An energy interpretation of exact solutions of Dirac equation in terms of standing waves modes

Patrick VAUDON* Alain REINEIX*

* Xlim Lab, University of Limoges, France

Abstract- Solutions of Dirac equation are well known in terms of bi-spinors, but the complete determination of these spinors is always derived using the assumption that the wave function is a plane wave. We investigate in this paper the search of general solutions under the form of standing wave modes, using methods issued from electromagnetism. It is shown that such solutions exist as solutions of a system of 64 equations and 64 unknowns, and that they must be associated with a relativistic relation of energy conservation. Complete exact solutions are presented. It is shown that these solutions highlight in a new manner the duality wave-particle, and the uncertainty principle.

Index Terms- Dirac equation – spinors – standing wave modes – quantum mechanics.

I. INTRODUCTION

Dirac equation is the equation which describes in the most accurate manner the behavior of particles in the infinite small world. In particular, it is able to take into account the spin phenomena, and to predict the existence of antimatter. These two properties are deduced from general solutions of the free Dirac equation, without any supplementary assumption. It’s not possible to progress towards more explicit solutions without adding some hypothesis, and the main one which is proposed is the assumption that the matter wave behaves like a plane wave. Free solutions of the Dirac equation are always presented under this condition as it can be seen in books at the state of art of the knowledge in this domain [1][2][3][4][5][6]. Papers linked to this subject [7][8][9] don’t explore the way of standing wave modes, no more than publications looking for exact solutions in electromagnetic or gravitational potentials [10][11][12][13].

Let us consider a particle at rest: it’s difficult to imagine how the matter wave works if it behaves like a plane wave inside the matter. A more pertinent description should be made using a wave working on standing wave modes. In such a description, the energy matter is supposed to be captive in a certain volume, in the same way as electromagnetic energy may be captive in a cavity. We know that in such a situation, the energy is exchanged inside the cavity between to kind of energy: electric energy and magnetic energy. Later in this paper, we will not wonder on the way in which the matter energy is confined in a region of space: the answer to this question is not known to us. But we will show that if we make the assumption that the internal energy is exchanged in stationary modes, this led to a quantum physics determinist and consistent with the actual probabilistic theory.

The following of this paper is composed with two sections. The first one explain how we can find exact solutions of Dirac equation in terms of standing wave modes and give explicit formulations of these exact solutions. The second one proposes energy interpretations of these new solutions and shows how they agree with the classical theory.

II. THE DIRAC SYSTEM FOR STANDING WAVES

We have to solve the Dirac system (1). No solutions of this system exists in terms of standing waves. We have then to formalize a method to found such a kind of solutions.

\[ \eta \Psi = j \frac{\partial \Psi_0}{\partial (t)} + j \frac{\partial \Psi_1}{\partial x} + j \frac{\partial \Psi_2}{\partial y} + j \frac{\partial \Psi_3}{\partial z} \]

\[ \eta \Psi_0 = j \frac{\partial \Psi_0}{\partial (t)} + j \frac{\partial \Psi_1}{\partial x} + j \frac{\partial \Psi_2}{\partial y} + j \frac{\partial \Psi_3}{\partial z} \]

\[ \eta \Psi_1 = j \frac{\partial \Psi_1}{\partial (t)} + j \frac{\partial \Psi_2}{\partial x} + j \frac{\partial \Psi_3}{\partial y} - j \frac{\partial \Psi_0}{\partial z} \]

\[ \eta \Psi_2 = j \frac{\partial \Psi_2}{\partial (t)} + j \frac{\partial \Psi_3}{\partial x} - j \frac{\partial \Psi_1}{\partial y} - j \frac{\partial \Psi_0}{\partial z} \]

\[ \eta \Psi_3 = j \frac{\partial \Psi_3}{\partial (t)} + j \frac{\partial \Psi_0}{\partial x} + j \frac{\partial \Psi_1}{\partial y} + j \frac{\partial \Psi_2}{\partial z} \]

We assume in a first time, that any solution of the Dirac system in Cartesian coordinates may be developed on a whole base of standing wave modes. This base is built with sixteen terms which represent all combinations of a product of four sine and cosine functions.
We adopt for each wave function $\psi_i$, for $i = 0, 1, 2, 3$ the following development:

$$
\psi_i = \left[a_S S_S S_S + b_S C_S S_S + c_S C_S S_S + d_S C_S S_S + e_S S_S S_S + f_S S_S C_S + g_S C_S C_S + h_S C_S C_S \right] C_i
\left[ + k_S S_S S_S + l_S S_S C_S + m_s S_S C_S + n_S S_S C_S + o_S C_S C_S + p_S C_S C_S \right] S_i
$$

(2)

In this expression, the following abbreviated notation has been used:

$$
S_x = \sin (k_x x) \quad S_y = \sin (k_y y) \quad S_z = \sin (k_z z) \quad S_t = \sin (k_t t)
$$

(3)

$$
C_x = \cos (k_x x) \quad C_y = \cos (k_y y) \quad C_z = \cos (k_z z) \quad C_t = \cos (k_t t)
$$

(4)

The wave vector is represented by its $k_x, k_y, k_z$ components, while for a homogeneous notation and consistent with relativity, the product $\omega t$ has been replaced with the expression $k_t t$, which allows to highlight in (3) and (4) the product of two four-vectors:

$$
\begin{pmatrix}
  x \\
  y \\
  z \\
  x_t = ct
\end{pmatrix}
\begin{pmatrix}
  k_x \\
  k_y \\
  k_z \\
  k_t = \frac{\omega}{c}
\end{pmatrix}
$$

(5)

Coefficients $a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, i_i, j_i, k_i, l_i, m_i, n_i, o_i, p_i$, for $i = 0, 1, 2, 3$, are real constants that weigh each of the modes and will serve as unknowns in the search for the wave functions $\psi_0, \psi_1, \psi_2, \psi_3$ solutions of the DIRAC system.

This leads, for each equation of the DIRAC system, to express the partial derivatives of the wave functions $\psi_0, \psi_1, \psi_2, \psi_3$ and to formulate a homogeneous system of 16 equations for the coefficients $a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, i_i, j_i, k_i, l_i, m_i, n_i, o_i, p_i$. Calculations are a bit laborious but without difficulties. They are presented in their entirety in [15].

The obtained global system is therefore a homogeneous system of 64 equations with 64 unknowns. It is presented in (6).

```
- jk_i a_i - jk_i k_1 - k_{j_2} - jk_i m_2 - \eta_i 0 = 0
- jk_i b_i - jk_i l_1 + k_{i_2} - jk_i n_2 - \eta_l 0 = 0
- jk_i c_i + jk_i i_2 + k_{j_3} - jk_i o_2 - \eta_k 0 = 0
- jk_i d_i + jk_i j_3 + k_{i_4} - jk_i p_2 - \eta_l 0 = 0
- jk_i e_i - jk_i o_3 + k_{i_5} + jk_i t_2 - \eta_m 0 = 0
- jk_i f_i - jk_i p_3 + k_{i_6} + jk_i j_2 - \eta_n 0 = 0
- jk_i g_i + jk_i m_3 - k_{i_7} + jk_i j_2 - \eta_o 0 = 0
- jk_i h_i - jk_i n_3 + k_{i_8} + jk_i l_2 - \eta_o 0 = 0
- jk_i i_2 - jk_i c_3 - k_{j_4} - jk_i e_2 - \eta_o 0 = 0
- jk_i j_3 - jk_i d_4 + k_{a_5} - jk_i f_2 - \eta_b 0 = 0
- jk_i k_3 + jk_i a_6 - k_{j_7} - jk_i g_2 - \eta_c 0 = 0
- jk_i l_4 + jk_i b_5 + k_{c_8} - jk_i h_2 - \eta_d 0 = 0
- jk_i m_4 - jk_i g_3 - k_{f_9} - jk_i a_2 - \eta_e 0 = 0
- jk_i n_4 - jk_i h_5 + k_{c_10} + jk_i b_2 - \eta_f 0 = 0
- jk_i o_3 + jk_i e_6 - k_{j_11} + jk_i c_2 + \eta_g 0 = 0
- jk_i p_5 + jk_i f_3 + k_{g_12} + jk_i d_2 - \eta_h 0 = 0
```

(6)
It is a homogeneous system that allows non-zero solution only if its determinant is zero. But the condition to cancel a determinant of a system of 64 equations with 64 unknowns is difficult to obtain.

This condition of cancellation may be reached in another way by searching the condition imposed by all standing wave modes to be solution of Klein-Gordon equation (7).

By substituting any function $\psi_i$ from (2) in (7), we get:

$$(-k_x^2 - k_y^2 - k_z^2 + k_i^2)\psi_i = \eta^2 \psi_i$$

(8)

This condition express that the pseudo norm of the four vector wave (5) is constant and ensure the nullity of the determinant of the Dirac system for standing wave modes (6).

Under condition (8), the Dirac system for standing wave modes has a great number of solutions whose presentation should need several pages and which may be consulted in [15]. One can exhibit a particular one (9), which is representative and which will serve for discussion in the next section.

$$\psi_0 = 0$$

$$\psi_1 = -k_x \sin(k_x x) \cos(k_y y) \cos(k_z z) \cos(k_x x) + j \eta \sin(k_x x) \cos(k_y y) \cos(k_z z) \sin(k_x x),$$

(9)

$$\psi_2 = jk_x \sin(k_x x) \sin(k_y y) \cos(k_z z) \sin(k_x x) + k_x \cos(k_x x) \cos(k_y y) \cos(k_z z) \sin(k_x x),$$

$$\psi_3 = k_x \sin(k_x x) \cos(k_y y) \sin(k_z z) \sin(k_x x),$$

The verification that (9) is solution of the Dirac system (1) under condition (8) is straightforward and needs only elementary calculus. To help the reader in this verification, the main needed expressions are reported in appendix.

III. AN ENERGY INTERPRETATION

From a mathematical point of view, we have any freedom to multiply all of the wave functions present in (9) by a constant, and one that seems indicated in this case is equal to $\frac{m \xi c}{\hbar}$. By replacing $\eta = \frac{m \xi c}{\hbar}$, we obtain:

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Each wave function has now an energy dimension, and one makes the observation that in this solution, each term contains an energy of different kind, considering that two pulse energies in orthogonal directions are differentiated.

- $(\hbar \omega)$ will be called wave energy
- $(m_0 c^2)$ will be called mass energy
- $(\hbar c k_x)$, $(\hbar c k_y)$, $(\hbar c k_z)$ will be called impulse energy along direction $x$, $y$, $z$

These energies exchange in the same way as electromagnetic energy in a cavity [14], but with some specificities. The first spinor $(\psi_0, \psi_1)$ carries exchanges between wave energy and mass energy, and the second spinor $(\psi_2, \psi_3)$ carries exchanges between impulse energies. We propose to analyze in more details these energy exchanges in the next paragraphs.

### A. The wave-particle duality

If we look at a point where the mass energy is extremum, we have to put in (10) $|\sin(k_x x)| = |\cos(k_y y)| = |\cos(k_z z)| = 1$. The solution (10) then takes the form:

\[
\psi_0 = 0
\]

\[
\psi_1 = -(\hbar \omega) \sin(k_x x) + (m_0 c^2) \sin(k_x x)
\]

\[
\psi_2 = 0
\]

\[
\psi_3 = 0
\]

where it is recognized the wave energy $(\hbar \omega)$ and the mass energy $(m_0 c^2)$. The result that teaches us the relationship (11) is that these energies are evolving in time quadrature, and that when one is extremum, the other is null. In other words, when the particle is in its total mass form, it has no wave energy, and when it occurs in its total wave form, it presents no mass energy. Energy present in the particle so alternates between mass and wave forms to the pulse $\omega$ defined by the equation of conservation of quantum energy (8) that for impulse energy equal to zero is simply written:

\[
\hbar \omega = (m_0 c^2)^{1/2}
\]

It may be assumed that it is in this ongoing exchange of energy that lies the mystery of the wave-particle duality which appears, in the light of the relationship (11), sometimes in the form of mass, sometimes in the form of wave.

### B. The uncertainty principle

This principle, enunciated by HEISENBERG, during the early days of quantum mechanics, was popularized in the expression: "it is impossible to know both the position and momentum of a particle". From a physical point of view, it is whole contained in a relationship that connects the uncertainty on position $\Delta x$ and the uncertainty on momentum $\Delta p_x$ of a particle in the quantum world:

\[
\Delta x \Delta p_x > \frac{\hbar}{2}
\]

In view of the exact solution (10), there is a question that naturally comes to mind. This solution is perfectly deterministic: each type of energy is known, in theory, with infinite precision for a position $(x, y, z)$ and an instant $(t)$ given. This state indeed seems in contradiction with the HEISENBERG uncertainty principle.

To remove this contradiction, we must first admit in the form of postulate the following conclusion: an observer can obtain information from a physical system only if it exchanges energy with this system. Let us place on a point in space $(x, y, z)$ where mass energy of the particle is extremum. The position of this mass energy can then be determined with any precision desired, according to the rules of classical mechanics. To ensure that this condition is achieved, the coordinates $x$, $y$, $z$ must check as previously:

\[
\sin^2(k_x x) = \cos^2(k_y y) = \cos^2(k_z z) = 1
\]

What requires:
\[ \cos(k_x x) = \sin(k_y y) = \sin(k_z z) = 0 \]  

(15)

It appears the following remarkable result: all impulse energy present in the particle are equal to zero at this location. In other words, if we move to a point where we can, through an exchange of energy with the energy of mass, know with precision the position of the particle, we cannot get any information on its momentum at this point because its impulse energy is zero at this place.

The reciprocal is expressed in the following way: if one moves to a place where the impulse energy following \( x \) is maximum, then mass energy and impulse energy along \( y \) and \( z \) are zero. A similar property is checked by permutation on the variables \( x, y, z \). These observations allow to understand how a completely deterministic theory built on exact stationary solutions to the DIRAC equation remains compatible with the HEISENBERG uncertainty principle.

**IV. CONCLUSION**

This paper shows how it is possible to find exact solutions of Dirac equation in terms of standing wave modes, upon a relativistic condition of energy conservation. These new solutions can be interpreted as internal energy exchanges within the particle, and it is shown how this determinist interpretation is fully compatible with the vision of Copenhagen school. It justifies the wave-particle duality in indicating in what manner the internal energy to the particle alternately goes in the form of mass energy and wave energy and it shows how the HEISENBERG uncertainty principle is interpreted by indicating how the mass energy and impulse energy are not simultaneously present in the same place.

**APPENDIX**

We give in appendix elements which allow a straightforward verification of the proposed solution (9). Verification of the sums is left to the reader care.

**First equation of the Dirac system (1):**

\[ \eta \psi_0 = j \frac{\partial \psi_0}{\partial x} + j \frac{\partial \psi_1}{\partial y} + j \frac{\partial \psi_2}{\partial z} \]

\[ \eta \psi_0 = 0 \]

\[ j \frac{\partial \psi_1}{\partial x} = 0 \]

\[ j \frac{\partial \psi_1}{\partial y} = jk_x k_z \cos(k_x x) \cos(k_y y) \cos(k_z z) \sin(k_x x) \]

\[ j \frac{\partial \psi_1}{\partial z} = -k_x k_z \sin(k_x x) \sin(k_y y) \sin(k_z z) \sin(k_x x) \]

\[ j \frac{\partial \psi_2}{\partial x} = k_x k_z \sin(k_x x) \sin(k_z z) \sin(k_x x) \sin(k_z z) \sin(k_x x) \]

\[ -jk_x k_z \cos(k_x x) \cos(k_y y) \cos(k_z z) \sin(k_x x) \]

\[ -jk_x k_z \sin(k_x x) \cos(k_y y) \cos(k_z z) \sin(k_x x) \]

\[ -j k_x k_z \cos(k_x x) \sin(k_y y) \cos(k_z z) \sin(k_x x) \]

\[ -j k_x k_z \sin(k_x x) \cos(k_y y) \sin(k_z z) \sin(k_x x) \]

\[ \text{The sum leads to the relation of energy conservation (8):} \]

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\[ \eta^2 = k_x^2 - k_y^2 - k_z^2 \]

Third equation of the Dirac system (1):
\[
\eta \psi_z = -i \frac{\partial \psi_z}{\partial x} - j \frac{\partial \psi_x}{\partial z} - j \frac{\partial \psi_z}{\partial y} - j \frac{\partial \psi_y}{\partial z}
\]
\[
\eta \psi_x = j \eta k_x \sin(k_x x) \cos(k_y y) \cos(k_z z) \sin(k_x x) + \eta k_x \cos(k_x x) \cos(k_y y) \cos(k_z z) \sin(k_x x)
\]
\[
- \frac{1}{j} \frac{\partial \psi_x}{\partial x} = k_x k_x \sin(k_x x) \cos(k_y y) \cos(k_z z) \cos(k_x x) - j k_x k_x \cos(k_x x) \cos(k_y y) \cos(k_z z) \sin(k_x x)
\]
\[
- \frac{1}{j} \frac{\partial \psi_y}{\partial y} = -k_x k_x \sin(k_x x) \cos(k_y y) \cos(k_z z) \cos(k_x x) + j \eta k_x \sin(k_x x) \cos(k_y y) \cos(k_z z) \sin(k_x x)
\]
\[
- \frac{1}{j} \frac{\partial \psi_z}{\partial z} = 0
\]

Fourth equation of the Dirac system (1):
\[
\eta \psi_z = -i \frac{\partial \psi_z}{\partial x} - j \frac{\partial \psi_x}{\partial z} + \frac{\partial \psi_y}{\partial y} + \frac{\partial \psi_z}{\partial z}
\]
\[
\eta \psi_x = \eta k_x \sin(k_x x) \cos(k_y y) \sin(k_z z) \sin(k_x x)
\]
\[
- \frac{1}{j} \frac{\partial \psi_x}{\partial x} = -j k_x k_x \sin(k_x x) \cos(k_y y) \cos(k_z z) \cos(k_x x)
\]
\[
- \frac{1}{j} \frac{\partial \psi_y}{\partial y} = 0
\]
\[
\frac{\partial \psi_z}{\partial z} = 0
\]
\[
\frac{\partial \psi_y}{\partial y} = j k_x k_x \sin(k_x x) \cos(k_y y) \sin(k_z z) \cos(k_x x) + \eta k_x \sin(k_x x) \cos(k_y y) \cos(k_z z) \sin(k_x x)
\]

REFERENCES

AUTHORS

First Author – Patrick Vaudon, Professor at Xlim, University of Limoges France, patrick.vaudon@unilim.fr
Second Author – Alain Reineix, Research Director at Xlim, University of Limoges France, alain.reineix@xlim.fr
Correspondence Author – Patrick Vaudon, Xlim, Faculté des sciences, 123 rue Albert Thomas 87000 Limoges France, patrick.vaudon@unilim.fr.