gr-compactness in topological spaces

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Abstract

In this paper, we introduce the new concepts gr-compactness in topological spaces and obtain some of their properties using gr-closed sets.

Keywords: gr-closed sets, gr-continuous maps and gr-compactness.

1 Introduction

The notions of compactness is useful and fundamental notions of not only general topology but also of other advanced branches of mathematics. Many researchers [1-7] have analyzed the basic properties of compactness. The notions of compactness resulted in motivating mathematicians to generalize these notions further.

Bhattacharya S. [8] introduced and studied the properties of gr-closed sets in topological spaces. The aim of this paper is to study gr-compactness using gr-closed set and also discuss some of their properties

2 Preliminaries

Throughout this paper (X, τ), (Y, σ) (or simply X and Y) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of (X, τ), cl(A) and Int(A) denote the closure of A and interior of A respectively.

Definition 2.1. Let (X, τ) be a topological space. Then, a subset A of (X, τ) is called gr-closed set [8] if rcl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).

The complement of the above mentioned gr-closed set is gr-open set.

Definition 2.2. A function f : (X, τ) → (Y, σ) is called

(i) gr-continuous [9] if the inverse image of every closed set in (Y, σ) is gr-closed in (X, τ).
(ii) gr-irresolute [9] if the inverse image of every gr-closed set in (Y, σ) is gr-closed in (X, τ).

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3  

gr-compactness

Definition 3.1. A collection \( \{A_i : i \in I\} \) of gr-open sets in a topological space \( X \) is called a gr-open cover of a subset \( B \) of \( X \) if \( B \subseteq \bigcup \{A_i : i \in I\} \) holds.

Definition 3.2. A topological space \( X \) is gr-compact if every gr-open cover of \( X \) has a finite subcover.

Definition 3.3. A subset \( B \) of a topological space \( X \) is said to be gr-compact relative to \( X \) if, for every collection \( \{A_i : i \in I\} \) of gr-open subsets of \( X \) such that \( B \subseteq \bigcup \{A_i : i \in I\} \) there exists a finite subcover \( I_0 \) of \( I \) such that \( B \subseteq \bigcup \{A_i : i \in I_0\} \).

Definition 3.4. A subset \( B \) of a topological space \( X \) is said to be gr-compact if \( B \) is gr-compact as a subspace of \( X \).

Theorem 3.1. Every gr-closed subset of a gr-compact space \( X \) is gr-compact relative to \( X \).

Proof. Let \( A \) be gr-closed subset of gr-compact space \( X \). Then, \( A^* \) is gr-open in \( X \). Let \( M = \{G_\alpha : \alpha \in I\} \) be a cover of \( A \) by gr-open sets in \( X \). Then, \( M^* = M \cup A^* \) is a gr-open cover of \( X \). Since \( X \) is gr-compact, \( M^* \) is reducible to a finite subcover of \( X \), say \( X = G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_m} \cup A^* \), \( G_{\alpha_k} \in M \). But, \( A \) and \( A^* \) are disjoint hence \( A \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_m} \), \( G_{\alpha_k} \in M \), which implies that any gr-open cover \( M \) of \( A \) contains a finite subcover. Therefore, \( A \) is gr-compact relative to \( X \). Thus, every gr-closed subset of gr-compact space \( X \) is gr-compact. \( \Box \)

Theorem 3.2. Every gr-compact space is compact.

Proof. Let \( X \) be a gr-compact space. Let \( \{A_i : i \in I\} \) be an open cover of \( X \). Then \( \{A_i : i \in I\} \) is a gr-open cover of \( X \) as every open set is gr-open set. Since \( X \) is gr-compact, the gr-open cover \( \{A_i : i \in I\} \) of \( X \) has a finite subcover, say \( \{A_i : i = 1, \ldots, n\} \) for \( X \). Hence \( X \) is compact. \( \Box \)

Definition 3.5. A function \( f : X \longrightarrow Y \) is said to be gr-continuous [9] if \( f^{-1}(F) \) is gr-closed in \( X \) for every closed set \( F \) of \( Y \).

Definition 3.6. A function \( f : X \longrightarrow Y \) is said to be gr- irresolute [9] if \( f^{-1}(F) \) is gr-closed in \( X \) for every gr-closed set \( F \) of \( Y \).

Theorem 3.3. Let \( f : X \rightarrow Y \) be surjective, gr-continuous function. If \( X \) is gr-compact, then \( Y \) is compact.

Proof. Let \( \{A_i : i \in I\} \) be an open cover of \( Y \). Since \( f \) is gr-continuous function, then \( \{f^{-1}(A_i) : i \in I\} \) is gr-open cover of \( X \) has a finite subcover, say \( \{f^{-1}(A_i) : i = 1, \ldots, n\} \). Therefore, \( X = \bigcup_{i=1}^{n} f^{-1}(A_i) \) which implies \( f(X) = \bigcup_{i=1}^{n} f(A_i) \). Since \( f \) is surjective, \( Y = \bigcup_{i=1}^{n} f(A_i) \). Thus, \( \{A_1, A_2, \ldots, A_n\} \) is a finite subcover of \( \{A_i : i \in I\} \) for \( Y \). Hence \( Y \) is compact. \( \Box \)

Theorem 3.4. If a map \( f : X \rightarrow Y \) is gr- irresolute and a subset \( B \) of \( X \) is gr-compact relative to \( X \), then the image \( f(B) \) is gr-compact relative to \( Y \).

Proof. Let \( \{A_\alpha : \alpha \in I\} \) be any collection of gr-open subsets of \( Y \) such that \( f(B) \subseteq \bigcup \{A_\alpha : \alpha \in I\} \). Then, \( B \subseteq \bigcup \{f^{-1}(A_\alpha) : \alpha \in I\} \) holds. From the hypothesis, \( B \) is gr-compact relative to \( X \). Then, there exists a finite subset \( I_0 \) of \( I \) such that \( B \subseteq \bigcup \{f^{-1}(A_\alpha) : \alpha \in I_0\} \). Therefore, we have \( f(B) \subseteq \bigcup \{A_\alpha : \alpha \in I_0\} \), which shows that \( f(B) \) is gr-compact relative to \( Y \). \( \Box \)

4  Conclusion

In this paper, we have introduced gr-compactness in the topological spaces by using gr-closed sets and their properties were studied.
References


