Relative basis and approximation Propetrty

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Abstract:

We investigate some structure properties of certain valued functions spaces by using new basis definitions of Banach spaces. We show that the space ℓ_∞ has a relative basis and so has an approximation property. Further the linear subspace $P(\mathbb{B},X)$ of $\ell_\infty(\mathbb{B},X)$ with a precompact rang has an X – Schauder basis. We prove that the Series of $\mathbb{C}$ – Schauder basis of $\ell_\infty$is not unconditional.

Key words: investigation, functional analysis, Banach spaces, oprator space, approximation, relative basis, unconditional.

1.Introduction:

Investigation of structural properties of some vector-valued function spaces often requires generalizations of classical topics of functional analysis. A recent example, for instance, is presented in [1,16] concerning extension of the classical adjoint operator notion. We used the new definition of adjoint in the characterization of some space of operators between certain vector-valued Banach function spaces. An important tool in such studies, and in structural and geometric investigations of classical Banach spaces, is the basis notion. For example, if Y is an arbitrary Banach space then the operator space $\mathcal{L}(\ell_1,Y)$ is equivalent, by the mapping $u \to \{u(e_n)\}$, to the space

$$\ell_\infty(Y) = \left\{\{y_n\} \subset Y\colon \sup_{1\le n<\infty} \|y_n\| < \infty\right\},$$

endowed with the norm $\|\{y_n\}\| = \sup_{1\le n<\infty}\|y_n\| < \infty$ where $\{e_n\}$ the unit vector basis of ℓ_1[2]. But, in spaces having no basis, these kind properties, and some important topics such as approximation property, separability, representation of operators and every topic in which the bases are used, can not be investigated thoroughly. In particular, in the representations of the spaces of continuous linear operators on some Banach spaces, bases play a crucial role. However, we see that this important tool doesn‘t exist in vector-valued function spaces in general. So one needs often an extended definition of Schauder bases, including a larger scale of these spaces. A natural question arising in this point is "what is the characterization of the operators on the Banach function spaces $\lambda(\mathbb{A},X)$?" where λ is the any of the c_0 or ℓ_p, $p \ge 1$, $\mathbb{A}$ is an infinite set, X is a Banach space and the definitions of the spaces $\lambda(\mathbb{A},X)$ are given in the following. We presented a particular answer to these questions in [3], a primitive form of this work. However, our further studies on the work bring more complete, and larger, definition of new basis notion, to our minds. By the novelty of the new definition, we prove that ℓ_∞, a nonseparable space, has a relative basis, and so we deduce a result, that it has approximation property. Note that $P(\mathbb{B},X) = \ell_\infty(\mathbb{B},X)$ for some finite-dimensional space X.

2.The unconditional bases:

Geometric and other structural properties of the vector-valued function spaces have been studied intensively in the last twenty years. In particular, the space of all bounded X-valued functions can be found. Some important references on $\ell_\infty(\mathbb{A},X)$ are [4-8]. In general, these spaces have derived properties with X. For example, there are many examples [5, 7], of normed spaces of $\ell_\infty(\mathbb{A},X)$ which are barrelled. Further, Ferrando and Lüdkovsky,[9] have considered the function space $c_0(\mathbb{A},X)$ which is an important extension of the classical Banach space c_0 and proved that $c_0(\mathbb{A},X)$ is either barrelled, ultrabornological or unordered baire-like if and only if X is, respectively. An analogous result for the X-valued sequence space $c_0(X) = c_0(\mathbb{N},X)$ was obtained earlier by Mendoza [10]. In [11], for a locally convex space X, we deal with the function spaces $\ell_1(\mathbb{A},X)$ and $c_0(\mathbb{A},X)$ and study some geometric and structural properties of them. This work may also be considered as a contribution to the efforts on the structural investigation of these vector-valued function spaces.

For normed spaces X and Y over $\mathbb{C}, \mathcal{L}(X,Y)$ denotes the space of all continuous linear operators from X into Y while $\mathcal{L}(X) = \mathcal{L}(X,X)$ and X^* denotes the continuous dual of X. Further, we denote by B_X and S_X the closed unit ball and sphere of X, respectively.

Let us now introduce the notion of (unordered) infinite sums in Banach Space. Let $\mathbb{A}$ be an infinite set, $\{ x_{a_\varepsilon}: a_\varepsilon \in \mathbb{A} \}$ be a family of vectors in a normed space X and let $\mathcal{F}$ denote the family of all finite subsets of $\mathbb{A}$. $\mathcal{F}$ is directed by the inclusion relation $\subseteq$ and, for each $F \in \mathcal{F}$, we can form the finite sum $S_F = \sum_{a_\varepsilon \in F} x_{a_\varepsilon}$. If the net $(S_F: \mathcal{F})$ converges to some x in X, then we say that the family $\{ x_{a_\varepsilon}: a_\varepsilon \in \mathbb{A} \}$ is summable, or that the sum $\sum_{a_\varepsilon \in \mathbb{A}} x_{a_\varepsilon}$ exists, and we write $\sum_{a_\varepsilon \in \mathbb{A}} x_{a_\varepsilon} = x$ in X. We mean by the convergence of the net $(S_F: \mathcal{F})$ to x in X that, for each $\varepsilon > 0$, there exists an $F_0 = F_0(\varepsilon) \in \mathcal{F}$ such that $\|S_F - x\| < \varepsilon$ whenever $F_0 \subseteq F$. The definition of summable family does not involve any ordering of the index set , and we may therefore say that the notion of a sum thus defined is commutative(unconditional). In case $\mathbb{A} = \mathbb{N}$, that the family $\{ x_n: n \in \mathbb{N} \}$ is summable to x is equivalent to that the series $\sum_{n \in \mathbb{N}} x_n$ is unconditio nally convergent to x. Note that a series $\sum_{n \in \mathbb{N}} x_n$ in a Banach space is said to be unconditionally convergent if and only if, for each permutation σ of $\mathbb{N}$, $\sum_{n \in \mathbb{N}} x_{\sigma(n)}$ is convergent. The definition of a convergent series essentially involves the order structure of $\mathbb{N}$. If the series $\sum_{n \in \mathbb{N}} x_n$ is convergent, and if σ is a permutation of $\mathbb{N}$, then the series $\sum_{n \in \mathbb{N}} x_{\sigma(n)}$ may not be convergent, that is, $\{ x_n: n \in \mathbb{N} \}$ may not be summable [12]. Hence, summability of the family $\{x_n: n \in \mathbb{N} \}$ in X is stronger than the convergence of the series $\sum_{n \in \mathbb{N}} x_n$ in this sense.

Let a family $\{ V_{a_\varepsilon}: a_\varepsilon \in \mathbb{A} \}$ of topological spaces be given. The product $\prod_{a_\varepsilon \in \mathbb{A}} V_{a_\varepsilon}$ is the set of all functions $x : \mathbb{A} \to \cup V_{a_\varepsilon}$ such that $x(a_\varepsilon) \in V_{a_\varepsilon}$, for each $a_\varepsilon \in \mathbb{A}$. Usually, we use the notation x_{a_ε} for $x(a_\varepsilon)$ since it is more convenient. By $V^{\mathbb{A}}$ we mean $\prod_{a_\varepsilon \in \mathbb{A}} V_{a_\varepsilon}$ with $V_{a_\varepsilon} = V$ for each $a_\varepsilon \in \mathbb{A}$. Let us define $P_{a_\varepsilon} = \prod_{a_\varepsilon \in \mathbb{A}} V_{a_\varepsilon} \to V_{a_\varepsilon}$ by $P_{a_\varepsilon}(x) = x_{a_\varepsilon}$. This is called the projection on the a_ε-th factor – or briefly, it is called a_ε-th projection.

Let X be a normed space over $\mathbb{C}$. Some important linear subspaces of $X^{\mathbb{A}}$ are given as follows. The space $\ell_\infty(\mathbb{A}, X)$ is the linear space of all functions $x: \mathbb{A} \to X$ such that

$$\sup\{\|x_{a_\varepsilon}\| : a_\varepsilon \in \mathbb{A} \} < \infty.$$

Moreover, $\ell_\infty(\mathbb{A}, X)$ is a normed space with the sup-norm

$$\|x\|_\infty = \sup \{\|x_{a_\varepsilon}\| : a_\varepsilon \in \mathbb{A} \}.$$

We denote by $\ell_\infty(\mathbb{A}, X)$, $1 \leq p < \infty$, the set of all functions $x: \mathbb{A} \to X$ such that $\{\|x_{a_\varepsilon}\|^p : a_\varepsilon \in \mathbb{A} \}$ is summable, i.e.,

$$\sum_{a_\varepsilon \in \mathbb{A}} \|x_{a_\varepsilon}\|^p < \infty,$$

and it is a normed space with the norm:

$$\|x_{a_\varepsilon}\|_p = \left(\sum_{a_\varepsilon \in \mathbb{A}} \|x_{a_\varepsilon}\|^p \right)^{1/p}$$

Further, the function space $c_0(\mathbb{A}, X)$ is the linear space of all functions $x: \mathbb{A} \to X$ such that, for each $\varepsilon > 0$, the set

$$\{a_\varepsilon \in \mathbb{A}: \|x_{a_\varepsilon}\| > \varepsilon \}$$

is finite or empty. It is a normed space with the sup norm. These function spaces are Banach spaces if and only if X is a Banach space. Moreover, if $\mathbb{A}$ is a directed set then $\lambda(\mathbb{A}, X), \lambda = \ell_\infty, c_0$ or ℓ_p, is a linear space of all nets with corresponding property. They are usually denote by $\lambda(X)$ in the case $\mathbb{A} = \mathbb{N}$ and are called X- valued sequence spaces.

3.The Relative bases & projection:

Definition (1):

Let X and Y be Banach spaces and $\mathbb{A}$ be a set. A family $\{\eta_{a_\varepsilon}: a_\varepsilon \in \mathbb{A} \}$ of continuous linear functions $\eta_{a_\varepsilon}: Y \to X$ is called Y-basis for X if the following condition is satisfied. There exists a direct subset $\mathfrak{D}$(by some relation $\eta_{a_\varepsilon} \ll$) of $\mathcal{F}$ satisfying the property; for each $a_\varepsilon \in \mathbb{A}$ there is some $F \in \mathfrak{D}$ such that $a_\varepsilon \in F$, and there exists a unique family $\{R_{a_\varepsilon}: a_\varepsilon \in \mathbb{A}\}$ of linear functions R_{a_ε} from X onto Y such that, for each $x \in X$, the net $(\pi_F(x): \mathfrak{D}, \ll)$ converges to x in X where

$$\pi_F(x) = \sum_{a_\varepsilon \in F} (\eta_{a_\varepsilon} \circ R_{a_\varepsilon})(x),$$

for each $F \in \mathfrak{D}$, and $\mathcal{F}$ is the family of all finite subsets of the index set $\mathbb{A}$. furthermore, $\{\eta_{a_\varepsilon}\}$ is called a Y- Schauder basis for X whenever each R_{a_ε} is continuous .

Definition (2):

The Y-basis $\{\eta_{a_\varepsilon}: a_\varepsilon \in \mathbb{A}\}$ in the definition is called unconditional whenever $\mathfrak{D} = \mathcal{F}$ with the inclusion relation $\subseteq$.

Definition (3):

The family $\{R_{a_\varepsilon}: a_\varepsilon \in \mathbb{A}\}$ is called associate family of functions (A.F.F.) to the Y-basis $\{\eta_{a_\varepsilon}: a_\varepsilon \in \mathbb{A}\}$.

Let $\{\eta_{a_\varepsilon}: a_\varepsilon \in \mathbb{A}\}$ be a Y- basis for X. Clearly, the finite summation $\pi_F(x)$ defines an operator $\pi_F(x)$ defines an operator π_F on X for each $F \in \mathfrak{D}$. This operator is called F- projection on X corresponding Y- basis and it is continuous whenever $\{\eta_{a_\varepsilon}\}$ is a Y- Schauder bases.

Theorem (4):

Let $\mathbb{B}$ be a set and X be a Banach space. Then $\ell_p(\mathbb{B}, X)$ and $c_0(\mathbb{B}, X)$ have unconditional X-Schauder bases.

Proof:

Take $\mathbb{A} = \mathbb{B}$ in the definition and define

$$\mathrm{I}_{a_\varepsilon}: X \to \lambda(\mathbb{A}, X), \lambda = c_0 \text{ or } \ell_p,$$

By $\mathrm{I}_{a_\varepsilon}(t) = y$ such that $y_\delta = t$, if $\delta = a_\varepsilon$ otherwise $y_\delta = 0$. Then the family $\{\mathrm{I}_{a_\varepsilon}: a_\varepsilon \in \mathbb{A}\}$ is an unconditional X-Schuader basis for $\lambda(\mathbb{A}, X)$. We prove the assertion only for $c_0(\mathbb{A}, X)$ since the proof for $\ell_p(\mathbb{A}, X)$ almost is the same. Take the family $\{\mathrm{R}_{a_\varepsilon}: a_\varepsilon \in \mathbb{A}\}$ as

$$\mathrm{R}_{a_\varepsilon}: c_0(\mathbb{A}, X) \to X; \qquad R_{a_\varepsilon}(x) = P_{a_\varepsilon}(x) = x_{a_\varepsilon},$$

and $\mathfrak{D} = \mathcal{F}$, directed by the inclusion $\subseteq$. Then we must show that the net $(\pi_F(x) : \mathcal{F})$ converges to x in $c_0(\mathbb{A}, X)$ where

$$\pi_F(x) = \sum_{a_\varepsilon \in F} (I_{a_\varepsilon} \circ P_{a_\varepsilon})(x) = \sum_{a_\varepsilon \in F} I_{a_\varepsilon}(x_{a_\varepsilon}).$$

For some $x \in c_0(\mathbb{A}, X)$, $\pi_F(x)$ is the function x_F such that $x_F(a_\varepsilon) = x_{a_\varepsilon}$ if $a_\varepsilon \in F$ and $x_F(a_\varepsilon) = 0$ if $a_\varepsilon \notin F$. Now, consider an arbitrary $\varepsilon > 0$. We must find a finite subset $F_0 = F_0(\varepsilon) \in \mathcal{F}$ such that, for each finite $F \supseteq F_0$,

$\|x - x_F\|_\infty \le \varepsilon.$

Since the set $\{ a_\varepsilon \in \mathbb{A}: \|x_{a_\varepsilon}\| > \varepsilon \}$ is finite or empty, taking F_0 as

$F_0 = \{ a_\varepsilon \in \mathbb{A}: \|x_{a_\varepsilon}\| > \varepsilon \},$

we have

$$\|x - x_F\|_\infty = \left\|\{x_{a_\varepsilon}: a_\varepsilon \in \mathbb{A} - F\}\right\|_\infty \le \varepsilon,$$

for each finite $F \supseteq F_0$.

For the uniqueness of the family $\{P_{a_\varepsilon}\}$ suppose

$$\sum_{a_\varepsilon \in F} (I_{a_\varepsilon} \circ P_{a_\varepsilon})(x) = \sum_{a_\varepsilon \in F} (I_{a_\varepsilon} \circ R'_{a_\varepsilon})(x).$$

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and

$$\pi_F^{\circ}(x) = \sum_{a_\varepsilon \in F}(I_{a_\varepsilon} \circ (P_{a_\varepsilon} - R'_{a_\varepsilon}))(x),\ F \in \mathcal{F}$$

Remember that

$\|\pi_F^{\circ}(x)\|_\infty = \sup_{a_\varepsilon \in F}\left\|\left(P_{a_\varepsilon} - R'_{a_\varepsilon}\right)(x)\right\|$ and $\|\pi_F^{\circ}(x)\|_\infty \leq \|\pi_G^{\circ}(x)\|_\infty$

for $F \subseteq G$. Since

$$\lim_{F \in \mathcal{F}} \|\pi_F^{\circ}(x)\|_\infty = 0,$$

We have that $\left(P_{a_\varepsilon} - R'_{a_\varepsilon}\right)(x) = 0$ for each a_ε and for every $x \in c_0(\mathbb{A}, X)$.

This implies, $P_{a_\varepsilon} = R'_{a_\varepsilon}$ for each a_ε.

Further, each P_{a_ε} is continuous because $\|x_{a_\varepsilon}\| \leq \|x\|_\infty$.

Theorem (5):

Let $\mathbb{B}$ be a set and X be a Banach space. Then the linear subset $P(\mathbb{B}, \mathrm{X})$ of $\ell_\infty(\mathbb{B}, \mathrm{X})$ of all those functions with precompact range has an X-Schauder basis.

Proof:

Take $\mathbb{A} = 2^{\mathbb{B}}$ in the definition and define

$\mu_{a_\varepsilon}: X \to P(\mathbb{B}, \mathrm{X})$,

for each $\alpha \in \mathbb{A}$ ($\alpha \subseteq$ with some desinguished point h_{a_ε}) by

$a_\varepsilon(t) = \sum_{a_\varepsilon \in a_\varepsilon} \mathrm{I}_{a_\varepsilon}(t)$, for $t \in X$,

where $\mathrm{I}_{a_\varepsilon}(t) = y$ such that $y_\delta = t$ if $\delta = a_\varepsilon$ other wise $y_\delta = 0$. Then the family $\{\mu_\alpha: \alpha \in \mathbb{A}\}$ is an X-Shuader basis for $P(\mathbb{B}, \mathrm{X})$. Describe $\mathfrak{D}$ as the set of all finite partition of $\mathbb{B}$. It is obviously a subfamily of $\mathcal{F}$, the family of all finite subsets of $\mathbb{A} = 2^{\mathbb{B}}$, and $\mathfrak{D}$ is directly by the relation $\rho_1 \ll \rho_2$, meaning that each $\alpha' \in \rho_2$ is included in some $\alpha \in \rho_1$, where $\rho_1 = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\rho_2 = (\alpha'_1, \alpha'_2, \dots, \alpha'_n)$ are partitions of $\mathbb{B}$ with distinguished points $h_i \in \alpha_i$ and $h'_j \in \alpha'_j$, respectively. Take the family $\{R_\alpha: \alpha \in \mathbb{A}\}$ as

$$R_\alpha: P(\mathbb{B}, \mathrm{X}) \to X;\ \ R_\alpha(x) = x_{h_\alpha}$$

Then it isn't hard to show that the net $(\pi_\rho(x): \mathfrak{D})$ converges to x in $P(\mathbb{B}, \mathrm{X})$ where

$$\pi_\rho(x) = \sum_{\alpha \in \rho}(\mu_\alpha \circ R_\alpha)(x) = \sum_{\alpha \in \rho} \mu_\alpha\left(x_{h_\alpha}\right).$$

Uniqueness is similar to that of the previous result.

Further, each $R_\alpha$is continuous.

Since for a finite-dimensional space X, one has $P(\mathbb{B}, \mathrm{X}) = \ell_\infty P(\mathbb{B}, \mathrm{X})$ then we have the following corollary.

Corollary (6):

The family

$$\mu_\alpha: \mathbb{C} \to \ell_\infty;\ \ \mu_\alpha(t) = t.X_\alpha,$$

is a $\mathbb{C}$- Schauder basis for ℓ_∞, where X_α is the characteristic function of $\alpha \subseteq \mathbb{N}$.

Proof:

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Take $\mathbb{B} = \mathbb{N}$ and $\mathrm{X} = \mathbb{C}$ from theorem (5) above. Then

$$\pi_\rho(x) = \sum_{\alpha\in\rho}(\mu_\alpha \circ R_\alpha)(x) = \sum_{\alpha\in\rho} X_\alpha . x_{h_\alpha},$$

and converges to x in ℓ_∞.

Theorem(7):

The $\mathbb{C}$- Schauder basis of ℓ_∞ determined in Corollary (6) is not unconditional.

Proof:

Let $\mathbb{A} = 2^{\mathbb{N}}$ and $\mathfrak{D} = \mathcal{F}$ with the inclusion relation $\subseteq$. We will show that the net $(\pi_\rho(x) : \mathcal{F})$ doesn't converge to x for $x = (1,1,\dots) \in \ell_\infty$. Consider the chain $\mathcal{C} = (F_1, F_2, \dots)$ with

$F_1 \{\{1\}\}$

$F_2 \{\{1\},\{1,2\}\}$

$F_3 \{\{1\},\{1,2\},\{1,2,3\}\}$

$\vdots$

in $(\mathcal{F}, \subseteq)$. Then the subnet $(\pi_\rho(x): \mathcal{C})$(a sequence, in fact) of $(\pi_F(x): \mathcal{F})$ isn't convergent to x. Indeed, for $F \in \mathcal{C}$,

$$\pi_F(x) = \sum_{\alpha\in F}(\mu_\alpha \circ R_\alpha)(x)$$

$$= \sum_{\alpha\in\rho} X_\alpha\left(x_{h_\alpha}\right) = \sum_{\alpha\in\rho} X_\alpha$$

$$= (|F|, |F|-1, \dots, 1,0,0,\dots)$$

where $|F|$ is the cardinality of F. Hence,

$$\lim_{F\in\mathcal{C}} \|x - \pi_F(x)\|_\infty \neq 0,$$

This implies $(\pi_F(x): \mathcal{F}) \nrightarrow x$ in ℓ_∞.

Theorem(8):

Let X be a Banach space for which a family $\{\eta_\alpha : a_\varepsilon \in \mathbb{A}\}$ is a Y-basis for some separable Banach space Y. Then, X is separable if $\mathbb{A}$ is countable.

Proof:

Let $\mathbb{A}$ be countable and W be a countable dense subspace of Y. Then the set $\bigoplus_{a_\varepsilon\in\mathbb{A}} \eta_{a_\varepsilon}(W)$ of all finite sums of the form $\sum_{a_\varepsilon\in F} \eta_{a_\varepsilon}(y_{a_\varepsilon})$ with F finite and $y_{a_\varepsilon} \in W$ for every $a_\varepsilon \in F$ is a countable dense subspace of X since the image of countable sets under a function is also countable, and a union of a countable family of countable sets is countable. Furthermore, for some $x \in X$ there exist a net $\{x_\delta^{a_\varepsilon}\}_\delta$ in W such that $x_\delta^{a_\varepsilon} \to R_{a_\varepsilon}(x)$ in Y. So

$$\eta_{a_\varepsilon}(x_\delta^{a_\varepsilon}) \to (\eta_{a_\varepsilon} \circ R_{a_\varepsilon})(x)$$

in X for each $a_\varepsilon \in \mathbb{A}$. This implies

$$\sum_{a_\varepsilon\in F} \eta_{a_\varepsilon}(x_\delta^{a_\varepsilon}) \to \sum_{a_\varepsilon\in F}(\eta_{a_\varepsilon} \circ R_{a_\varepsilon})(x) = \pi_F(x), \quad \text{for each } F \in \mathcal{D}.$$

Hence each neighborhood $U_{\pi_F(x)}$ of $\pi_F(x)$ includes an element $\sum_{a_\varepsilon\in F} \eta_{a_\varepsilon}(x_\delta^{a_\varepsilon})$ of $\bigoplus_{a_\varepsilon\in\mathbb{A}} \eta_{a_\varepsilon}(W)$. On the other hand, each neighborhood U_x of x includes a neighborhood $U_{\pi_{F_0}(x)}$ of some $\pi_{F_0}(x)$ since the net $(\pi_F(x), \mathcal{D})$ converges to x in X where

http://dx.doi.org/10.29322/IJSRP.8.11.2018.p8334

$\mathcal{D} \subset \mathcal{F}$ is a directed family (by some relation$\ll$) corresponding to the basis $\{\eta_{a_\varepsilon}: a_\varepsilon \in \mathbb{A}\}$. Consequently, U_x includes an element of $\bigoplus_{a_\varepsilon \in \mathbb{A}} \eta_{a_\varepsilon}(W)$. This shows that $\bigoplus_{a_\varepsilon \in \mathbb{A}} \eta_{a_\varepsilon}(W)$ is dense in X.

Theorem (9):

Let X be a Banach Space and $\{\eta_{a_\varepsilon}: a_\varepsilon \in \mathbb{A}\}$ be a Y-basis for X with some Banach space Y. Then, X is linearly homeomorphic with a subspace Z of $Y^{\mathbb{A}}$ such that $\{I_{a_\varepsilon}: a_\varepsilon \in \mathbb{A}\}$ is a Y-basis for Z where I_{a_ε} is defined by $I_{a_\varepsilon}(t) = \omega$ such that

$$\omega(b) = \begin{cases} R_{a_\varepsilon}(x), & \text{for some } x \in X\,, if\ a_\varepsilon = b \\ 0\,, & \text{other wise} \end{cases}$$

and $\{R_{a_\varepsilon}\}$ is the $A.F.F.$ to the Y- basis $\{\eta_{a_\varepsilon}\}$. Further $\{\eta_{a_\varepsilon}\}$ is Y-Schauder basis if and only if $\{I_{a_\varepsilon}\}$ is.

Proof:

Let $Z = \{z \in Y^{\mathbb{A}}: z_{a_\varepsilon} = R_{a_\varepsilon}(x) \text{ for some x} \in \text{X},\}$

and define $T: X \to Z$ by $T_X = z$. Clearly T is a linear isomorphism. Further, by imposing on Z the quotient topology $Q\mathcal{T}$ by the operator T, we identified X with Z topologically since T is continuous in $Q\mathcal{T}$. Moreover,

$$z = T_X = \sum_{a_\varepsilon \in F} (T \circ \eta_{a_\varepsilon} \circ R_{a_\varepsilon})(x) = \sum_{a_\varepsilon \in F} (T \circ \eta_{a_\varepsilon}) \left(R_{a_\varepsilon}(x)\right)$$

$$= \sum_{a_\varepsilon \in F} (T \circ \eta_{a_\varepsilon})(z_{a_\varepsilon}) = \sum_{a_\varepsilon \in F} (T \circ \eta_{a_\varepsilon} \circ P_{a_\varepsilon})(z) = \sum_{a_\varepsilon \in F} (I_{a_\varepsilon} \circ P_{a_\varepsilon})(z)$$

where P_{a_ε} is the a_ε-th – projection on the product space $Y^{\mathbb{A}}$ and $I_{a_\varepsilon} = T \circ \eta_{a_\varepsilon}$. Hence $\{I_{a_\varepsilon}\}$ is a Y- basis. Further, P_{a_ε} is continuous from $(Z; Q\mathcal{T})$ to Y if and only if R_{a_ε} is continuous from X to Y.

Corollary(10):

The series of $\mathbb{C}$-Schauder basis of ℓ_∞ is not unconditional.

Proof:

Let $\mathbb{A} = 2^{\mathbb{N}}$ and $\mathfrak{D} = \mathcal{F}$ with the inclusion relation $\subseteq$. We will show that the series of net $(\sum_{i=1}^{n} \pi_\rho(x_i) : \mathcal{F})$ doesn't converge to $\sum_{i=1}^{n} x_i$ for $x_i = (1,1,\dots) \in \ell_\infty$. Consider the chain stated in Theorem (7), then the series of subnet $(\sum_{i=1}^{n} \pi_\rho(x_i) : \mathcal{C})$ of $(\sum_{i=1}^{n} \pi_\rho(x_i) : \mathcal{F})$ is not convergent to $\sum_{i=1}^{n} x_i$. Therefore $\tilde{F} \in \mathcal{C}$,

$$\sum_{i=1}^{n} \pi_{\tilde{F}}(x_i) = \sum_{i=1}^{n} \sum_{\alpha \in \tilde{F}} (\mu_\alpha \circ R_\alpha)(x_i)$$

$$= \sum_{i=1}^{n} \sum_{\alpha \in \rho} X_\alpha\left((x_i)_{h_\alpha}\right) = \sum_{\alpha \in \rho} X_\alpha$$

$$= (|\tilde{F}|, |\check{F}| - 1, \dots, 1,0,0, \dots)$$

where $|\tilde{F}|$ is the cardinality of F. Hence,

$$\lim_{\tilde{F} \in \mathcal{C}} \left\| \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \pi_{\tilde{F}}(x_i) \right\|_\infty \neq 0,$$

This implies that

$$\sum_{i=1}^{n} \pi_{\tilde{F}}(x_i) : \mathcal{F}) \nrightarrow \sum_{i=1}^{n} x_i \text{ in } \ell_\infty.$$

Now, let us introduce relative biorthogonal systems in Banach spaces.

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Definition (11):

Let X and Y be Banach spaces, $\{\eta_{a_\varepsilon}: a_\varepsilon \in \mathbb{A}\} \in \mathcal{L}(X,Y)^{\mathbb{A}}$ and $\{R_{a_\varepsilon}: a_\varepsilon \in \mathbb{A}\} \in \mathcal{L}(X,Y)^{\mathbb{A}}$. If $R_{a_\varepsilon} \circ \eta_{a_\varepsilon} = I_Y$ for $a_\varepsilon = b$, and $R_{a_\varepsilon} \circ \eta_{a_\varepsilon} = 0$ for $a_\varepsilon \neq b$, then the family

$$\{(\eta_{a_\varepsilon}, R_{a_\varepsilon}): a_\varepsilon \in \mathbb{A}\}$$

is called a Y- biorthogonal system for X.

Proposition (12):

Let a family $\{(\eta_{a_\varepsilon}, R_{a_\varepsilon}): a_\varepsilon \in \mathbb{A}\}$ be as in the above defntion. Then $\{(\eta_{a_\varepsilon}, R_{a_\varepsilon}): a_\varepsilon \in \mathbb{A}\}$ is a Y- biorthogonal system for X if and only if $\{\eta_{a_\varepsilon}: a_\varepsilon \in \mathbb{A}\}$ is Y-basis for X with $\{R_{a_\varepsilon}\}$ being $A.F.F.$ to $\{\eta_{a_\varepsilon}: a_\varepsilon \in \mathbb{A}\}$.

Proof:

For an arbitrary $v \in Y, \eta_b(v) \in X$, and so

$$\eta_b(v) = \sum_{a\in\mathbb{A}} (\eta_{a_\varepsilon} \circ R_{a_\varepsilon})(\eta_b(v)) = \sum_{a_\varepsilon\in\mathbb{A}} \eta_{a_\varepsilon}[(R_{a_\varepsilon} \circ \eta_b)(v)].$$

Hence, by the uniqueness of the family $\{R_{a_\varepsilon}\}$, this equality holds if and only if $R_{a_\varepsilon} \circ \eta_b = I_Y$ for $a_\varepsilon = b$, and $R_{a_\varepsilon} \circ \eta_b = 0$, $a_\varepsilon \neq b$.

It is well-known that a Banach space having a basis also has the approximation property (AP). However, this depend directly on the space Y here. Note that a Banach space X is said to be have AP if, for every compact set K in X and every $\varepsilon > 0$, there exists an operator $T: X \to X$ of finite rank (i.e. $T(x) = \sum_{i=1}^{n} x_i^*(x) x_i$, for some $\{x_i\}_{i=1}^{n} \subset X$ and $\{x_i^*\}_{i=1}^{n} \subset X^*$) so that $\|T(x) - x\| \leq \varepsilon$, for every $x \in K$.

Theorem (13):

Let X be a Banach space having a Y-Schauder basis for some Banach spaces Y. Then X has the approximation property if and only if Y has.

Proof:

It follows from Theorem (9) that we may consider X to be a subspace of $Y^{\mathbb{A}}$ with the Y-basis $\{I_{a_\varepsilon}\}$ and with the $.F.F.$ $\{P_{a_\varepsilon}\}$. Let X has the AP and fix some $a_\varepsilon \in \mathbb{A}$. Suppose some $K \subset Y$ is compact and let $\varepsilon > 0$. Since $I_{a_\varepsilon}(K)$is also compact in X, there exists a finite rank operator T on X such that

$$\|T(I_{a_\varepsilon}(y)) - I_{a_\varepsilon}(y)\| \leq \varepsilon$$

for every $y \in K$. This implies

$$\|(P_{a_\varepsilon} \circ T \circ I_{a_\varepsilon})(y) - y\| = \|(P_{a_\varepsilon} \circ T \circ I_{a_\varepsilon})(y) - (P_{a_\varepsilon} \circ I_{a_\varepsilon})y\|$$

$$\leq \|P_{a_\varepsilon}\| \left\| T\left(I_{a_\varepsilon}(y)\right) - I_{a_\varepsilon}(y) \right\| \leq \varepsilon.$$

Hence $P_{a_\varepsilon} \circ T \circ I_{a_\varepsilon}$ is the desired finite rank operator (corresponding to K and ε).

Conversely, let Y has AP. It isn't hard to see that any finite direct sum of Banach spaces has AP if and only if each component space has this property. Let $K \subset X$ be compact and $\varepsilon > 0$. There exists an $F_0(\varepsilon) \in \mathcal{D}, \mathcal{D} \subset \mathcal{F}$ is a directed family (by some relation $\ll$) corresponding to the Y-basis, such that

$$\|\pi_F(x) - x\| \leq \varepsilon/2, \text{for each finite } \mathrm{F}_0 \ll F\,,$$

Since the net $(\pi_F(x), \mathcal{D})$ converges to x in X. Further, for each $F \in \mathcal{D}$, $\pi_F(K)$ is a compact subset of $\oplus_{a_\varepsilon \in F} Y$ and so there exists a finite rank operator T on $\oplus_{a_\varepsilon \in F} Y$ such that

$$\|T(\pi_F(x)) - \pi_F(x)\| \leq \varepsilon/2,$$

 http://dx.doi.org/10.29322/IJSRP.8.11.2018.p8334

for every $x \in K$. Thus, for each finite $F \supseteq F_0$,

$$\|(T \circ \pi_F)(x) - x\| = \|T(\pi_F(x)) - \pi_F(x) + \pi_F(x) - x\|$$

$$\leq \|T(\pi_F(x)) - \pi_F(x)\| + \|\pi_F(x) - x\|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This implies that each $T \circ \pi_F$ such that $F_0 \ll F$ is the desired finite rank operator.

Corollary (14):

The space $\ell_p(\mathbb{B}, X), P(\mathbb{B}, X)$ of $\ell_\infty(\mathbb{B}, X)$ and $c_0(\mathbb{B}, X)$ have AP whenever X has. In particular, ℓ_∞, c_0 and ℓ_p have AP.

Theorem (15):

Let X and Y be a Banach spaces and $\{\eta_{a_\varepsilon}\}$ be a Y-basis for X. Then

$$\|x\|^\circ = \sup_{F \in \mathcal{D}} \{\|\pi_F(x)\|\}$$

defines a norm on X, equivalent to the initial norm of X, and

$$\|x\| \leq \|x\|^\circ \leq K\|x\|; \ x \in X,$$

for some $K > 0$.

The proof is quite similar to its classical analogue ([13,2]).

Example (16):

The X-bases of $\ell_\infty(\mathbb{B}, X)$ and $c_0(\mathbb{B}, X)$ given in Theorem (4) are monotone. Let us prove the assertion only for $c_0(\mathbb{B}, X)$. Fix some $v_{a_\varepsilon} \in S_X$ and define $x^{a_\varepsilon}: \mathbb{A} \to X$, for some $a_\varepsilon \in \mathbb{A}$, by

$$x^{a_\varepsilon}(b) = \begin{cases} v_{a_\varepsilon}, b = a_\varepsilon \\ 0, \ b \neq a_\varepsilon \end{cases}$$

Then, for some $F \in \mathcal{D}$ containing a_ε, we have

$$\pi_F(x^{a_\varepsilon}) = \sum_{b \in F} (I_b \circ P_b)(x^{a_\varepsilon}) = x^{a_\varepsilon}.$$

Thus, for some $x^{a_\varepsilon} \in B_{c_0(\mathbb{B},X)}$,

$$\|\pi_F(x^{a_\varepsilon})\|_\infty = \|x^{a_\varepsilon}\|_\infty$$

Also, for every $x \in B_{c_0(\mathbb{B},X)}$,

$$\|\pi_F(x)\|_\infty = \left\| \sum_{a \in F} (I_b \circ P_b)(x^{a_\varepsilon}) \right\|_\infty$$

$$\leq \left\| \sum_{a_\varepsilon \in \mathbb{A}} (I_b \circ P_b)(x) \right\|_\infty = \|x\|_\infty,$$

Whence, we have, $\|\pi_F\| = 1$ for each $F \in \mathcal{D}$.

We now characterize continuous operators on $c_0(\mathbb{B}, X)$ as an application of relative bases. These spaces have a fundamental role in the theory of operator spaces for $\mathbb{B} = \mathbb{N}$, especially in the investigation of the separable extension property (see [14], [15]). The techniques used here can be applied to all spaces having a suitable relative basis.

Theorem (17):

Let $\mathbb{B}$ be a set and X be a Banach space. then, the operator space $\mathcal{L}(c_0(\mathbb{B},X))$ is equivalent (isometrically isomorphic) by the mapping

$$T \to \{ T \circ I_{a_\varepsilon} : a_\varepsilon \in \mathbb{B} \}$$

to the Banch space Λ, the space of all $\varphi \in \mathcal{L}(X, c_0(\mathbb{B},X))^{\mathbb{B}}$ such that

$$\sup_{g \in B_{\ell_1(\mathbb{B},X^*)}} \sum_{a_\varepsilon \in \mathbb{B}} \|g \circ \varphi_{a_\varepsilon}\| < \infty,$$

endowed with the norm

$$\|\varphi\| = \sup_{g \in B_{\ell_1(\mathbb{B},X^*)}} \sum_{a_\varepsilon \in \mathbb{B}} \|g \circ \varphi_{a_\varepsilon}\|$$

where $\{ I_{a_\varepsilon} : a_\varepsilon \in \mathbb{B} \}$ is the X-Shauder basis for $c_0(\mathbb{B},X)$ in the Theorem (1).

Proof:

By using the X-basis $\{ I_{a_\varepsilon} : a_\varepsilon \in \mathbb{B} \}$ for $c_0(\mathbb{B},X)$, let us write $\varphi_{a_\varepsilon} = T \circ I_{a_\varepsilon}$ for some $T \in \mathcal{L}(c_0(\mathbb{B},X))$, and define

$$\Psi : \mathcal{L}(c_0(\mathbb{B},X)) \to \Lambda \text{ by } = \Psi(T) = \varphi = \left\{ \varphi_{a_\varepsilon} : a_\varepsilon \in \mathbb{B} \right\}.$$

Our first task is to show that Ψ really defines a mapping from$\mathcal{L}(c_0(\mathbb{B},X))$ to Λ. Let $T \in \mathcal{L}(c_0(\mathbb{B},X))$ and for some $F \in \mathcal{D} = \mathcal{F}$ write

$\pi'_F(x) = \sum_{a_\varepsilon \in \mathbb{A}} (\varphi_{a_\varepsilon} \circ P_{a_\varepsilon})(x), \;\; x \in c_0(\mathbb{B},X).$

It is continuous linear operator on $c_0(\mathbb{B},X)$ because $\pi'_F(x) = (T \circ \pi_F)(x)$ where $\pi_F(x) = \sum_{a_\varepsilon \in F} (I_{a_\varepsilon} \circ P_{a_\varepsilon})(x)$. By taking into account the equality

$c_0(\mathbb{B},X)^* = \ell_1(\mathbb{B},X^*)$

we have

$$\sup_{g \in B_{\ell_1(\mathbb{B},X^*)}} \sum_{a_\varepsilon \in \mathbb{B}} \|g \circ \varphi_{a_\varepsilon}\| = \sup_{g \in B_{\ell_1(\mathbb{B},X^*)}} \sup_{F \in \mathcal{D}} \sum_{a_\varepsilon \in F} \|g \circ \varphi_{a_\varepsilon}\|$$

$$= \sup_{g \in B_{\ell_1(\mathbb{B},X^*)}} \sup_{F \in \mathcal{D}} \sup_{x \in B_{c_0(\mathbb{B},X)}} \left| \sum_{a_\varepsilon \in F} (g \circ \varphi_{a_\varepsilon})(x_{a_\varepsilon}) \right|$$

$$= \sup_{F \in \mathcal{D}} \sup_{x \in B_{c_0(\mathbb{B},X)}} \sup_{g \in B_{\ell_1(\mathbb{B},X^*)}} \left| g\left(\sum_{a_\varepsilon \in F} \varphi_{a_\varepsilon}(x_{a_\varepsilon}) \right) \right|$$

$$= \sup_{F \in \mathcal{D}} \sup_{x \in B_{c_0(\mathbb{B},X)}} \sup_{g \in B_{c_0(\mathbb{B},X)^*}} \left| g\left(\sum_{a \in F} (\varphi_{a_\varepsilon} \circ P_{a_\varepsilon})(x) \right) \right|$$

$$= \sup_{F \in \mathcal{D}} \sup_{x \in B_{c_0(\mathbb{B},X)}} \left\| \sum_{a_\varepsilon \in F} (\varphi_{a_\varepsilon} \circ P_{a_\varepsilon})(x) \right\|$$

$$= \sup_{F \in \mathcal{D}} \|\pi'_F\|.$$

Also, $\sup_{F \in \mathcal{D}} \|\pi'_F\| < \infty$ since the net $(\pi'_F(x : F \in \mathcal{D})$ converges to Tx for each $x \in c_0(\mathbb{B},X)$. Hence,

$$\sup_{F \in \mathcal{D}} \|\pi'_F\| < \infty$$

By the uniform boundedness principle. This means $\varphi \in \Lambda$ and so Ψ is well- defined.

http://dx.doi.org/10.29322/IJSRP.8.11.2018.p8334

If $\|\varphi\| = 0$, then

$$\|g \circ \varphi\| = 0$$

for each $g \in B_{\ell_1(\mathbb{B}, X^*)}$ and for each $a_\varepsilon \in \mathbb{B}$. This implies each $\varphi_{a_\varepsilon} = 0$ and so $\varphi = 0$. Other conditions of norm can easily be verified.

Further

$$\|Tx\| = \left\| T\left(\sum_{a \in F} (I_{a_\varepsilon} \circ P_{a_\varepsilon})(x) \right) \right\|$$

$$= \left\| \sum_{a_\varepsilon \in \mathbb{B}} (\varphi_{a_\varepsilon} \circ P_{a_\varepsilon})(x) \right\|$$

$$\leq \sup_{F \in \mathcal{D}} \|\pi'_F(x)\|$$

$$\leq \|x\| \sup_{F \in \mathcal{D}} \|\pi'_F\|$$

$$\leq \|x\| \, \|\varphi\|.$$

That is, $\|T\| \leq \|\varphi\|$. On the other hand, for each $F \in \mathcal{D}$,

$$\sum_{a_\varepsilon \in F} \|g \circ \varphi_{a_\varepsilon}\| \leq \sup_{x \in B_{c_0(\mathbb{B}, X)}} \left| g\left(\sum_{a_\varepsilon \in F} \varphi_{a_\varepsilon}(x_{a_\varepsilon}) \right) \right|$$

$$\leq \|g\| \sup_{x \in B_{c_0(\mathbb{B}, X)}} \left\| \sum_{a_\varepsilon \in \mathbb{B}} (\varphi_{a_\varepsilon} \circ P_{a_\varepsilon})(x) \right\|.$$

This implies

$$\|\varphi\| \leq \sup_{x \in B_{c_0(\mathbb{B}, X)}} \left\| \sum_{a_\varepsilon \in \mathbb{B}} (\varphi_{a_\varepsilon} \circ P_{a_\varepsilon})(x) \right\|.$$

$$= \sup_{x \in B_{c_0(\mathbb{B}, X)}} \|Tx\| = \|T\|.$$

This show that Ψ is an isometric isomorphism. Now let us prove that Ψ is also surjective. Let $\varphi \in \Lambda$ be arbitrary and write

$$Tx = \sum_{a_\varepsilon \in \mathbb{A}} (\varphi_{a_\varepsilon} \circ P_{a_\varepsilon})(x), \text{for } x \in c_0(\mathbb{B}, X).$$

We shall show that this formula defines a bounded linear operator on

$c_0(\mathbb{B}, X)$. Let us first show that it is well-defined.

This is obviously equivalent to the summability of the family $\{(\varphi_{a_\varepsilon} \circ P_{a_\varepsilon})(x) : a_\varepsilon \in \mathbb{B}\}$ to Tx, and so is equivalent to the convergence of the net $(\pi'_F(x) : \mathcal{D})$ to Tx for each $x \in c_0(\mathbb{B}, X)$. From the fact that the family $\{(I_{a_\varepsilon}, P_{a_\varepsilon}) : a_\varepsilon \in \mathbb{B}\}$ is an X-biorothogonal system, we have

$$\pi'_{F \cup F_1}(\pi_F(x)) = \pi'_F(\pi_F(x)), \qquad F, F_1 \in \mathcal{D}.$$

This implies $(\pi'_F(\pi_F(x)) : \mathcal{D})$ is a Cauchy net in $c_0(\mathbb{B}, X)$, and so it has a limit. For the set of all $\pi_F(x)$ is dense in $c_0(\mathbb{B}, X)$, and $c_0(\mathbb{B}, X)$ is complete, and we have $(\pi'_F(x) : \mathcal{D})$ converges to Tx, whence, T is well-defined. Linearity is obvious and T is continuous since

$$\sup_{x \in B_{c_0(\mathbb{B}, X)}} \|Tx\| \leq \|\varphi\|.$$

Example (18):

Cleary,

$$\ell_1(\mathbb{B}, \mathcal{L}(X, c_0(\mathbb{B}, X))) \subseteq \Lambda$$

in the theorem and the inclusion relation may be strict. Let us give an example to verify the last sentence. Let $\mathbb{B} = \mathbb{N}$ and $X = \ell_1$. Take

$$\varphi_1 = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix}, \dots \right\}_{n=1}^{\infty}$$

$$\varphi_2 = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix}, \dots \right\}_{n=1}^{\infty}$$

$\vdots$

Then, for each $n \in \mathbb{N}$,

$$\varphi_n(x) = (\varphi_1(x), \varphi_2(x), \dots) = (0, \dots, 0, x_n e_n, 0, \dots)$$

Where 0 is the null vector of ℓ_1 and $e_n$is the nth-vector of the canonical basis of ℓ_1, defines clearly a continuous linear operator from ℓ_1 into $c_0(\ell_1)$, that is, each

$\varphi_n \in \mathcal{L}(\ell_1, c_0(\ell_1))$,

and also $\|\varphi_n\| = 1$. This means

$$\varphi = (\varphi_n)_{n=1}^{\infty} \notin \ell_1(\mathcal{L}(\ell_1, c_0(\ell_1))).$$

However, the following discussion shows that $\varphi = (\varphi_n)_{n=1}^{\infty} \in \Lambda$. Note that $c_0(\ell_1)^* = \ell_1(\ell_\infty)$ so that each $\mathrm{g} \in c_0(\ell_1)^*$ has the form $\mathrm{g} = (\mathrm{g}_1, \mathrm{g}_2, \dots)$ and

$$\mathrm{g}(x) = \sum_{n \in \mathbb{N}} \mathrm{g}_n \cdot x_n = \sum_{n \in \mathbb{N}} \sum_{n \in \mathbb{N}} \mathrm{g}_{nk} \cdot x_{nk}$$

for $(x_n)_{n=1}^{\infty} \in c_0(\ell_1)$, where $x_n = (x_{nk})_{k=1}^{\infty} \in \ell_1$ and $\mathrm{g}_n = (\mathrm{g}_{nk})_{k=1}^{\infty} \in \ell_\infty$ for each $n \in \mathbb{N}$. Hence, for $y \in B_{c_0}$,

$$|(\mathrm{g} \circ \varphi_n)(y)| = |\mathrm{g}(\varphi_{n1}(y), \varphi_{n2}(y), \dots)|$$

$$= \left| \sum_{k \in \mathbb{N}} \mathrm{g}_k \quad \cdot \varphi_{nk}(y) \right| = |\mathrm{g}_{nn} y_n| \leq |\mathrm{g}_{nn}|,$$

where $\varphi_{nn}^{n}(y) = y_n$ and $\varphi_{nk}^{i}(y) = 0$ otherwise. This shows that

$$\sup_{\mathrm{g} \in B_{c_0(\ell_1)^*}} \sum_{n \in \mathbb{N}} \|\mathrm{g} \circ \varphi_n\| \leq \sup_{\mathrm{g} \in B_{\ell_1(\ell_\infty)}} \sum_{n \in \mathbb{N}} \|\mathrm{g}_{nn}\| \leq \|\mathrm{g}\| \leq 1,$$

that is, $\varphi \in \Lambda$.

Corollary (19):

For $\epsilon > 0$, let a family $I_{a_\epsilon} = \{(n_{a_\epsilon}, R_{a_\epsilon}): a_\epsilon \in \mathbb{A}\}$ be given, (see Definition (11). Then $I_{a_\epsilon}$is a Y- biorthogonal system for X if and only if $\{n_{a_\epsilon}: a_\epsilon \in \mathbb{A}\}$ is Y-basis for X with $\{R_{a_\epsilon}\}$ being $A.F.F.$ to $\{n_{a_\epsilon}: a_\epsilon \in \mathbb{A}\}$.

Proof:

For any $\nu \in Y, n_{a_\epsilon+\epsilon}(\nu) \in X$, and so

$$n_{a_\epsilon+\epsilon}(\nu) = \sum_{a_\epsilon \in \mathbb{A}} \left(n_{a_\epsilon} \circ R_{a_\epsilon}\right)\left(n_{a_\epsilon+\epsilon}(\nu)\right).$$

$$= \sum_{a_\epsilon \in \mathbb{A}} n_{a_\epsilon}\left[\left(R_{a_\epsilon} \circ n_{a_\epsilon+\epsilon}\right)(\nu)\right].$$

Hence, by the uniqueness of the family $\{R_{a_\epsilon}\}$, this equality holds if and only if $R_{a_\epsilon} \circ n_{a_\epsilon+\epsilon} = I_y$ for $a_\epsilon = a_\epsilon$, and $R_{a_\epsilon} \circ n_{a_\epsilon+\epsilon} = 0, a_\epsilon \neq a_\epsilon + \epsilon$.

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