)$$

$$-j\frac{\partial\psi_0}{\partial z} = 0$$

Fourth equation of the Dirac system (1):

$$\eta\psi_3 = -j\frac{\partial\psi_3}{\partial x_t} - j\frac{\partial\psi_0}{\partial x} + \frac{\partial\psi_0}{\partial y} + j\frac{\partial\psi_1}{\partial z}$$

$$\eta\psi_3 = \eta k_z \sin(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t)$$

$$-j\frac{\partial\psi_3}{\partial x_t} = -jk_z k_t \sin(k_x x)\cos(k_y y)\sin(k_z z)\cos(k_t x_t)$$

$$-j\frac{\partial\psi_0}{\partial x} = 0$$

$$\frac{\partial\psi_0}{\partial y} = 0$$

$$j\frac{\partial\psi_1}{\partial z} = jk_t k_z \sin(k_x x)\cos(k_y y)\sin(k_z z)\cos(k_t x_t) + \eta k_z \sin(k_x x)\cos(k_y y)\sin(k_z z)\sin(k_t x_t)$$

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