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A Study of Random Operators on the Tensor Product of Banach Spaces

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Abstract- Let (Ω,β,μ) be a complete probability measure space and X_1 and X_2 be separable Banach spaces. For Banach spaces Y_1 and Y_2, let F_1: $\Omega\times X_1\to Y_1$ and F_2: $\Omega\times X_2\to Y_2$ be random operators. Using F_1 and F_2, we construct a random operator F: $\Omega\times(X_1\otimes_\gamma X_2)\to Y_1\otimes_\gamma Y_2$, which is continuous if F_1 and F_2 are continuous. We prove that if F_1 and F_2 are stochastically continuous, then F is also stochastically continuous. Similar result is also established in case of separability of random operators. The fixed points of such random operators on the tensor product of Banach spaces is also studied here.

***Keywords*:** random operator/ tensor product / stochastically continuous operator/ fixed point

I. Introduction

The systematic study of random operator equations using the methods of functional analysis was introduced around 1955. In recent times, random nonlinear analysis together with the random fixed point theory has emerged into an active research field (refer to [1], [2], [6], [7]) due to its tremendous applications in mathematical modeling of many real life problems. In this paper, we study random operators on the projective tensor product of two Banach spaces and also discuss about the fixed points of such tensor product random operators.

II. Preliminary Ideas

Before discussing the main results, we first present some preliminary definitions (refer to [1], [4], [5], [6]).

Let (Ω,β,μ) be a complete probability measure space and (X, β_X) be a measurable space, where X is a separable Banach space and β_X is the σ-algebra of all Borel subsets of X.

Definition 2.1 A mapping f: $\Omega\to X$ is said to be X-valued random variable if the inverse image under the mapping f of every Borel set B belongs to β, i.e., $f^{1}(B)\in\beta$ for all $B\in\beta_X$.

Definition 2.2 Let Y be a Banach space. An operator F: $\Omega\times X\to Y$ is said to be a random operator if $F(\omega)x$ is a Y-valued random variable for all $x\in X$. [For a fixed $\omega\in\Omega$, $F(\omega)$ denotes a deterministic operator from X into Y].

Definition 2.3 A random operator F: $\Omega\times X\to Y$ is said to be continuous at x if $x_n\to x$ implies $F(\omega)x_n\to F(\omega)x$ almost surely.
F: $\Omega\times X\to Y$ is said to be stochastically continuous at x if for every $\varepsilon>0$, we have,

$$\|x_n - x\| \to 0 \Rightarrow \lim_{n\to\infty} \mu\{\omega : \|F(\omega)x_n - F(\omega)x\| > \varepsilon\} = 0 .$$

Definition 2.4 A random operator F: $\Omega\times X\to Y$ is called separable if there exists a countable dense set $S\subseteq X$ and $N\in\beta$ with $\mu(N)=0$ such that for every closed set $K\subseteq Y$ and open set $G\subseteq X$, we have,

$$\{\omega\in\Omega : F(\omega)(G\cap S)\subseteq K\}\ \{\omega\in\Omega : F(\omega)G\subseteq K\}\subset N.$$

Any such S is called a separant of F.

Definition 2.5 Let X and Y be two normed spaces. The projective tensor norm $\|.\|_\gamma$ on $X\otimes Y$ is defined as:

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$$\|u\|_{\gamma} = \inf\{ \sum_i \|x_i\|.\|y_i\| : u = \sum_i x_i \otimes y_i \}$$

where the infimum is taken over all (finite) representations of u.

The completion of $(X\otimes Y, \|.\|_{\gamma})$ is called the projective tensor product of X and Y, and is denoted by $X\otimes_{\gamma}Y$.

III. Main results

First, we construct a random operator for the projective tensor product of the Banach spaces X_1 and X_2.

Theorem 3.1 Let X_1 and X_2 be separable Banach spaces and $F_1: \Omega\times X_1\to Y_1$ and $F_2: \Omega\times X_2\to Y_2$ be random operators. We consider $F: \Omega\times(X_1\otimes_{\gamma}X_2)\to Y_1\otimes_{\gamma}Y_2$ such that

$$F(\omega)(\sum_n x_{1n}\otimes x_{2n}) = \sum_n F_1(\omega)x_{1n}\otimes F_2(\omega)x_{2n}, \text{ for } \sum_n x_{1n}\otimes x_{2n} \in X_1\otimes_{\gamma}X_2.$$

Then F is a random operator on $X_1\otimes_{\gamma}X_2$, and F is continuous if both F_1 and F_2 are continuous.

Proof. For some fixed n, let $F_1(\omega)x_{1n} = y_{1n}(\omega)$ and $F_2(\omega)x_{2n} = y_{2n}(\omega)$. Now, $y_{1n}(\omega)$ and $y_{2n}(\omega)$ are respectively Y_1 and Y_2 valued random variables.

We take $y_{1n}\otimes y_{2n} : \Omega \to Y_1\otimes_{\gamma} Y_2$ be such that $(y_{1n}\otimes y_{2n})(\omega) = y_{1n}(\omega)\otimes y_{2n}(\omega)$.

Let $B\in \beta_{Y_1\otimes_{\gamma}Y_2}$, (the σ-algebra of all Borel subsets of $Y_1\otimes_{\gamma}Y_2$) be arbitrary.

If $\omega\in (y_{1n}\otimes y_{2n})^{-1}(B)$, then $y_{1n}(\omega)\otimes y_{2n}(\omega)\in B$. Let $y_{1n}(\omega) = t_1$ and $y_{2n}(\omega) = t_2$.
Now, $t_1\otimes t_2\in B$. Then there exists an open sphere $S_r(t_1\otimes t_2) \subseteq B$. Let $x_1\otimes x_2\in S_r(t_1\otimes t_2)$, for some non-zero x_1 and x_2.
It can be shown that $x_1\in S_{r1}(t_1)$ and $x_2\in S_{r2}(t_2)$, for some suitably chosen r_1 and r_2. So, $t_1\in B_1$, an open set in β_{Y_1} and $t_2\in B_2$, an open set in β_{Y_2}.
But $y_{1n}^{-1}(B_1)\in\beta$ and $y_{2n}^{-1}(B_2)\in\beta \Rightarrow \omega = y_{1n}^{-1}(t_1)\in N_1$ and $\omega = y_{2n}^{-1}(t_2)\in N_2$, for some $N_1, N_2\in\beta$.
Hence, $\omega\in N_1\cap N_2\in\beta$ and thus, $(y_{1n}\otimes y_{2n})^{-1}(B)\subseteq N_1\cap N_2\in\beta$.
So, $y_{1n}\otimes y_{2n}$ is a $Y_1\otimes_{\gamma} Y_2$-valued random variable.

Thus, $F: \Omega\times(X_1\otimes_{\gamma}X_2)\to Y_1\otimes_{\gamma}Y_2$ is a random operator.

To show the continuity of F, we take a sequence $\{\sum_i x_{1in}\otimes x_{2in}\}$ in $X_1\otimes_{\gamma}X_2$ which converges to $\sum_i x_{1i}\otimes x_{2i}$.
Then $x_{1in}\to x_{1i}$ and $x_{2in}\to x_{2i}$, for each i.
Since F_1 is continuous, $F_1(\omega)x_{1in}\to F_1(\omega)x_{1i}$ almost surely. So, there exists some $A_1\in\beta$ with $\mu(A_1)=0$ such that

$$\lim_{n\to\infty}\|F_1(\omega)x_{1in} - F_1(\omega)x_{1i}\| = 0 \ \forall\,\omega\notin A_1. \ldots\ldots\ldots\ldots..(3.1)$$

Similarly, since F_2 is continuous, so, there exists some $A_2\in\beta$ with $\mu(A_2)=0$ such that

$$\lim_{n\to\infty}\|F_2(\omega)x_{2in} - F_2(\omega)x_{2i}\| = 0 \ \forall\,\omega\notin A_2. \ldots\ldots\ldots\ldots(3.2)$$

Let $A=A_1\cup A_2$. Then $\mu(A)=\mu(A_1)+\mu(A_2)=0$. Also $\omega\notin A\Rightarrow\omega\notin A_1$ and $\omega\notin A_2$.

Now, $$\left\|F(\omega)(\sum_i x_{1in}\otimes x_{2in}) - F(\omega)(\sum_i x_{1i}\otimes x_{2i})\right\|$$

$$= \left\| \sum_i F_1(\omega)x_{1in} \otimes F_2(\omega)x_{2in} - \sum_i F_1(\omega)x_{1i} \otimes F_2(\omega)x_{2i} \right\|$$

$$= \left\| \sum_i [(F_1(\omega)x_{1in} - F_1(\omega)x_{1i}) \otimes F_2(\omega)x_{2in} + F_1(\omega)x_{1i} \otimes (F_2(\omega)x_{2in} - F_2(\omega)x_{2i})] \right\|$$

$$\leq \sum_i [\|F_1(\omega)x_{1in} - F_1(\omega)x_{1i}\|.\|F_2(\omega)x_{2in}\| + \|F_1(\omega)x_{1i}\|.\|F_2(\omega)x_{2in} - F_2(\omega)x_{2i}\|]$$

$\to 0$ as $n\to\infty$ $\forall$ $\omega \notin A$ (using (3.1) and (3.2))

Thus $F(\omega)(\sum_i x_{1in} \otimes x_{2in}) \to F(\omega)(\sum_i x_{1i} \otimes x_{2i})$ almost surely.

So, F is a continuous random operator. □

Theorem 3.2 If the random operators F_1 and F_2 as given in the above theorem are stochastically continuous, then F is also stochastically continuous.

Proof. Let $\sum_i x_{1i} \otimes x_{2i} \in X_1 \otimes_\gamma X_2$ and $\{ \sum_i x_{1in} \otimes x_{2in} \}$ is a sequence in $X_1 \otimes_\gamma X_2$ which converges to $\sum_i x_{1i} \otimes x_{2i}$.

Let $\omega \in \Omega$ be such that $\left\| F(\omega)(\sum_i x_{1in} \otimes x_{2in}) - F(\omega)(\sum_i x_{1i} \otimes x_{2i}) \right\| > \varepsilon$, where $\varepsilon > 0$ is arbitrary.

Now, $\varepsilon < \left\| \sum_i F_1(\omega)x_{1in} \otimes F_2(\omega)x_{2in} - \sum_i F_1(\omega)x_{1i} \otimes F_2(\omega)x_{2i} \right\|$

$$\leq \sum_i [\|F_1(\omega)x_{1in} - F_1(\omega)x_{1i}\|.\|F_2(\omega)x_{2in}\| + \|F_1(\omega)x_{1i}\|.\|F_2(\omega)x_{2in} - F_2(\omega)x_{2i}\|],$$ (as shown in Theorem 3.1).

We choose some $\varepsilon_1 < \varepsilon\,(> 0)$ such that $\|F_1(\omega)x_{1in} - F_1(\omega)x_{1i}\|.\|F_2(\omega)x_{2in}\| > \varepsilon_1$

and $\|F_1(\omega)x_{1i}\|.\|F_2(\omega)x_{2in} - F_2(\omega)x_{2i}\| > \varepsilon_1$, for each i.

Taking $\varepsilon_2 = \frac{1}{\|F_2(\omega)x_{2in}\|}\varepsilon_1$, ($\|F_2(\omega)x_{2in}\| \neq 0$) and $\varepsilon_3 = \frac{1}{\|F_1(\omega)x_{1i}\|}\varepsilon_1$, ($\|F_1(\omega)x_{1i}\| \neq 0$) respectively, we get,

$\|F_1(\omega)x_{1in} - F_1(\omega)x_{1i}\| > \varepsilon_2$ and $\|F_2(\omega)x_{2in} - F_2(\omega)x_{2i}\| > \varepsilon_3$.

Since F_1 and F_2 are stochastically continuous, so $\mu(\omega) = 0$ as $n\to\infty$.

Thus, $\lim_{n\to\infty} \mu\left\{ \omega : \left\| F(\omega)(\sum_i x_{1in} \otimes x_{2in}) - F(\omega)(\sum_i x_{1i} \otimes x_{2i}) \right\| > \varepsilon \right\} = 0$.

Hence F is stochastically continuous. □

In [5], we have the following result regarding separability of random operators.

Lemma 3.3 [5] Let F: $\Omega \times X \to Y$ be a random operator and S be a countable dense subset of X. Then F is separable with separant S if and only if there exists $N \in \beta$ with $\mu(N)=0$ such that for all $\omega \notin N$ and $x \in X$, there exists $\{x_n\} \subset S$ such that
$x_n \to x$ and $F(\omega)x_n \to F(\omega)x$.

Here, we prove:

Theorem 3.4 If F_1 and F_2 are separable random operators, then F is also a separable random operator.

Proof. Let $\sum_i x_{1i} \otimes x_{2i} \in X_1 \otimes_\gamma X_2$ be arbitrary. Then $x_{1i} \in X_1$ and $x_{2i} \in X_2$, for each i.

Let F_1: $\Omega \times X_1 \to Y_1$ be separable with separant S_1. So, using Lemma 3.3, there exists $N_1 \in \beta$ with $\mu(N_1)=0$ such that for all $\omega \notin N_1$ and $x_{1i} \in X_1$, there exists a sequence $\{x_{1in}\}$ in S_1 such that $x_{1in} \to x_{1i}$ and $F_1(\omega)x_{1in} \to F_1(\omega)x_{1i}$.

Again, let F_2: $\Omega \times X_2 \to Y_2$ be be separable with separant S_2. So, there exists $N_2 \in \beta$ with $\mu(N_2)=0$ such that for all $\omega \notin N_2$ and $x_{2i} \in X_2$, there exists a sequence $\{x_{2in}\}$ in S_2 such that $x_{2in} \to x_{2i}$ and $F_2(\omega)x_{2in} \to F_2(\omega)x_{2i}$.

Now, $S=S_1 \otimes S_2$ is a countable dense subset of $X_1 \otimes_\gamma X_2$. Let $N=N_1 \cup N_2$. Then $N \in \beta$ and $\mu(N_1 \cup N_2) = \mu(N_1)+\mu(N_2)=0$.
Let $\omega \notin N_1 \cup N_2$. Then $\omega \notin N_1$ and $\omega \notin N_2$.

Now, $\{ \sum_i x_{1in} \otimes x_{2in} \}$ is a sequence in S converging to $\sum_i x_{1i} \otimes x_{2i}$.
Also, we can show that $F(\omega)(\sum_i x_{1in} \otimes x_{2in}) \to F(\omega)(\sum_i x_{1i} \otimes x_{2i})$.
Hence F is separable. □

It is stated in [5] that under certain conditions, every random operator is equivalent to a separable random operator.

Lemma 3.5 Let X be a separable Banach space and K be a compact subset of another Banach space Y. If F: $\Omega \times X \to K$ is a random operator, then there exists a separable random operator F_S: $\Omega \times X \to Y$ such that $\mu(\{\omega : F(\omega)x=F_S(\omega)x\}) =1$.

So, using the above Lemma in Theorem 3.4, we have,

Corollary 3.6 Let X_1 and X_2 be separable Banach spaces and K_1 and K_2 be compact subsets of Y_1 and Y_2 respectively. If F_1: $\Omega \times X_1 \to K_1$ and F_2: $\Omega \times X_2 \to K_2$ be random operators, then there exists a separable random operator F_S: $\Omega \times (X_1 \otimes_\gamma X_2) \to Y_1 \otimes_\gamma Y_2$ such that

$$\mu(\{\omega : \sum_n F_1(\omega)x_{1n} \otimes F_2(\omega)x_{2n} = F_S(\omega)(\sum_n x_{1n} \otimes x_{2n})\}) = 1$$

The study of random fixed points attracted much attention of the research workers since late 90's. Different classical fixed point theorems are extended to random operators by different research workers (refer to [1], [2], [6], [7]). A random variable x: $\Omega \to X$ is a random fixed point of a random operator F: $\Omega \times X \to X$ if $F(\omega)x(\omega)=x(\omega)\ \forall \omega \in \Omega$.

Regarding fixed point of the tensor product of random operators, we establish the following.

Theorem 3.7 Let x_1 and x_2 be fixed points of the random operators F_1: $\Omega \times X_1 \to X_1$ and F_2: $\Omega \times X_2 \to X_2$ respectively. Then $x_1 \otimes x_2$: $\Omega \to X_1 \otimes_\gamma X_2$ is a fixed point of F.

Proof. Since x_1 and x_2 are fixed points of F_1 and F_2, so, $F_1(\omega)x_1(\omega)=x_1(\omega)\ \forall \omega \in \Omega$

and $F_2(\omega)x_2(\omega)=x_2(\omega)\ \forall \omega \in \Omega$.

Now, $\|F(\omega)(x_1 \otimes x_2)(\omega) - (x_1 \otimes x_2)(\omega)\| = \|F_1(\omega)x_1(\omega) \otimes F_2(\omega)x_2(\omega) - x_1(\omega) \otimes x_2(\omega)\| = 0$.

Hence, $F(\omega)(x_1 \otimes x_2)(\omega) = (x_1 \otimes x_2)(\omega)\ \forall \omega \in \Omega$, and thus, $x_1 \otimes x_2$ is a fixed point of F. □

Example 3.8 Let $\Omega=[0,1]$ and β be the σ-algebra of all Lebesgue measurable subsets of Ω.
Let $X_1=R$ (the set of reals), and F_1: $\Omega \times R \to R$ be such that $F_1(\omega)x=\omega^2-\omega x-1$. Let x_1: $\Omega \to R$ be such that $x_1(\omega)=\omega-1\ \forall \omega \in \Omega$. Then x_1 is a fixed point for F_1.

Let $X_2=C$ (the set of complex numbers), and F_2: $\Omega\times C\to C$ be such that $F_2(\omega)x=4\omega^2-2i\omega x+1$. Let $x_2:\Omega\to C$ be such that $x_2(\omega)=1-2i\omega$ $\forall\,\omega\in\Omega$. Then x_2 is a fixed point for F_2.

Now, $x_1\otimes x_2$: $\Omega\to R\otimes_\gamma C$, defined by $(x_1\otimes x_2)(\omega)=(\omega-1)\otimes(1-2i\omega)$ is a fixed point for the random operator F: $\Omega\times(R\otimes_\gamma C)\to R\otimes_\gamma C$. □

It can be shown that the converse of the Theorem 2.7 also holds.

Theorem 3.9 Let F_1: $\Omega\times X_1\to X_1$ and F_2: $\Omega\times X_2\to X_2$ be two random operators and F be the corresponding random operator on $X_1\otimes_\gamma X_2$. If $x_1\otimes x_2:\Omega\to X_1\otimes_\gamma X_2$ is a fixed point for F, then x_1 and x_2 are fixed points of F_1 and F_2 respectively.

Proof. We have, $F(\omega)(x_1\otimes x_2)(\omega)=(x_1\otimes x_2)(\omega)\ \forall\,\omega\in\Omega.$

$$\Rightarrow F_1(\omega)x_1(\omega)\otimes F_2(\omega)x_2(\omega)=x_1(\omega)\otimes x_2(\omega)$$
$$\Rightarrow (F_1(\omega)x_1(\omega)\otimes F_2(\omega)x_2(\omega))(f,g)=(x_1(\omega)\otimes x_2(\omega))(f,g)\quad \forall\, f\in X_1^*,\ g\in X_2^*.$$
$$\Rightarrow f(F_1(\omega)x_1(\omega))g(F_2(\omega)x_2(\omega))=f(x_1(\omega))g(x_2(\omega))\quad \forall\, f\in X_1^*,\ g\in X_2^*.$$

In particular, we take f be such that $\mathrm{Ker}(f)=\{0\}$ and g be a non zero constant function.
Then it follows that $F_1(\omega)x_1(\omega)=x_1(\omega)\ \forall\,\omega\in\Omega$.
Similarly, taking f as a non zero constant function and Ker $(g)=\{0\}$, we can show that
$F_2(\omega)x_2(\omega)=x_2(\omega)\ \forall\,\omega\in\Omega$. □

Remark

We consider the converse parts of Theorems 3.1 and 3.4. Here, we can raise the following problems:

(i) From a given random operator F: $\Omega\times(X_1\otimes_\gamma X_2)\to Y_1\otimes_\gamma Y_2$, can we construct random operators F_1: $\Omega\times X_1\to Y_1$ and F_2: $\Omega\times X_2\to Y_2$? If F is separable, what can be said about the separability of F_1 and F_2?

(ii) Let S be a non empty subset of a Banach space X. The random operator F: $\Omega\times S\to S$ is said to be (k, n) rotative random operator for $k<n$, if for every $\omega\in\Omega$,

$\left\|x(\omega)-F^n(\omega)x(\omega)\right\|\le k.\left\|x(\omega)-F(\omega)x(\omega)\right\|$, where x: $\Omega\to S$ is an S-valued random variable, and $n\in N$ (refer to [1]).

Now, considering F_1: $\Omega\times X_1\to X_1$ and F_2: $\Omega\times X_2\to X_2$ as (k_1, n_1) and (k_2, n_2) rotative random operators respectively, what can be said about the random operator F on $X_1\otimes_\gamma X_2$?

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