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Pricing a European Put Option by Numerical Methods

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Abstract
This paper aims at introducing the concept of pricing options by applying numerical methods. In particular we focus on the pricing of a European Put Option by two numerical techniques, that is, the Monte-Carlo simulation and the Crank-Nicolson finite difference method. In the Monte-Carlo simulation method, the concept of a random walk is used in the simulation of the path followed by the underlying stock price. The Black-Scholes partial differential equation is approximated by using the Crank-Nicolson algorithm to obtain the Put Option price. The explicit price of the European Put Option is known, thus we will at the end of the exercise, compare the numerical prices obtained using these two techniques to the closed form price.

Keywords
Black-Scholes model, Stochastic Differential Equation, Options pricing, Monte-Carlo simulation, Crank-Nicholson algorithm

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1. Introduction

Numerical methods play an important role in the pricing of options, especially when there is no closed form solution or when the problem itself is too complicated to be solved analytically [1]. In this paper, we will discuss two crucial techniques used in option pricing: the Crank-Nicolson finite difference method and Monte-Carlo simulation. There are other forms of finite difference methods, for example the implicit and explicit methods, but the Crank-Nicolson method is considered because it is more accurate, unconditionally stable and converges to the solution faster [2].

According to [3], an option on a stock is a contract in which the owner is granted the right but not the obligation to trade on a given number of shares of a common stock at a fixed price K, called strike price, and at a predetermined date T, called the expiry. The holder will then have to decide whether to exercise the right or not, depending on the market price of the stock at that time as compared to the strike price K, and to the type of the option. The writer, who is the person selling the option contract, will then have no choice but to honour the agreement when the holder of the option decides to exercise the right.

Other examples of options include the American, Asian, Barrier and Bermudan Options. Each of them can be categorised as a Call or a Put Option. A call option is a financial contract in which the holder has the right but not the obligation to buy a certain number of shares of a stock at a predetermined price K, whereas a Put Option gives the holder the right but not the obligation to sell a certain amount of shares of stock at a price K. The main difference between the European and the American option is in the exercising period. The holder of a European option has to wait until maturity of the option in order to exercise the right, while the American option holder has the privilege of exercising the contract at any time up to the maturity date. The payoff for a European Put Option is

$$P(S,t) = \max(K - S_T, 0) \tag{1}$$

where S_T is the asset price at the expiry date T and K is the strike price. The strike price, also known as the exercise price, is the price at which the two parties on an option contract agree upon, such that the holder of the option will either buy or sell an underlying asset on the expiry date T at that price. The payoff diagram for a Put Option is given in figure 1 below.

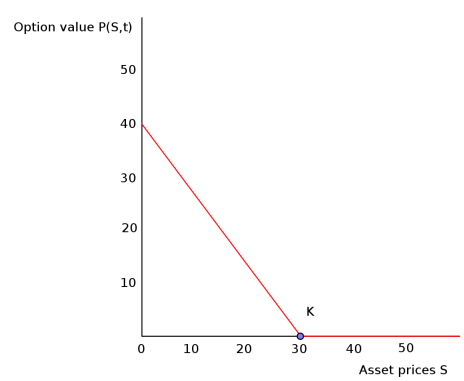

Figure 1. Payoff for a Put Option.

To enable the reader obtain a clear understanding of the objective of this paper, the work is organized as follows:

In Chapter 2, the reader is introduced to the Black-Scholes model, which plays a crucial role in the pricing of options. We also deal with the derivation of the Black-Scholes linear parabolic partial differential equation and we then state the Black-Scholes equation used for evaluating the explicit solution for the option.

Chapter 3 deals with pricing of a European Put Option by numerical methods. This is the central point of this paper, as the reader is taken through the derivation of the Crank-Nicolson scheme, discussion about Monte-Carlo simulation and the implementation of both schemes.

The comparison of the solutions obtained by the numerical methods to the closed form solution is shown in Chapter 4. Finally in Chapter 5 we present a brief conclusion and future problems.

2. Preliminaries

Here we present some key terms and mathematical definitions used in the construction of financial models.

2.1 Definition: σ-algebra

Let Ω be a non-empty set and let $\mathscr{F}$ be a collection of subsets of Ω. Then $\mathscr{F}$ is said to be a σ-algebra if ;

i) $\emptyset \in \mathscr{F}$,

ii) given that a set $A \in \mathscr{F}$ then the complement of A i.e $A^c \in \mathscr{F}$,

iii) whenever the sequence $\{A_i\} \in \mathscr{F}$ for $i = 1, 2, \cdots$, then $\cup_{n=1}^{\infty} A_n \in \mathscr{F}$. [4]

2.2 Definition: Probability measure

Let Ω be a non-empty set and let $\mathscr{F}$ be a σ-algebra of subsets of Ω. Then a function that assigns every set $A \in \mathscr{F}$ to a number in $[0,1]$ is called a probability measure $\mathbb{P}$. For this case we denote by $\mathbb{P}(A)$ the probability of A and it is such that ;

i) $\mathbb{P}(\Omega) = 1$,

ii) if $A_1, A_2, \cdots$ is a sequence of disjoint sets in $\mathscr{F}$ then

$$\mathbb{P}(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n) .$$

The pair $(\Omega, \mathscr{F})$ is called a measurable space while the triple $(\Omega, \mathscr{F}, \mathbb{P})$ is called a probability space. [4]

2.3 Definition: Filtration

Let Ω be a non-empty set and T be a fixed positive number and assume that for each $t \in [0,T]$ there is a $\sigma-$algebra $\mathscr{F}(t)$. Suppose $s \leq t$ then every set $\mathscr{F}(s)$ is also in $\mathscr{F}(t)$. A filtration is a collection of $\sigma-$algebras $\mathscr{F}(t)$ for $0 < t < T$. [4]

2.4 Definition: Adapted Process

Let ω be a non-empty sample space equipped with a filtration $\mathscr{F}(t)$, $0 < t < T$. Let $X(t)$ be a collection of random variable indexed by $t \in [0,T]$. Then given that for each t, the random variable $X(t)$ is $\mathscr{F}$-measurable, then the collection of these random variables is said to be an adapted stochastic process. [4]

2.5 Theorem: Girsanov Theorem

Let $B(t)$ where $0 < t < T$ be a Brownian motion on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ where $\mathscr{F}(t)$, is assumed to be the filtration for this Brownian motion. Let $\mathscr{A}(t)$ be an adapted process. Define

$$Z(t) = \exp\left(-\int_0^t \mathscr{A}(u)\mathrm{d}B(u) - \frac{1}{2}\int_0^t \mathscr{A}^2(u)\mathrm{d}u \right) ,$$

$$\tilde{B}(t) = B(t) + \int_0^t \mathscr{A}(u)\mathrm{d}u ,$$

and assume that

$$\mathbb{E}\int_0^T \mathscr{A}^2(u)Z^2(u)\mathrm{d}u < \infty .$$

Set $Z = Z(t)$. Then $\mathbb{E}(Z) = 1$ and under the probability measure $\tilde{\mathbb{P}}$, where,

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega)\mathrm{d}\mathbb{P}(\omega), \ \forall A \in \mathscr{F} ,$$

the process $\tilde{B}(t)$, $0 < t < T$ is a Brownian motion on a new probability space $(\Omega, \mathscr{F}, \tilde{\mathbb{P}})$. [4]

2.6 Definition: Stochastic Process

A stochastic process $\{X(t)\}_{t\in[0,T]}$ is a family of random variables parametrized by time t, i.e, $X(t)$ is a random variable for each $t \in [0,T]$. [5]

2.7 Definition: Martingale

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let T be a positive fixed number and let $\mathscr{F}(t)$, $0 \leq t \leq T$ be a filtration of sub-σ-algebra of $\mathscr{F}$. Then an adapted stochastic process $M(t)$, $0 \leq t \leq T$ is called a martingale process if

$$\mathbb{E}\Big[M(t)|\mathscr{F}(s)\Big] = M(s)$$

for all $0 \leq s \leq t \leq T$. [4]

2.8 Definition: Brownian Motion/Wiener process

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space where $\mathbb{P}$ is the probability measure, $\mathscr{F}$ is a collection of sets whose probabilities are influenced by the measure $\mathbb{P}$ and Ω the sample space. Then $W(t)$ $t \geq 0$ is a standard Brownian motion or Wiener process if the following conditions are satisfied ;

i) $W(t)$ is a continuous function.

ii) $W(0) = 0$.

iii) The increments $W(t_1)-W(t_0), W(t_2)-W(t_1), \cdots, W(t_n)-W(t_{n-1})$ are independent for all $0=t_0<t_1<\cdots<t_n$.

iv) $W(t_{k+1})-W(t_k) \sim N(0, t_{k+1}-t_k)$. [4]

Figure 2 below demonstrates the sample path followed by Brownian motion of stock prices.

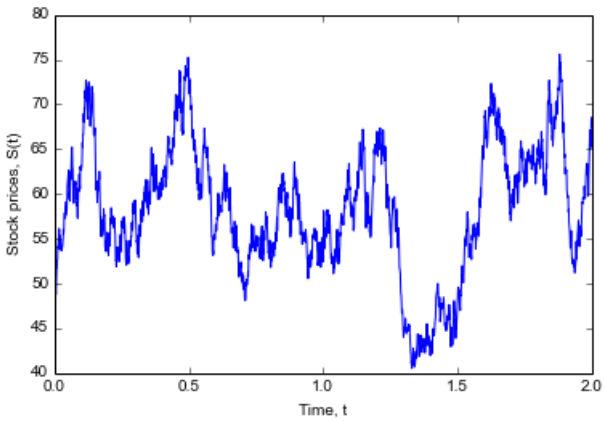

Figure 2. Brownian motion.

2.9 Definition: Geometric Brownian Motion

A geometric brownian motion is a stochastic process $S(t)$ which satisfies the stochastic differential equation (SDE)

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t \ ,$$

where W_t is the Wiener process, μ and σ are constant parameters called the drift and rate of volatility respectively. Given that $S(0)=S_0$, the solution of the SDE above is

$$S_t = S_0 \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W_t\right) \ ,$$

where W_t is the Wiener increment. [6]

2.10 Definition: Arbitrage

Arbitrage is the process where an investor has the possibility of gaining profit from zero investment without taking any risk. [5]

2.11 Definition: Normal Distribution

A random variable X is said to be normally distributed with mean μ and variance σ^2 if it's probability density function is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right), \ -\infty < x < \infty \ .$$

[7]

2.12 Theorem: The Central Limit Theorem

Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed random variables each with mean $\mu < \infty$ and variance $\sigma^2 < \infty$. Let $S_n = \sum_{i=1}^{n} X_i$. Then as $n \to \infty$, $S_n \sim N(n\mu, n\sigma^2)$. Hence for any x we have that

$$P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right) \sim \Phi(x) \text{ as } n \to \infty \ .$$

[8]

2.13 Definition: Log-normal Distribution

A random variable X is said to be log-normally distributed if $\log(X)$ is normally distributed and the probability density function of the random variable X is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}x\sigma} \exp\left(\frac{-(\log(x)-\mu)^2}{2\sigma^2}\right), \ x > 0 \ .$$

[9]

2.14 Definition: European Put Option

This is a type of financial contract giving the holder the right but not the obligation to sell an underlying asset at a predetermined strike price K at the expiry time T. [10]

3. The Black-Scholes Model

The greatest success achieved in the pricing of European stock options was made in the early 1970's by Fisher Black, Myron Scholes and Robert Merton. This then led to the development of what is now famously known as the Black-Scholes-Merton (Black-Scholes) model which has greatly assisted traders in the pricing and hedging of derivatives. Their work led to their acknowledgement by the award of the Nobel Prize for Economics in 1997 [11].

Let us now derive the Black-Scholes partial differential equation and state the Black-Scholes formula for the pricing of European options.

3.1 Stochastic Differential Equation

A stochastic differential equation (SDE) is a differential equation in which one of the terms in the equation follows a random process.

Consider the following geometric Brownian motion which represents the price dynamics of a non-dividend paying stock

$$\mathrm{d}S(t) = S(t)\left(\mu \mathrm{d}t + \sigma \mathrm{d}B(t)\right) \ , \tag{2}$$

where S is the asset value, μ is the appreciation rate (drift), which is a measure of the average rate of the growth of the asset price and σ is the volatility coefficient which is a measure of standard deviation of the returns. The price is defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. The term $B(t)$ is a standard Brownian motion with respect to $\mathbb{P}$. We now apply the

concept of Girsanov change of probability measure, which is described in [12], to obtain a new probability distribution $\widehat{\mathbb{P}}$ which is an equivalent martingale measure.

The drift component in equation (2) above can be split into its risk-free component r and $(\mu - r)$, therefore we have

$$\frac{\mathrm{d}S(t)}{S(t)} = r\mathrm{d}t + (\mu - r)\mathrm{d}t + \sigma \mathrm{d}B(t) \ . \tag{3}$$

Hence applying the Girsanov transformation we have

$$\sigma\widehat{B}(t) = (\mu - r)t + \sigma B(t) \ ,$$
$$\mathrm{d}\widehat{B}(t) = \frac{(\mu - r)}{\sigma}\mathrm{d}t + \mathrm{d}B(t) \ ,$$

and substituting this into equation (3) we have

$$\begin{aligned}\frac{\mathrm{d}S(t)}{S(t)} &= r\mathrm{d}t + (\mu - r)\mathrm{d}t + \sigma\left(\frac{\sigma\mathrm{d}\widehat{B}(t) - (\mu - r)\mathrm{d}t}{\sigma}\right) \ , \\ &= r\mathrm{d}t + (\mu - r)\mathrm{d}t + \sigma\mathrm{d}\widehat{B}(t) - (\mu - r)\mathrm{d}t \ .\end{aligned}$$

This results in the driftless stochastic differential equation

$$\mathrm{d}S(t) = S(t)\left(r\mathrm{d}t + \sigma\mathrm{d}\widehat{B}(t)\right) \ . \tag{4}$$

3.2 Itô's Process

An Itô process is a stochastic process $\{X_t, t \geq 0\}$ given by

$$X_t = X_0 + \int_0^t a(\tau,\omega)\mathrm{d}\tau + \int_0^t b(\tau,\omega)\mathrm{d}B_\tau \ .$$

The corresponding SDE is given by

$$\mathrm{d}X_t = a\mathrm{d}\tau + b\mathrm{d}B_t \ ,$$

where $a(\tau,\omega)$ and $b(\tau,\omega)$ are adapted random functions. [13]

3.3 Itô's Lemma

[13] Let $f(S,t)$ be a twice continuously differentiable function on $[0,\infty) \times \mathbb{R}$ and let S_t be an Itô process

$$\mathrm{d}S_t = a_t\mathrm{d}t + b_t\mathrm{d}B_t \ , t \geq 0 \ .$$

Taking the Taylor series expansion of f we have

$$\begin{aligned}\mathrm{d}f_t &= \frac{\partial f}{\partial S_t}\mathrm{d}S_t + \frac{\partial f}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2 F}{\partial S_t^2}(\mathrm{d}S_t)^2 \\ &\quad + \text{higher order terms} \ .\end{aligned}$$

Hence ignoring higher order terms and substituting for $\mathrm{d}S_t$ we obtain

$$\mathrm{d}f_t = \frac{\partial f}{\partial S_t}(a_t\mathrm{d}t + b_t\mathrm{d}B_t) + \tag{5}$$
$$\frac{\partial f}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}(a_t\mathrm{d}t + b_t\mathrm{d}B_t)^2 \ ,$$
$$= \frac{\partial f}{\partial S_t}(a_t\mathrm{d}t + b_t\mathrm{d}B_t) + \frac{\partial f}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}b_t^2\mathrm{d}t \ , \tag{6}$$
$$= \left(\frac{\partial f}{\partial S_t}a_t + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}b_t^2\right)\mathrm{d}t + \frac{\partial f}{\partial S_t}b_t\mathrm{d}B_t \ . \tag{7}$$

This is the Itô's formula, where we have used the relation $\mathrm{d}t \cdot \mathrm{d}t = \mathrm{d}B_t \cdot \mathrm{d}t = \mathrm{d}t \cdot \mathrm{d}B_t = 0$ and $\mathrm{d}B_t \cdot \mathrm{d}B_t = \mathrm{d}t$. Equation (7) plays a very important role in the field of mathematical modelling in finance and specifically in the pricing of derivatives as it is used in the derivation of the Black-Scholes PDE. When this PDE is solved, the value of f will represent the price of the derivative.

We conclude that f_t follows the Itô's process and the drift rate is given by

$$\mu = \left[\frac{\partial f}{\partial S_t}a_t + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}b_t^2\right] \ ,$$

and the variance is

$$\sigma^2 = \frac{\partial^2 f}{\partial S_t^2}b_t^2 \ .$$

Given that the variable $S(t)$, representing stock price, follows a geometric Brownian motion, then it follows the stochastic differential equation (2). Therefore, for a general function $F(S,t)$, Itô's lemma will give

$$\mathrm{d}F = \left[\mu S\frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial S^2}\right]\mathrm{d}t + \sigma S\frac{\partial F}{\partial S}\mathrm{d}B_t \ , \tag{8}$$

where μ and σ are constants representing the drift rate and the rate of volatility respectively.

Example

Consider a stock price S which follows the random process

$$\mathrm{d}S(t) = S(t)\Big(\mu\mathrm{d}t + \sigma\mathrm{d}B(t)\Big) \ .$$

Let $F(S,t) = \log S$ then,

$$\frac{\partial F}{\partial S} = \frac{1}{S}, \ \frac{\partial^2 F}{\partial S^2} = \frac{-1}{S^2} \text{ and } \frac{\partial F}{\partial t} = 0 \ .$$

Substituting these values into (8) we obtain

$$\mathrm{d}(\log S) = (\mu - \frac{\sigma^2}{2})\mathrm{d}t + \sigma\mathrm{d}B_t \ .$$

This result implies that $\log S$ is a Brownian motion whose drift parameter is $(\mu - \frac{\sigma^2}{2})$ and variance is σ^2. Taking the integral from 0 to T we have

$$\int_0^T \mathrm{d}(\log S) = \int_0^T (\mu - \frac{\sigma^2}{2})\mathrm{d}t + \sigma\int_0^T \mathrm{d}B_t \ ,$$
$$\log S_T - \log S_0 = (\mu - \frac{\sigma^2}{2})T + \sigma(B(T) - B(0)) \ ,$$
$$S_T = S_0 \exp\left((\mu - \frac{\sigma^2}{2})T + \sigma Z\sqrt{T}\right) \ ,$$

where $Z \sim N(0,1)$. This clearly shows that stock prices obeys the log-normal distributions [6].

3.4 Black-Scholes Partial Differential Equation

To obtain the Black-Scholes partial differential equation we follow the process described in [14] and consider the SDE (2). The term $B(t)$ is a random variable which follows a normal distribution whose mean is 0 and variance is $\mathrm{d}t$. Therefore we can write

$$dB(t) = Z\sqrt{dt}\ ,$$

where Z is a standardised normal distribution, i.e, $Z \sim N(0,1)$ with probability density function given by

$$f(Z) = \frac{1}{\sqrt{2}}\exp(-\frac{1}{2}Z^2)\ ,$$

for $-\infty < Z < \infty$. The expectation can be defined for any function F by

$$\mathbb{E}[F(\cdot)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(Z)\exp(-\frac{1}{2}Z^2)\mathrm{d}Z\ .$$

Hence we have that

$$\mathbb{E}[Z] = 0\ ,$$
$$\mathbb{E}[Z^2] = 1\ .$$

If there is no uncertainty about the price, that is when $\sigma = 0$, and μ is a constant, then we get an ODE. When solved we obtain an exponential growth in the value of the asset, i.e,

$$\frac{\mathrm{d}S}{S} = \mu\mathrm{d}t\ ,$$
$$\implies S = S_0\exp^{\mu(t-t_0)}\ ,$$

where S_0 is the value of the asset at $t = t_0$. This means that when $\sigma = 0$, the asset price is totally deterministic and thus the future price can be predicted.

To obtain the Black-Scholes PDE, let $f(S)$ be a smooth function of the asset price S and let us assume that S is not stochastic. Taking the Taylors series expansion we have

$$\begin{aligned}
\mathrm{d}f &= \frac{\mathrm{d}f}{\mathrm{d}S}\mathrm{d}S + \frac{1}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}S^2}\mathrm{d}S^2 + \text{ higher order terms}\ ,\\
&= \frac{\mathrm{d}f}{\mathrm{d}S}(\sigma S\mathrm{d}B(t) + \mu S\mathrm{d}t) + \frac{1}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}S^2}(\sigma S\mathrm{d}B(t) + \mu S\mathrm{d}t)^2,\\
&= \frac{\mathrm{d}f}{\mathrm{d}S}(\sigma S\mathrm{d}B(t) + \mu S\mathrm{d}t)\\
&+ \frac{1}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}S^2}(\sigma^2 S^2\mathrm{d}B(t)^2 + 2\mu\sigma S^2\mathrm{d}t\ \mathrm{d}B(t) + \mu^2 S^2\mathrm{d}t^2)\ ,\\
&= \frac{\mathrm{d}f}{\mathrm{d}S}(\sigma S\mathrm{d}B(t) + \mu S\mathrm{d}t) + \frac{1}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}S^2}(\sigma^2 S^2\mathrm{d}t), \qquad (9)\\
&\text{since } \mathrm{d}B(t)^2 \longrightarrow \mathrm{d}t \ \text{ as } \ \mathrm{d}t \longrightarrow 0 \text{ (Itô's lemma)}\\
&= \sigma S\frac{\mathrm{d}f}{\mathrm{d}s}\mathrm{d}B(t) + (\mu S\frac{\mathrm{d}f}{\mathrm{d}S} + \frac{1}{2}\sigma^2 S^2\frac{\mathrm{d}^2 f}{\mathrm{d}S^2})\mathrm{d}t\ . \qquad (10)
\end{aligned}$$

The general case for equation (10) above is obtained by considering a function $F(S,t)$ of the random variable S and time t. Since S and t are independent variables we will use partial derivatives in the expansion. Therefore taking the expansion of $F(S+dS, t+dt)$ in Taylor series about (S,t) we have

$$\mathrm{d}F = \frac{\partial F}{\partial S}dS + \frac{\partial F}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\mathrm{d}S^2 + \cdots\ . \qquad (11)$$

By neglecting the remainder terms and using the *Itô's lemma* and the SDE (2), we can write equation (11) above as

$$\mathrm{d}F(S,t) = \sigma S\frac{\partial F}{\partial S}\mathrm{d}B(t) + \left(\mu S\frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial S^2}\right)\mathrm{d}t\ . \qquad (12)$$

Here we determine a portfolio π consisting of one option F and a number $-\Delta$ of the underlying asset [14]. Therefore this portfolio would have value

$$\pi = F - \Delta S\ , \qquad (13)$$

hence we have

$$\mathrm{d}\pi = \mathrm{d}F - \Delta\mathrm{d}S\ . \qquad (14)$$

Substituting equations (2) and (12) into (14) above we find that the portfolio π follows the random walk

$$\begin{aligned}
\mathrm{d}\pi &= \sigma S\frac{\partial F}{\partial S}\mathrm{d}B(t) + \left(\mu S\frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial S^2}\right)\mathrm{d}t\\
&- \Delta(\mu S\mathrm{d}t + \sigma S\mathrm{d}B(t))\ , \qquad (15)\\
&= \sigma S\left(\frac{\partial F}{\partial S} - \Delta\right)\mathrm{d}B(t) +\\
&\left(\mu S\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial S^2} + \frac{\partial F}{\partial t} - \mu S\Delta\right)\mathrm{d}t\ . \qquad (16)
\end{aligned}$$

We choose Δ to eliminate the random component $B(t)$ in this random walk. Hence

$$\Delta = \frac{\partial F}{\partial S}\ , \qquad (17)$$

which represents the rate of change of the value of our option with respect to S. The portfolio now, no longer follows a random walk and we have

$$\mathrm{d}\pi = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial S^2}\right)\mathrm{d}t\ . \qquad (18)$$

This implies that the increment in this portfolio is wholly deterministic.

Black and Scholes then considered the no-arbitrage principle which simply states that the value of a portfolio must be equal (on average) to the value of the portfolio at a risk-free interest rate, r. If this was not the case then some individuals, knowledgeable about the market would risklessly profit. Thus the return on an amount π invested in a riskless asset would yield a growth of $r\pi\mathrm{d}t$ in time $\mathrm{d}t$.

Hence we have that

$$r\pi \mathrm{d}t = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}\right) \mathrm{d}t \ . \tag{19}$$

Substituting equations (13) and (17) into (19) and dividing through by dt we obtain

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + rS\frac{\partial F}{\partial S} - rF = 0 \ . \tag{20}$$

This is the Black-scholes partial differential equation. We note that this PDE is a parabolic equation and it does not contain the growth parameter μ, meaning that the value of an option is independent of the nature of the growth of the asset.

3.5 The Black-Scholes Equation

The Black-Scholes PDE, equation (20) has an infinite number of solutions. Therefore to obtain a unique solution, hence avoiding the possibilities of arbitrage, boundary conditions must be imposed. These boundary conditions specify how the solution of the problem behaves in some region of the solution domain. We note that the highest derivative with respect to S is a second order derivative and the highest derivative with respect to t is a first order derivative. Hence we impose two conditions about the behaviour of the solution in S and one in t [14].

Let us now introduce the boundary conditions of the PDE as described in [14] for a European Put Option whose payoff is denoted by $P(S,t)$. The final condition for the option, that is, the value at the terminal point where time $t = T$, is given by

$$\begin{aligned} P(S,T) &= (K-S)^+ \ , \\ &= \max(K-S,0) \ , \end{aligned}$$

where K is the strike price and S is the asset value.

Now suppose that the value of the asset is zero. Then the payoff for the option will automatically be K and the present value of the option received at time T is

$$P(0,t) = Ke^{-r(T-t)} \ .$$

Finally, we consider the case when the asset value is large enough. Therefore as S increases, the option value tend to be worthless,

$$P(S,t) = 0 \quad \text{as } S \to \infty \ .$$

The closed form solution for the European Put Option problem (20) together with the boundary conditions described above is fully explained in [10] where the following assumptions were made in deriving the solution equation ;

i) The risk free interest rate r, and asset volatility are constant.

ii) The underlying asset prices follow a lognormal random walk.

iii) There are no arbitrage opportunities hence all portfolios would earn equal returns.

iv) During delta hedging of a portfolio, no transaction costs are incurred.

v) The short selling of securities is allowed and the assets are divisible.

vi) No dividends are paid during the entire period of the contract.

vii) There is continuous trading of the underlying assets.

Therefore the explicit solution for the European Put Option is given by

$$P(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1) \ ,$$

where

$$\begin{aligned} d_1 &= \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \ , \\ d_2 &= \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \ . \end{aligned}$$

The value of d_2 can also be approximated as $d_2 \approx d_1 - \sigma\sqrt{T-t}$.

The parameter $N(\cdot)$ is the cumulative probability distribution function for a standardised normal random variable

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy \ . \tag{21}$$

To obtain the value of $N(x)$ we use the idea described in [15], and we proceed to compute the error function as

$$erf(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-y^2} dy \ ,$$

which simplifies equation (21) to

$$N(x) = \frac{1}{2}\left(1 + erf(\frac{x}{\sqrt{2}})\right) \ .$$

4. Pricing The European Put Option

4.1 Finite-Difference Methods

The concept of finite difference methods as described in [11] has major applications in the valuation of derivatives. This method aims at solving the differential equation satisfying the underlying derivative. For this case, we attempt to obtain the numerical solution of the Black-Scholes PDE by converting it into a set of difference algebraic equations which can then be solved iteratively.

The most common forms of finite difference methods used in the computation of this PDE are the implicit method, the explicit method and the Crank-Nicolson scheme. The three methods are similar in computation and implementation,

they only differ in the accuracy, stability and execution speed. In this paper, we consider the Crank-Nicolson method in pricing a European Put Option because it is more accurate to $O\Big((\Delta t)^2, (\Delta S)^2\Big)$, unconditionally stable and converges faster than the other two methods of finite difference [2].

Therefore to achieve this objective, the main things to consider in the formulation of the partial differential equation problem are ;

i) the PDE,

ii) the region of space-time to which the PDE is defined,

iii) the ancillary boundary and initial conditions to be met.

This means that the solution of this problem is defined on the (S,t) plane.

Equation Discretization

The PDE and the boundary conditions need to be discretized either by using forward or backward forms of approximations. The option value is a function of two independent variables S and t and thus we discretize the equation with respect to S and t.

Now according to [11] we suppose that the option will mature after time T. We then divide this time into N equally spaced intervals each of length Δt. Thus we have $N+1$ points

$$0, \Delta t, 2\Delta t, \cdots, N\Delta t \ ,$$

where $\Delta t = \frac{T}{N}$.

Similarly we need to discretize the stock price. Assume that the highest price of the stock is $S_{\max}$, at which point, the Put Option becomes worthless. We also know that the stock prices cannot go below zero and and since we need the highest stock price to be large enough, we assume that $S_{\max} = 2S_0$ [1]. Dividing $S_{\max}$ into M equally spaced intervals of length ΔS, we have $M+1$ points,

$$0, \Delta S, 2\Delta S, \cdots, M\Delta S \ ,$$

where $\Delta S = \frac{S_{\max}}{M}$. The (S,t) plane will have the axis ranges $(0,\ S_{\max})$ and $(0,\ T)$ and thus the grid will have $(M+1)\times(N+1)$ points.
The (S,t) plane is shown in figure 3 below [11].

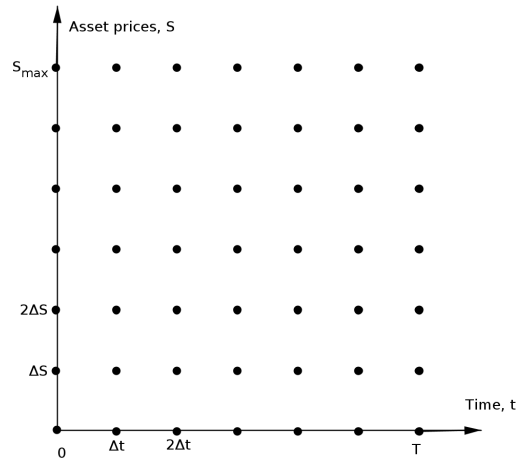

Figure 3. The (S,t) plane.

The point (n,m) on the grid corresponds to the stock price $m\Delta S$ where $m = 0,1,2,\cdots,M$, at time $n\Delta t$ where $n = 0,1,2,\cdots,N$. Therefore in this method we denote by $F_{n,m}$ the value of the option at time t_n when the asset price is S_m. That is,

$$F_{n,m} = F(n\Delta t, m\Delta S) = F(t_n, S_m) \ .$$

From the Black-Scholes PDE, we need to replace the partial derivatives by approximations based on the Taylor series expansion about points of interest. Therefore we need to obtain the approximation for the first order partial derivative with respect to t and S and the second order partial derivative with respect to S. By considering the Taylor series expansions we end up with the relevant finite difference approximations.

The expansion of $F(t, S+\Delta S)$ in Taylor series gives

$$F(t,S+\Delta S) = F(t,S) + F_S(t,S)\Delta S + \frac{1}{2}F_{SS}(t,S)(\Delta S)^2 + O\Big((\Delta S)^3\Big) \ , \tag{22}$$

and thus the first partial derivative is

$$F_S(t,S) = \frac{F(t,S+\Delta S) - F(t,S)}{\Delta S} + O(\Delta S) \ , \approx \frac{F_{n,m+1} - F_{n,m}}{\Delta S} \ . \tag{23}$$

This is the forward difference approximation.

Similarly the expansion of $F(t, S-\Delta S)$ is given by

$$F(t,S-\Delta S) = F(t,S) - F_S(t,S)\Delta S + \frac{1}{2}F_{SS}(t,S)(\Delta S)^2 - O\Big((\Delta S)^3\Big) \ , \tag{24}$$

this implies that the first partial derivative is

$$F_S(t,S) = \frac{F(t,S) - F(t,S-\Delta S)}{\Delta S} + O(\Delta S) \ , \approx \frac{F_{n,m} - F_{n,m-1}}{\Delta S} \ . \tag{25}$$

This results to backward difference form of approximation. Subtracting equation (24) from equation (22) and taking the first order partial derivative we obtain the central difference approximation

$$F_S(t,S) = \frac{F(t,S+\Delta S) - F(t,S-\Delta S)}{2\Delta S} + O\Big((\Delta S)^2\Big) , \approx \frac{F_{n,m+1} - F_{n,m-1}}{2\Delta S} . \quad (26)$$

To obtain the approximation of the second order partial derivative of the stock price, we add equations (22) and (24) thereby obtaining

$$F_{SS}(t,S) = \frac{F(t,S+\Delta S) - 2F(t,S) + F(t,S-\Delta S)}{\Delta^2 S} + O\Big((\Delta S)^2\Big) , \approx \frac{F_{n,m+1} - 2F_{n,m} + F_{n,m-1}}{(\Delta S)^2} . \quad (27)$$

For the time derivatives we have the approximations

$$F(t+\Delta t,S) = F(t,S) + \Delta t F_t(t,S) + (\Delta t)^2 F_{tt}(t,S) + O\Big((\Delta t)^3\Big) . \quad (28)$$

Hence the forward difference derivative

$$F_t(t,S) = \frac{F(t+\Delta t,S) - F(t,S)}{\Delta t} + O(\Delta t) , \approx \frac{F_{n+1,m} - F_{n,m}}{\Delta t} . \quad (29)$$

Similarly for backward difference in time we have

$$F(t-\Delta t,S) = F(t,S) - \Delta t F_t(t,S) + (\Delta t)^2 F_{tt}(t,S) - O\Big((\Delta t)^3\Big) , \quad (30)$$

and therefore we have

$$F_t(t,S) = \frac{F(t,S) - F(t-\Delta t,S)}{\Delta t} + O(\Delta t) , \quad F_t(t,S) \approx \frac{F_{n,m} - F_{n-1,m}}{\Delta t} . \quad (31)$$

Substituting these approximations into the PDE yields a difference equation which will be used in obtaining the approximation for the option value $F(t,S)$.

5. Boundary Conditions

The Black-Scholes PDE has an infinite number of solutions. Therefore since the price of the European option whose payoff is given by $\max(K - S_T, 0)$ must be unique, then we need to impose some boundary and initial conditions.

It is clear that when the stock value is zero, the Put Option will be worth the strike price K at time T, and discounting to time $t = 0$ we have

$$F_{n,0} = Ke^{-r(N-n)\Delta t} \quad \text{for } n = 0,1,2,\cdots,N . \quad (32)$$

We can also see that the value of the option will tend to zero if the value of the underlying asset increases. If we take the maximum value of the underlying asset $S_{max} = S_M$, then

$$F_{n,M} = 0 \quad \text{for} \quad n = 0,1,2,\cdots,N . \quad (33)$$

Since the value of the Put Option at time T is known, we can impose the initial condition

$$F_{N,m} = \max(K - m\Delta S, 0) \quad \text{for} \quad m = 0,1,2,\cdots,M . \quad (34)$$

The initial condition gives us the value of the option F at the maturity time T and not at the beginning. Therefore we need to discount to the initial time zero. The price of the option at time $t = 0$ for the initial price S_0 is given by $F_{0,\frac{M}{2}}$ and $F_{0,\frac{M+1}{2}}$ when the number of steps M is even and odd respectively [1]. This is because we took the assumption that $S_{\max} = 2S_0$.

6. Explicit Method

The explicit method is obtained by taking the backward difference approximation in time, that is, equation (31) and together with equations (26) and (27), substitute into the PDE taking note that $S = m\Delta S$. Therefore we have

$$\frac{F_{n,m} - F_{n-1,m}}{\Delta t} + rm\Delta S \frac{F_{n,m+1} - F_{n,m-1}}{2\Delta S} + \frac{1}{2}\sigma^2 m^2 (\Delta S)^2 \frac{F_{n,m+1} - 2F_{n,m} + F_{n,m-1}}{(\Delta S)^2} - rF_{n,m} = 0 . \quad (35)$$

We can rewrite this so that the present values of the underlying asset depends on the future values. Hence we have

$$F_{n-1,m} = aF_{n,m-1} + bF_{n,m} + cF_{n,m+1} ,$$

where

$$a = \frac{\Delta t}{2}(\sigma^2 m^2 - rm) ,$$
$$b = 1 - \Delta t(\sigma^2 m^2 + r) ,$$
$$c = \frac{\Delta t}{2}(rm + \sigma^2 m^2) ,$$

for $n = N-1, N-2, \cdots, 1, 0$ and $m = 1, 2, \cdots M-1$.

If we assume that the values of $\frac{\partial F}{\partial S}$ and $\frac{\partial^2 F}{\partial S^2}$ at points (n,m) are the same as at point $(n+1,m)$, then equations (26) and (27) respectively become

$$F_S(t,S) \approx \frac{F_{n+1,m+1} - F_{n+1,m-1}}{2\Delta S} , \quad (36)$$

$$F_{SS}(t,S) \approx \frac{F_{n+1,m+1} - 2F_{n+1,m} + F_{n+1,m-1}}{\Delta^2 S} . \quad (37)$$

Therefore substituting (36), (37) and (29) into the PDE (20) we have

$$\frac{F_{n+1,m}-F_{n,m}}{\Delta t}+rm\Delta S\frac{F_{n+1,m+1}-F_{n+1,m-1}}{2\Delta S}-rF_{n,m}$$
$$+\frac{1}{2}\sigma^2m^2(\Delta S)^2\frac{F_{n+1,m+1}-2F_{n+1,m}+F_{n+1,m-1}}{(\Delta S)^2}=0\,. \tag{38}$$

We can rearrange this as

$$F_{n,m}=\frac{1}{1+r\Delta t}\Big(a_mF_{n+1,m-1}+b_mF_{n+1,m}+c_mF_{n+1,m+1}\Big)\,, \tag{39}$$

where the coefficients

$$a_m=\frac{\sigma^2m^2\Delta t}{2}-\frac{rm\Delta t}{2}\,,$$
$$b_m=1-\sigma^2m^2\Delta t\,,$$
$$c_m=\frac{rm\Delta t}{2}+\frac{\sigma^2m^2\Delta t}{2}\,,$$

for $n=N-1,N-2,\cdots,1,0$ and $m=1,2,\cdots M-1$. This is the explicit method which is accurate to $O\Big(\Delta t,(\Delta S)^2\Big)$. The coefficients above represent the risk neutral probabilities of the asset prices $S+\Delta S, S$ and $S-\Delta S$ at time $t+\Delta t$ and their sum is 1. If they are all non-negative, then they represent the probability that the underlying asset prices increases, decreases or remain constant. However at times these coefficients are negative. This brings numerical instability hence the results never converge to the required solution of the differential equation. They are negative when $m^2\sigma^2\Delta t>1$ and $m<\frac{r}{\sigma^2}$. [16]. Figure 4 below describes the Explicit method [11].

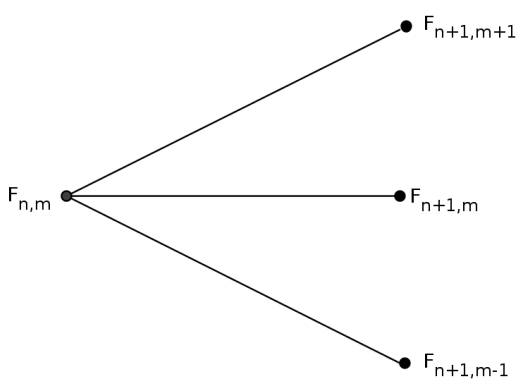

Figure 4. Explicit finite difference.

7. Implicit Method

We have seen that the problem with the explicit method is its instability condition. The implicit method tries to overcome this challenge. It aims at approximating the future prices of the underlying asset using the present values. It is obtained by substituting the forward difference approximation of the time derivative, equation (29), and the first and second order partial derivatives of the stock price given by equations (26) and (27) into the Black-Scholes PDE. Taking note that $S=m\Delta S$ we have

$$\frac{F_{n+1,m}-F_{n,m}}{\Delta t}+rm\Delta S\frac{F_{n,m+1}-F_{n,m-1}}{2\Delta S}$$
$$+\frac{1}{2}\sigma^2m^2(\Delta S)^2\frac{F_{n,m+1}-2F_{n,m}+F_{n,m-1}}{(\Delta S)^2}-rF_{n,m}=0\,. \tag{40}$$

This can be represented as

$$F_{n+1,m}=a^*F_{i,j-1}+b^*F_{i,j}+c^*F_{i,j+1}\,,$$

where

$$a^*=\frac{\Delta t}{2}(rj-\sigma^2m^2)\,,$$
$$b^*=1+\Delta t(\sigma^2m^2+r)\,,$$
$$c^*=\frac{-\Delta t}{2}(rj+\sigma^2m^2)\,,$$

for $n=N-1,N-2,\cdots,1,0$ and $m=1,2,\cdots M-1$.

This is the **implicit finite difference method** which is accurate to $O\Big(\Delta t,(\Delta S)^2\Big)$. The Implicit method is shown in figure 5 below [11].

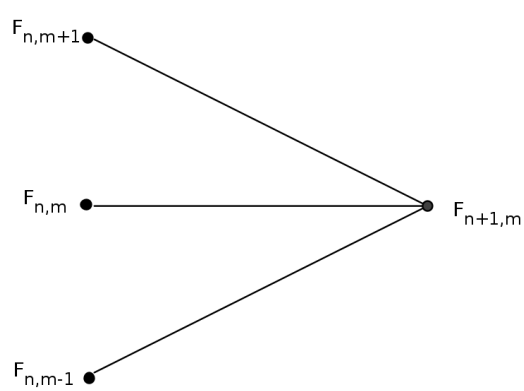

Figure 5. Implicit finite difference.

8. Crank-Nicolson Method

The Crank-Nicolson method is the average of the explicit and implicit methods .Therefore we have

$$\frac{1}{2}\Bigg(\frac{F_{n+1,m}-F_{n,m}}{\Delta t}+rm\Delta S\frac{F_{n+1,m+1}-F_{n+1,m-1}}{2\Delta S} + \frac{1}{2}\sigma^2m^2(\Delta S)^2\frac{F_{n+1,m+1}-2F_{n+1,m}+F_{n+1,m-1}}{(\Delta S)^2} - rF_{n+1,m}+\frac{F_{n+1,m}-F_{n,m}}{\Delta t}+rm\Delta S\frac{F_{n,m+1}-F_{n,m-1}}{2\Delta S} + \frac{1}{2}\sigma^2m^2(\Delta S)^2\frac{F_{n,m+1}-2F_{n,m}+F_{n,m-1}}{(\Delta S)^2}-rF_{n,m}\Bigg)=0\,. \tag{41}$$

This can be written as

$$F_{n+1,m}-F_{n,m} = \frac{-rm\Delta t}{4}\Big(F_{n,m+1}-F_{n,m-1}+F_{n+1,m+1}-F_{n+1,m-1}\Big) + \frac{r\Delta t}{2}(F_{n,m}+F_{n+1,m})-\frac{\sigma^2m^2\Delta t}{4}\Big(F_{n,m+1}-2F_{n,m} + F_{n,m-1}+F_{n+1,m+1}-2F_{n+1,m}+F_{n+1,m-1}\Big)\,.$$

Rearranging the result is

$$-\left(\frac{\sigma^2m^2\Delta t}{4}-\frac{rm\Delta t}{4}\right)F_{n,m-1} + \left(1+\frac{\sigma^2m^2\delta t}{2}+\frac{r\Delta t}{2}\right)F_{n,m} - \left(\frac{\sigma^2m^2\Delta t}{4}+\frac{rm\Delta t}{4}\right)F_{n,m+1} = \left(\frac{\sigma^2m^2\Delta t}{4}-\frac{rm\Delta t}{4}\right)F_{n+1,m-1} + \left(1-\frac{\sigma^2m^2\Delta t}{2}-\frac{r\Delta t}{2}\right)F_{n+1,m} + \left(\frac{\sigma^2m^2\Delta t}{4}+\frac{rm\Delta t}{4}\right)F_{n+1,m+1}\,. \tag{42}$$

We can rewrite equation (42) as

$$-\alpha_mF_{n,m-1}+(1-\beta_m)F_{n,m}-\gamma_mF_{n,m+1} = \alpha_mF_{n+1,m-1}+(1+\beta_m)F_{n+1,m}+\gamma_mF_{n+1,m+1}\,, \tag{43}$$

where

$$\alpha_m=\frac{\Delta t}{4}(\sigma^2m^2-rm)\,,$$
$$\beta_m=\frac{-\Delta t}{2}(\sigma^2m^2+r)\,,$$
$$\gamma_m=\frac{\Delta t}{4}(\sigma^2m^2+rm)\,,$$

for $n=N-1,N-2,\cdots,1,0$ and $m=1,2,\cdots M-1$. This equation can be represented in a tri-diagonal matrix as

$$\begin{pmatrix}(1-\beta_1) & -\gamma_1 & 0 & \cdots & 0\\ -\alpha_2 & (1-\beta_2) & -\gamma_2 & \cdots & 0\\ 0 & -\alpha_3 & (1-\beta_3) & \ddots & 0\\ \vdots & \vdots & \ddots & \ddots & -\gamma_{M-2}\\ 0 & 0 & 0 & -\alpha_{M-1} & (1-\beta_{M-1})\end{pmatrix} \times\begin{pmatrix}f_{n,1}\\ f_{n,2}\\ f_{n,3}\\ \vdots\\ f_{n,M-1}\end{pmatrix} = \begin{pmatrix}(1+\beta_1) & \gamma_1 & 0 & \cdots & 0\\ \alpha_2 & (1+\beta_2) & \gamma_2 & \cdots & 0\\ 0 & \alpha_3 & (1+\beta_3) & \ddots & 0\\ \vdots & \vdots & \ddots & \ddots & \gamma_{M-2}\\ 0 & 0 & 0 & \alpha_{M-1} & (1+\beta_{M-1})\end{pmatrix} \times\begin{pmatrix}F_{n+1,1}\\ F_{n+1,2}\\ F_{n+1,3}\\ \vdots\\ F_{n+1,M-1}\end{pmatrix}+d$$

where d is given by

$$d=[\alpha_1(F_{n,0}+F_{n+1,0}),0,\cdots,\gamma_{M-1}(F_{n,M}+F_{n+1,M})]^T\,.$$

The diagram for a Crank-Nicolson method is shown in figure 6 below [2].

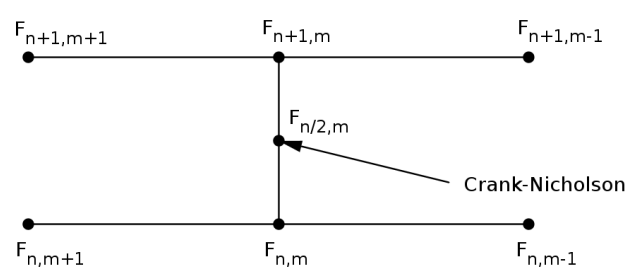

Figure 6. Crank-Nicolson finite difference.

8.1 Accuracy of the Crank-Nicolson Method

The accuracy of the three methods is mainly affected by the truncation errors arising from the finite difference approximation in the Taylor series expansions. The Crank-Nicolson approximation is more accurate than either the implicit or explicit finite difference approximations. It is accurate up to $O\Big((\Delta t)^2,(\Delta S)^2\Big)$ thus it has faster convergence than the other two methods. As described in [17], it can be shown that by equating the central difference and the symmetric central difference at

$$F_{n+\frac{1}{2},m} \equiv F(t+\frac{\Delta t}{2},S)\ ,$$

one would end up with the Crank-Nicolson method with an accuracy of $O\Big((\Delta t)^2,(\Delta S)^2\Big)$.

9. Monte-Carlo Simulation

The Monte Carlo simulation, as discussed in [11], is another important approach used in the valuation of derivatives. Under the risk-neutral assumption, the value of the option is obtained by calculating the average of the option payoff then discounting it to the present value under a risk-free interest rate, r .

Therefore Monte Carlo simulation is a process by which we generate a large sample of random asset paths which follow the SDE from equation (4). The option payoff for each path is computed and the arithmetic mean is evaluated, then the value is discounted under a risk-free interest rate to obtain an approximation of the option value.

The mean is obtained by using the concept of the law of large numbers. It states that if $X_1,X_2,\cdots,X_n$ are independent identically distributed sequences of random variables with finite expectation having similar distribution as the random variable X, then as $n\to\infty$,

$$\overline{X_n} \equiv \frac{1}{n}(X_1+X_2+\cdots,X_n) \to E(X)\ .$$

We also note that $\overline{X_n}$ is itself a random variable [16].

In this section, we follow the concepts discussed in [11]. Therefore, in order to simulate the path followed by S, under risk-neutral measure, we subdivide the life of S into N equal intervals of Δt. Thus the SDE from equation (4), that is,

$$\mathrm{d}S(t) = S(t)\Big(r\mathrm{d}t+\sigma d\widehat{B}(t)\Big)\ , \tag{44}$$

becomes

$$S(t+\Delta t)-S(t) = rS(t)\Delta t+\sigma S(t)Z\sqrt{\Delta t}\ . \tag{45}$$

Here $S(t)$ is the price of S at time t and $Z\sim N(0,1)$. Therefore from this, the value of S at time Δt is obtained using the initial value of S, the value of S at $2\Delta t$ is obtained using the value of S at Δt and so on. The path followed by S is then simulated using the M random sample from a normal distribution.

We simulate $\ln S$, because it is more accurate than simulating S, thus from Itô's lemma, the process followed by $\ln S$ is given by

$$\begin{aligned}\ln S &= (r-\frac{\sigma^2}{2})\Delta t+\sigma d\widehat{B}(t)\ ,\\ \implies \ln S(t+\Delta t)-\ln S(t) &= (r-\frac{\sigma^2}{2})\Delta t+\sigma Z\sqrt{\Delta t}\ ,\\ S(t+\Delta t) &= S(t)\exp\left((r-\frac{\sigma^2}{2})\Delta t+\sigma Z\sqrt{\Delta t}\right)\ .\end{aligned} \tag{46}$$

Equation (46) is the equation used in constructing the path of S.

Since r and σ are constants, then

$$\ln S(T)-\ln S(0) = (r-\frac{\sigma^2}{2})\Delta t+\sigma Z\sqrt{\Delta t}\ , \tag{47}$$

$$\therefore\quad S(T) = S(0)\exp\left((r-\frac{\sigma^2}{2})T+\sigma Z\sqrt{T}\right)\ . \tag{48}$$

The accuracy of the option value obtained using the Monte Carlo simulation depends on the number of sample paths, M. The higher the number of trials, the more accurate the results obtained. Therefore as $M\to\infty$, the option value converges to the known value of the Black-Scholes equation.

The mean μ which is the estimate of the of value of the derivative and the standard deviation β is calculated from the discounted payoff. From the central limit theorem, the standard error of the approximation, which is itself a random variable is given by

$$\frac{\beta}{\sqrt{M}}\ .$$

Thus a 95% confidence interval for the option price f is given by

$$\mu-\frac{1.96\beta}{\sqrt{M}} < f < \mu+\frac{1.96\beta}{\sqrt{M}}\ . \tag{49}$$

One of the advantages of the Monte Carlo simulation is that it is easy to implement. It is also good in computing the price of a derivative where the payoff depends on the path followed by the asset price.

The key disadvantages of the method is that it takes more computational time in implementation. It is also not efficient in pricing of American options where early exercise is permitted [11].

10. Results

In this chapter, we present the results obtained in the calculation of the European Put Option by using a Monte-Carlo simulation and the Crank-Nicolson method. These results are then compared to the closed form solution, that is, the solution obtained from the Black-Scholes formula. These numerical solutions were obtained by implementing an R program written for both methods [18].

10.1 Solutions by Monte Carlo Simulation

In using Monte Carlo simulation, the results tend to approach the explicit solution as we increase the number of simulation paths. The main steps followed in the Monte carlo simulation are ;

i) Generate a path for the underlying asset prices by a random walk under a risk-neutral world.

ii) Evaluate the payoff.

iii) Repeat the steps above several times to obtain more sample values for the payoff.

iv) Evaluate the average of these payoffs to obtain an estimate of the expected payoff.

v) The expected payoff is then discounted at a risk-free rate to get the required present value of the option.

Example 1: Monte-Carlo simulation

Consider the pricing of a European Put Option with $S_0 = 30,\ K = 60,\ r = 0.1,\ \sigma = 0.2$ and $T = \frac{3}{2}$. The option price obtained by the Black-Scholes equation is **21.68705**. Therefore when using the Monte-Carlo simulation method, as the number of simulations $M \to \infty$ the results become more accurate. The problem with this is that it takes more computational time when M is very large.

The table 1 below compares the accuracy with increasing the number of Monte-Carlo trials.

Number of simulations M	European Put value
100	21.54301
1000	21.77054
10000	21.66821
100000	21.69118
1000000	21.6794
10000000	**21.68596**

Table 1. Monte-Carlo simulation.

11. Solutions by Crank-Nicolson Method

Example 2: Crank-Nicolson method We need to compute the European Put value for a non-dividend paying stock whose initial price $S_0 = 50,\ K = 60,\ r = 0.1,\ \sigma = 0.2$ and $T = \frac{3}{2}$. Here we assume $S\max = 100$. The Black-Scholes price for the Put Option is 5.817974. The solution set for the European Put Option is shown in figure 7 below.

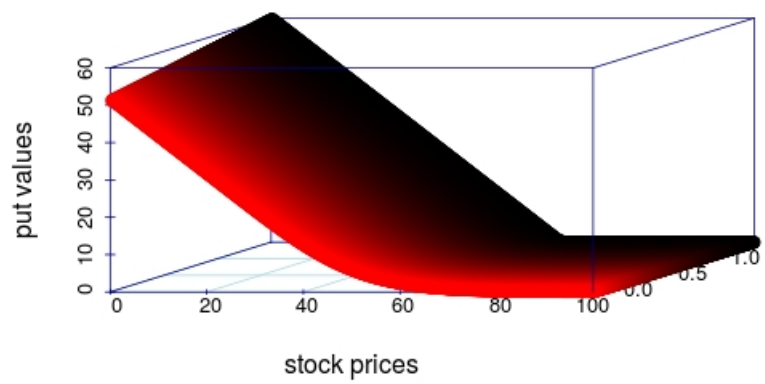

Figure 7. The price of a European Put Option by Crank-Nicolson Algorithm.

Consider the value of the European Put Option when $S = 0$. This is on the boundary condition. From the diagram above, the option will be worth the strike price at terminal time $t = T$, that is, $P(0,t) = 60$ where $t = 1.5$. To obtain the present value of the option, that is, the value at time $t = 0$ we have

$$\begin{aligned} P(0,t) &= K\exp(-r(T-t)) \ , \\ &= 60 * \exp(-0.1 * 1.5) \ , \\ &= 51.64248 \ . \end{aligned}$$

This value is shown on the diagram when $t = 0$.

Similarly, one can easily approximate the value of the Put Option on the diagram for different asset prices between 0 and S_{max} at any time from t_0 to T.

Example 2.1: Crank-Nicolson method with $M = N$

Considering example 2 above, it is clear that as we increase the number of the step sizes for both time and stock, that is, $M, N \to \infty$, the value obtained approaches the Black-Scholes price. However, it will take more computational time when the values of M and N are large $(M, N > 500)$.

Table 2 below shows the results obtained.

N = M	European Put
20	5.74192
50	5.805766
80	5.813204
100	5.814921
200	5.817209
300	5.817633
400	5.817782
500	5.81785
700	5.81791
1000	5.817942

Table 2. Option values when M=N.

Example 2.2: Crank-Nicolson method with $M \neq N$

Using example 2 above, we now compute the Put Option price when $M \neq N$. Similarly, we observe that as we increase the step sizes the results obtained tend to be accurate (approaches the Black-Scholes price). It will also take more time for the computer program to compute for bigger values of M and N.

The results obtained are shown in table 3 below, and it can be seen that the solution converges faster when $M \neq N$ than when we use $M = N$ [17].

N	M	European Put
20	40	5.798591
40	80	5.813127
60	120	5.815819
80	160	5.816761
100	200	5.817197
200	400	5.817779
300	600	5.817886
400	800	5.817924
500	1000	5.817941

Table 3. Option values when $M \neq N$.

12. Comparison of the results to the closed form solution

Example 3: Comparison

In this example we make the comparison of the European Put prices obtained by using the Black-Scholes formula, Monte-Carlo simulation and the Crank-Nicolson algorithm. Consider example 2 above where we have the parameters $K = 60$,$r = 0.1$,$\sigma = 0.2$,$T = \frac{3}{2}$. We will vary the stock prices and evaluate the corresponding put price and in the case of Crank-Nicolson algorithm we assume that $S\max = 2S_0$. We take the step sizes for the time and stock as $N = 200$ and $M = 400$ respectively.

The values obtained are shown in the table 4 below.

S	Black-Scholes	Monte carlo	Crank-Nicolson
20	31.64258	31.64224	31.37008
30	21.68705	21.68596	21.66453
40	12.49253	12.4918	12.4924
50	5.817974	5.817048	5.817779
60	2.244794	2.244683	2.244479
70	0.7541777	0.754244	0.7541207
80	0.2309951	0.2310311	0.2308693
90	0.06678	0.06691529	0.06678432
100	0.0186759	0.01874987	0.01866138

Table 4. Comparison of solutions.

13. Conclusion

The pricing of derivatives has been made easier by the development of the Black-Scholes model as discussed in this paper. The implementation of the Monte Carlo simulation and the Crank-Nicolson method made it easier to make a comparison of the results obtained by these numerical methods to the explicit solution obtained by using the Black-Scholes formula. We observed that the results obtained were approximately equal to the explicit solution.

These results were obtained by implementing the numerical schemes in R. The R codes for both numerical schemes are provided in [18]. However we only focused on the case where the asset is non-dividend paying. Therefore future work, the pricing of options where the underlying stock yields dividends will be incorporated.

14. Recommendation

The Black-Scholes model assumes that the underlying asset follows a normal distribution. In reality, asset returns are known to follow heavy-tailed distributions. Therefore for future work, we will consider the use of models which will incorporate the heavy-tailedness for example the variance gamma processes.

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