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Theory on Dimensions

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Abstract- A study of dimensions and hypercubes including ways to calculate faces of different dimensions in a hypercube, along with calculating the maximum numerical value of the respective dimensions present in the structure. To have a different way of viewing dimensions and to take dimensions themselves into algebraic calculations. These include three main topics:

i. Viewing higher dimensions with respect to lower dimensions, and defining **actual capacity** (bulk) and **relative capacity** of any **dimensional structure**.
ii. Calculating number of vertices, faces, edges....... in **HYPERCUBES**with help of Euler's polyhedron-**polytope** formula and forming relations between various dimensional faces in hypercubes.
iii. Max.D-f relations to calculate dimension containing maximum faces in any higher dimensional **HYPERCUBE** and putting the relations to basic and advanced mathematical operators.

Index Terms- **Dimensional Structure:** A shape/structure in any arbitrary dimension.
Hypercube: Higher dimensional representation of a regular cube (in 3-D).
MaxD-f: Relations to calculate the numerical value of dimensions which is present in the maximum number in any hypercube.
Capacity: Volume of a structure in any dimension.
Polytope: A dimensional structure in more than 3 dimensions.

I. VIEWING HIGHER DIMENSIONS

Note: Zeroth dimension has always a unitary value. It does not affect the capacity of any dimensional structure.
On Earth, the known dimensions are:

1. Zeroth (0-D)
2. 1st (1-D) where length(l) ∈ +ve
3. 2nd(2-D) where l, breadth(b) ∈ +ve
4. 3rd(3-D) where l, b, height(h) ∈ +ve

We define dimensions geometrically by using axes which are mutually perpendicular to each other.
We define the dimensions from zero where the respective axes are:

1) 0-D is represented by O (Origin- as called in Cartesian plane)
2) 1-D = O * X [X→ ∞]
3) 2-D = O * X * Y [Y→ ∞, Y ⊥ X]
4) 3-D = O * X * Y * Z [Z→ ∞, Z ⊥ X,Y]
5) So, to imagine 4-D, we consider an axis T such that [T→ ∞, T ⊥ X,Y,Z]

For viewing higher dimensions from lower dimensions, one has only the axes to perceive the structure that one's dimension corresponds to. For example like a square can be thought of an array of infinite straight lines, all equal in length. Similarly, a cube can be imagined as a row of infinite squares glued to each other along their faces such that the structure formed (a cube) is perpendicular to their length and breadth.

Q1) How many dimensions can be imagined?

Ans: In Physics, the string theory mentions that there can be a maximum of 10 dimensions. However, in mathematics, there can be any axis q where q ⊥ O, x, y, ... (q-1). So, infinite dimensions can be imagined.

So now comes the fact on how we see higher dimensions.

Indicating any dimension viewed by lower dimensions is the maximum length of the 'perceived' dimension "less than or equal to" the view frame (dimension).

This view can be represented as R.View (*dimension*).

- **R.View can be defined as,**

"The relative view of the 'perceivable' dimensions of a 'hyper-structure' with respect to the view frame such that the view frame's dimension is equal to or less than the viewed dimension."

Hence, for any q dimensional object viewed by a g dimensional object

$$R.View\ (q) = max.|\{g, (g-1), \ldots\ldots\ldots, (g-g)\}D|[V^{fr}_{g-D}] \ldots\ldots (\mathbf{1.1})$$

Example; $R.View\ (square) = max.|(length), \ldots\ldots\ldots, (zeroth)| \quad [V^{fr}_{1D}]$

If we can specify the view of higher dimensions, we can also specify their relative capacity (or relative bulk/dimensional volume).

- **R.Cap is nothing but the:**

"Product of the 'perceived' dimensions to specify the relative (partial) capacity of the viewed 'hyper-structure' multiplied with an arbitrary constant depending on the geometrical shape of the structure."

$$R.Cap(q) = max.|\{g(g-1) \ldots\ldots\ldots (g-g)\}D|q_c[V^{fr}_{g-D}]q_c \rightarrow constant(any) \ldots\ldots (\mathbf{1.2})$$

It is important to note that the 'D' is mandatory as without it *g (g-1)... (g-g)* = 0

But, due to the 'D', signifying dimension, we can reframe the equation as

$$R.Cap(q) = max.|\{gD * (g-1)D * \ldots\ldots\ldots * (g-g)D\}|q_c[V^{fr}_{gD}]$$

The arbitrary constant varies with the shape of the structure. For example, in case of a square, it is 1, while in case of a sphere; it is π{Since $4\pi r^2 = \pi*2r*2r$ and 2r=max(length) and 2r = max(breadth) (noting that the view frame is 2-D)}.

Example: Let a cube (3D) of side 'x' be viewed along any of its edges by a straight line (1D) such that the length of the line is also 'x'.

Then, R.Cap (Cube) = max (1) D * max (1-1) D * $q_c[V^{fr}_{1D}]$ = x * q_c

But here for the cube, $q_c = 1 \therefore the\ relative\ capacity\ is\ x\ (Answer)$

<u>ACTUAL CAPACITY (A.CAP)</u>:

- **Actual capacity of any structure is:**

"***The relative capacity of the structure from its own view frame <u>such that the maximum lengths of the dimension are measured, multiplied by an arbitrary constant".</u>***

'It can be also expressed as the relative capacity of the respective dimension from a non-zero viewpoint multiplied by the maximum length (relative term to express dimensions) of the dimensions unperceivable by the view frame.

$$A.Cap(x) = max.|x(x-1)(x-2) \ldots\ldots (x-n_0+1)D| * R.Cap(x) * q_c\left[V^{fr}_{(x-n_0)D}\right] \ldots\ldots (\mathbf{1.3})$$

$$= max.|x(x-1)(x-2) \ldots\ldots (x-x)D| * q_c\left[V^{fr}_{(x)D}\right]$$

$$= A.Cap(x) max.|(x-x)D| * q_c[V^{fr}_{0D}] \quad = A.Cap(x)\ [Since, (x-x)D \rightarrow unitary]$$

II. HYPERCUBES

EULER'S POLYTOPE FORMULA – Euler's formula states that number of i-dimensional faces in an n-dimensional solid is given by the formula:

$$2^{n-i}\binom{n}{n-i}$$

This is the basis of all functions and calculations for hypercubes.

i)VERTICES IN HYPERCUBES

The number of vertices in a d-dimensional hypercube is given by the formula 2^d.
Now let us give the notation V_m to the number of vertices in a hypercube of dimension m.

$$\therefore \frac{V_m}{V_n} = 2^{m-n} \dots\dots (i) V_m * V_n = 2^{m+n} \dots\dots (ii)$$

$$(V_m)^p = 2^{mp} \dots\dots (iii) (V_m)^{\frac{1}{p}} = 2^{\frac{m}{p}} \dots\dots (iv) \dots\dots (\mathbf{2.1\ eqns.})$$

ii)EDGES IN HYPERCUBES

The number of edges in a d-dimensional hypercube is given by the formula
$d * 2^{d-1}$.

Let us denote number of edges in a sD hypercube $as\ E_s$.

$$\frac{E_m}{E_n} = \left(\frac{m}{n}\right) * 2^{m-n} = \frac{m}{n} * \frac{V_m}{V_n}$$

$$or, nV_nE_m = mV_mE_n \quad also,\ 2E_m = mV_m \dots\dots (\mathbf{2.2\ eqns.})$$

$Finally, we\ can\ also\ have\ an\ expression \quad E_{m+1} = 2E_m + 2^m$

VERIFICATION:

$$\mathrm{E_m = m * 2^{m-1} \therefore E_{m-1} = (m-1)2^{m-2} \qquad or,\ 2E_{m-1} = m * 2^{m-1} - 2^{m-2}}$$

$$\mathrm{or, 2E_{m-1} + 2^{m-1} = m * 2^{m-1};\ 2E_m + 2^m = (m+1)2^m = E_{m+1} (verified)}$$

$$\therefore \mathrm{\frac{E_{m+1}}{E_m} = 2\frac{(m+1)}{m}}$$

$$Also, \qquad \frac{xV_yE_y}{yV_xE_x} = 4^{y-x} \dots\dots (\mathbf{2.3})$$

iii)OTHER DIMENSIONS IN HYPERCUBES

Similarly we can denote:

i) **Faces ($F_{dimension}$):** $F_x = 2^{x-2} * \frac{x(x-1)}{2} \dots\dots (i) F_x = 2^{x-y} F_y \frac{x(x-1)}{y(y-1)} \dots\dots (ii)$

$$8F_x = x(x-1)V_x \dots\dots (iii) 4F_x = (x-1)E_x \dots\dots (iv) \dots\dots (\mathbf{2.4})$$

ii) **Cubes** $(\boldsymbol{C_x})$

iii) **Tesseracts** $(\boldsymbol{T_x})$......................... And so on.

Generally, we can write the ratio of two different dimensions D and E in two different dimensional hypercubes x and y as:

$$(1)\frac{D_X}{E_y} = 2^{x-n_d-y+n_e} * \frac{x(x+1)\ldots\ldots(x+n_d-1)}{y(y+1)\ldots\ldots(y+n_e-1)} * \frac{n_e!}{n_d!}$$

[D_x – Any dimension's total face number in x-dimension {same for E_y}; n_d, n_e – Numerical value of dimensions D and E (e.g.n_C = 3)]

III. MAX-D FACE RELATIONS IN HYPERCUBES

Another interesting thing comes up when talking about hypercubes.

As noticed, in a cube, there are 6 faces, 12 edges and 8 vertices. Hence the number of edges is the maximum in a cube. However, in a square there are four vertices and four edges. So, both the edge and face in a square are equal and maximum.

Surprisingly, there is a particular pattern to this. If the dimension containing the maximum faces in a structure be represented by $f^{max.D}$, then we would get the following table {**3.1**}:

Hypercube of dimension	
0	**0**
1	**0**
2	**0,1**
3	**1**
4	**1**
5	**1,2**
6	**2**
7	**2**
8	**2,3**
9	**3**
10	**3**
11	**3,4**
......	**.....and so on**

Hence, there is a definite pattern in which the dimensions are in a hypercube.

A) **If any dimension M is of the form 3q or 3q+1,** $\boldsymbol{f^{max.D}[M] = q.}$

B)**If M is of form 3q+2,** $\boldsymbol{f^{max.D}[M] = q, (q+1). \ldots\ldots (3.1)}$

These are the two fundamental relations.

Dimensions belong to whole numbers (non-negative integers); so any dimension is of the form 3q, 3q+1 or 3q+2, where q is any whole number.

The proofs to these functions include Euler's polytope formula:
i) Consider M = 3m

a) We have to show that Face (m) > Face (m+q) [such that m + q $\leq$ 3m and m, q ∊ Z also **q>0**]
In other words,

$$2^{M-m}\binom{M}{m} > 2^{M-(m+q)}\binom{M}{m+q}$$

: We know that $2m + 2 > 2m - q + 1$

Similarly, $2m + 4 > 2m - q + 2;\ \ 2m + 6 > 2m - q + 3\ and\ so\ on\ till\ \ \ 2m + 2q > 2m$

Multiplying all the inequalities in order of their magnitude, we get:

$$\frac{(2m-q+1)(2m-q+2)\dots\dots\dots(2m)}{(2m+2)(2m+4)(2m+6)\dots\dots\dots(2m+2q)} < 1$$

Taking all the 2's common from the denominator and changing the products into factorial functions, we get:

$$\frac{(2m)!\,m!}{2^q(2m-q)!\,(m+q)!} < 1$$

$$or, \frac{1}{(3m-(m+q))!\,(m+q)!} < \frac{2^q}{(3m-m)!\,m!}$$

Multiplying both sides by (3m)!

$$\frac{(3m)!}{(3m-(m+q))!\,(m+q)!} < 2^q\frac{(3m)!}{(3m-m)!\,m!}$$

$$or,\ \ 2^{-q}\binom{3m}{m+q} < \binom{3m}{m}$$

$$or,\ \ 2^{3m-m-q}\binom{3m}{m+q} < 2^{3m-m}\binom{3m}{m}$$

Substituting 3m by M, we get

$$2^{M-(m+q)}\binom{M}{m+q} < 2^{M-m}\binom{M}{m}\ (\boldsymbol{proved})\ \dots\dots(\mathbf{3.2})$$

ii) Similarly, if we consider M = 3m+1

$$To\ prove\text{:}\ 2^{M-(m+q)}\binom{M}{m+q} < 2^{M-m}\binom{M}{m}$$

We have to say that $2m+2 > 2m+2-q;\ \ 2m+4 > 2m+3-q$ all the way up to $2m+2q > 2m+1$; then to follow the same steps as in the first case, just to multiply the two terms of the inequality with $2^{3m+1-m}(3m+1)!$ And then we can replace 3m+1 with M to **prove the result**.

For, M = 3m+2, we have a special case, i.e. Face (m) = Face (m+1).

We can see that if we go in the same way as case i and case ii.

$$2m+2 \geq 2m+3-q$$
$$But, 2m+4 > 2m+4-q;\ \ 2m+6 > 2m+5-q; \dots But\ \ 2m+2q \geq 2m+2$$

So, it can be clearly seen if q = 1, then $2m+2q = 2m+2.$

$$\therefore 2m+2 = 2(m+1); \quad or, \frac{(2m+2)!}{(2m+1)!} = \frac{2(m+1)!}{m!}$$

Following the steps of simplifying the expression into a combination;

$$\binom{3m+2}{m} = 1/2\binom{3m+2}{m+1}$$
$$or, 2^{M-m}\binom{M}{m} = 2^{M-(m+1)}\binom{M}{m+1} \textbf{(proved)} \ldots\ldots \textbf{(3.3)}$$

In this way, we can also prove Face (m) > Face (m-q) [such that m-q $\geq$ 0]

After this proof is done, it is easier to understand and implement complex Max.D relations and formulae.

But first, let us denote an expression by E_p; and if A is of form B, let us denote it by $A \equiv\equiv B$.
Compound examples include:

$$i)\ {}^{max.D}{}_f[3q(3q+1)] = {}^{max.D}{}_f[3(q(3q+1))] \equiv\equiv {}^{max.D}{}_f[3M] = q(q+1)$$

$$ii)\ {}^{max.D}{}_f[3q(3q+2)] = {}^{max.D}{}_f[3(q(3q+1))] \equiv\equiv {}^{max.D}{}_f[3M] = q(q+2)$$

$$iii)\ {}^{max.D}{}_f[(3q+2)(3q+1)] = {}^{max.D}{}_f[9q^2+9q+2] \equiv\equiv {}^{max.D}{}_f[3M+2]$$
$$= 3q(q+1), (3q(q+1)+1) \ldots\ldots \textbf{(3.4)}$$

To Note:**It is not possible to solve a Max.D – f relation if the variable has no coefficient or if the coefficient cannot be factorized in terms of 3q, 3q+1 or 3q+2.**

E.g. Find ${}^{max.D}{}_f[5p+1]$

A: This is not possible since $5q+1 \equiv\equiv 3q_0 + t \ [0 \leq t \leq 2, t \in Z]$

This is only possible when $q \equiv\equiv 3q_0 + t$.

E.g.2 Find ${}^{max.D}{}_f[p]$.

This is not possible unless we know the form of p and can express it in the form 3q, 3q+1, 3q+2.
But, for these problems, we can say that p is of form 3q, 3q+1 or 3q+2 and show the respective cases.

- **Multiple (or looped) ${}^{max.D}{}_f$ relations can be denoted by ${}^{max.D^t}{}_f[M]$** , ***where t denotes the number of times Max.D - f relation is to be carried out.***

$${}^{max.D^t}{}_f[M] = {}^{max.D}{}_f[{}^{max.D}{}_f[{}^{max.D}{}_f[\ldots(t\ times)\ldots\ldots {}^{max.D}{}_f[M]]]]]]\ldots\ldots]]]$$
$$Now,\ {}^{max.D}{}_f[3p] = p;\ {}^{max.D^2}{}_f[9p] = {}^{max.D}{}_f[3p] = p$$
$$Also,\ {}^{max.D^3}{}_f[27p] = p$$

So, a generalization of this will be: $\boldsymbol{{}^{max.D^t}{}_f[3^t p] = p}$

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More precisely, $\boldsymbol{{}^{max.D}{}_f^{\,t}[3^t p] = [3^{t-t} p] = p}$

$\boldsymbol{or,\ {}^{max.D}{}_f^{\,t}[3^q p] = [3^{q-t} p] \ldots\ldots (3.5)}$

It may be an enquiry as to what will happen if t > q. As a dimension is always a whole number, 3 raised to a negative power may make the result a fraction, which is not acceptable.

Hence, third brackets [ ] are used to denote dimensions, which in mathematics denotes greatest integral function. E.g. [7.256843] = 7

In hypercubes however, there is a bit modification of this rule for dimensions of the form 3q+2.

$If\ any\ dimension\ is\ represented\ by\ [a_1 a_2 \ldots a_n . b_1 b_2 \ldots b_n], there\ can\ be\ 2\ cases.$

$i)\ If\ b_1 < 5, b_1 \in Z, then\ E_p = (a_1 a_2 \ldots\ldots a_n)$

$ii)\ If\ b_1 \geq 5, b_1 \in Z, then\ E_p = (a_1 a_2 \ldots\ldots a_n), (a_1 a_2 \ldots\ldots a_n + 1)$

To note: Here $0 \leq\ a_1, a_2, \ldots, a_n, b_1, b_2, \ldots\ldots, b_n \leq 9 \ldots\ldots (\mathbf{3.6\ eqns.})$

Hence, ${}^{max.D}{}_f[2] = \left[\frac{2}{3}\right] = [0.6666 \ldots\ldots \infty] = 0{,}1\ (verified)$

Or, ${}^{max.D}{}_f[1] = \left[\frac{1}{3}\right] = [0.333 \ldots\ldots \infty] = 0\ (verified)$

This leads us to Loop Simplification and decimal estimation method.

If, in general, we have

${}^{max.D}{}_f^{\,c}[A] = {}^{max.D}{}_f^{\,d}[B]\ provided\ \boldsymbol{c \geq d\ and\ A \equiv\equiv B \equiv\equiv}$
$\boldsymbol{3t}, we\ can\ do\ a\ general\ subtraction\ of\ the\ common\ loops\ to\ get:$
${}^{max.D}{}_f^{\,c-d}[A] = [B] \ldots\ldots (\mathbf{3.7})$

Note: $\boldsymbol{{}^{max.D}{}_f[A,B] = {}^{max.D}{}_f[A], {}^{max.D}{}_f[B]}$

- **Loop Simplification:**

“*Loop Simplification is the process of step-by-step simplification by finding out the value of a loop one at a time.”*

${}^{max.D}{}_f^{\,t}[M] = {}^{max.D}{}_f[{}^{max.D}{}_f[{}^{max.D}{}_f[\ldots(t\ times)\ldots\ldots {}^{max.D}{}_f[M]]]]]\ldots\ldots]]]\ (\mathbf{3.8})$
$And\ then\ clearly\ we\ can\ solve\ the\ looped\ relation.$
$E.g. Find\ {}^{max.D}{}_f^{\,3}[32]$
$This\ can\ be\ rewritten\ as:\ {}^{max.D}{}_f[{}^{max.D}{}_f[{}^{max.D}{}_f[32]]]$
$= {}^{max.D}{}_f[{}^{max.D}{}_f[[10{,}11]]]$
$= {}^{max.D}{}_f[[3, (3{,}4)]]$

$= {}^{max.D}{}_{f}[3,4] \quad (Since, only\ unique\ values\ are\ counted, so\ 3,3\ becomes\ just\ 3).$
$= [1,1] = [1] = 1 = {}^{max.D}{}^{3}_{f}[32]$

But, as it is seen, Loop Simplification is a very lengthy process. If the loop is a large number such as 6 or 8, loop simplification will take a lot of space and time.
As an alternative to this, we can have the decimal estimation problem.

For this, we need decimal estimation**(Refer to 3.6 eqns.)**.

- **Decimal Estimation:**

(With reference to 3.6 eqns.) *"It is an estimation made on a Max.D-f function such that if the decimal value, when estimated to the nearest whole number, gives the lower limit as the answer, the actual answer is the lower limit. In case it tends to the upper limit, both the lower and the upper limits will be the answer and will be separated by a comma representing that both dimensions are maximum in that particular loop of the specified dimension."*
In decimal estimation method, we have:

$${}^{max.D}{}^{t}_{f}[M] = \left[\frac{M}{3^t}\right] \dots \dots (\mathbf{3.9})$$

Now, we can give examples and then verify it by Loop Simplification method.

$E.g.\ Find\ {}^{max.D}{}^{3}_{f}[46]$

Ans: $E_p = \left[\frac{46}{3^3}\right] = \left[\frac{46}{27}\right] = [1.703704 \dots .] = 1,2 \quad [Since\ b_1 > 5]$

Verification: ${}^{max.D}{}^{3}_{f}[46] = {}^{max.D}{}^{2}_{f}[15] = {}^{max.D}{}_{f}[5] = 1,2(verified)$

Another example may include:

$Find\ {}^{max.D}{}^{4}_{f}[354]$

Ans: $E_p = \left[\frac{354}{3^4}\right] = \left[\frac{354}{81}\right] = [4.37037 \dots .] = 4$

Verification:

$${}^{max.D}{}^{4}_{f}[354] = {}^{max.D}{}^{3}_{f}[118] = {}^{max.D}{}^{2}_{f}[39] = {}^{max.D}{}_{f}[13] = 4(verified)$$

- **Fundamental operations on Max.D-f relations:**

Addition (Subtraction requires similar functions):

$i)\ {}^{max.D}{}_{f}[3(a+c)] = {}^{max.D}{}_{f}[3a] + {}^{max.D}{}_{f}[3c]$

$ii)\ {}^{max.D}{}_{f}[3(a+c)+1] = {}^{max.D}{}_{f}[3a] + {}^{max.D}{}_{f}[3c] + {}^{max.D}{}_{f}[1]$
$\quad = {}^{max.D}{}_{f}[3a+1] + {}^{max.D}{}_{f}[3c]\ (or\ vice\ versa)$

$iii)\ {}^{max.D}{}_{f}[3(a+c)+2] = {}^{max.D}{}_{f}[3a] + {}^{max.D}{}_{f}[3c] + {}^{max.D}{}_{f}[2]$
$\quad = {}^{max.D}{}_{f}[3a+2] + {}^{max.D}{}_{f}[3c](or\ vice\ versa) \dots \dots (\mathbf{3.10\ eqns.})$

But, ${}^{max.D}{}_{f}[3(a+c)+2] \neq {}^{max.D}{}_{f}[3a+1] + {}^{max.D}{}_{f}[3c+1]$

Here we see a constant break. **"For any constant break in a Max.D-f relation, *providedit is in the simplest form,* ${}^{max.D}{}_{f}[2]$ needs to be added to the sum."**

$E.g.\ {}^{max.D}{}_{f}[3(a+c)+2] = {}^{max.D}{}_{f}[3a+1] + {}^{max.D}{}_{f}[3c+1] + {}^{max.D}{}_{f}[2]$

$= a + c + (0,1) = (a + c + 0), (a + c + 1) = (a + c), (a + c + 1)\ (correct)$

To Note: We cannot break non-simplified constants.

$E.g.\ {}^{max.D}{}_f[3(a + c) + 10] \neq {}^{max.D}{}_f[3a + 5] + {}^{max.D}{}_f[3c + 5] + {}^{max.D}{}_f[2]$

We can solve this by first simplifying 3(a+c) +10 into 3(a+c+3) +1, then following the respective steps mentioned.

$If\ a \equiv\equiv 3s, then\ {}^{max.D}{}_f[a + 2] = {}^{max.D}{}_f[a] + {}^{max.D}{}_f[2]$

$If\ a \equiv\equiv 3s + 1, then\ {}^{max.D}{}_f[a + 2] = {}^{max.D}{}_f[a - (1)] + {}^{max.D}{}_f[2 + (1)]$

$If\ a \equiv\equiv 3s + 2, then\ {}^{max.D}{}_f[a + 2] = {}^{max.D}{}_f[a - (2)] + {}^{max.D}{}_f[2 + (2)]$

$If\ a \equiv\equiv 3s + p, then\ {}^{max.D}{}_f[a + 2] = {}^{max.D}{}_f[a - p] + {}^{max.D}{}_f[2 + p][0 \leq p \leq 3]$

$\dots\dots(\mathbf{3.11})$

Let this be called the Constant Operation Rule for Max.D-f Hypercubes (COR-HC).

Some proofs: $\{Saying\ that\ n, p, s, t, k \in |I|\}$

$i)\ (1 + n)^p \equiv\equiv 3t + 1\ if\ n \equiv\equiv 3s$

$ii)\ (1 + n)^p \equiv\equiv 3t + 1, \quad if\ n \equiv\equiv 3s + 1, \quad and\ if\ p \equiv\equiv 2k$

$iii)\ (1 + n)^p \equiv\equiv 3t + 2, \quad if\ n \equiv\equiv 3s + 1, \quad and\ if\ p \equiv\equiv 2k + 1$

$iv)\ (1 + n)^p \equiv\equiv 3t, \quad if\ n \equiv\equiv 3s + 2$

- **PROOFS:**

CASE (i):

$(1 + n)^p \equiv\equiv (1 + 3s)^p = 1 + M(3)[M(n) = multiple\ of\ n]\ \equiv\equiv 3t + 1$

CASE (ii):

$(1 + n)^p \equiv\equiv (3s + 2)^{2k} = M(3) + 4^k = M(3) + (3 + 1)^k$

$= M(3) + M(3) + 1 = M(3) + 1 \equiv\equiv 3t + 1$

CASE (iii):

$(1 + n)^p \equiv\equiv (3s + 2)^{2k+1} = M(3) + 2^{2k+1} = M(3) + 2^{2k} * 2$

$= M(3) + [M(3) + 1] * 2 = M(3) + M(3) + 2 = M(3) + 2 \equiv\equiv 3t + 2$

CASE (iv):

$(1 + n)^p \equiv\equiv (3 + 3s)^p = M(3) + 3^p = M(3) + M(3) = M(3) \equiv\equiv 3t$

$\dots\dots(\boldsymbol{Proofs\ 3.1})$

Multiplication:

- **Two Important theorems:**

${}^{max.D}{}_f[3p] * {}^{max.D}{}_f[3q] = p * q = pq = {}^{max.D}{}_f[3pq] = {}^{max.D}{}_f[3] * {}^{max.D}{}_f[3pq]$

As we can see both ${}^{max.D}{}_f[3pq]\ and\ {}^{max.D}{}_f[3] * {}^{max.D}{}_f[3pq]$ yield the same result pq, but the pattern of writing is different. In the first case, we eliminate the common coefficient of the variable, while in the next case; we separate one of the coefficients.

Let the first case be called Common Coefficient Elimination (CCEl) and the next case be called Broken Coefficient Separation (BCS).

Hence; ${}^{max.D}{}_f[3a_1]{}^{max.D}{}_f[3a_2]{}^{max.D}{}_f[3a_3]\dots\dots{}^{max.D}{}_f[3a_n]$

$= {}^{max.D}{}_f[3a_1 a_2 a_3 \dots\dots a_n]\{By\ CCEl\}$

$= \{{}^{max.D}{}_f[3]\}^{n-1} * {}^{max.D}{}_f[3a_1 a_2 a_3 \dots\dots a_n]\ \{By\ BCS\}$

Now,

${}^{max.D}{}_f[3p + 1] * {}^{max.D}{}_f[3q + 1] = pq$

$= \left({}^{max.D}{}_f[3p] + {}^{max.D}{}_f[1]\right) * \left({}^{max.D}{}_f[3q] + {}^{max.D}{}_f[1]\right)$
$= {}^{max.D}{}_f[3p] * {}^{max.D}{}_f[3q] + {}^{max.D}{}_f[3p] * {}^{max.D}{}_f[1] + {}^{max.D}{}_f[3q] * {}^{max.D}{}_f[1] {}^{max.D}{}_f[1] * {}^{max.D}{}_f[1]$
$= {}^{max.D}{}_f[3p] * {}^{max.D}{}_f[3q] + {}^{max.D}{}_f[1] * {}^{max.D}{}_f[1] = {}^{max.D}{}_f[3p] * {}^{max.D}{}_f[3q] = pq$

And, ${}^{max.D}{}_f[3p+2] * {}^{max.D}{}_f[3q+2] = p,(p+1) * q,(q+1)$
$= \left({}^{max.D}{}_f[3p] + {}^{max.D}{}_f[2]\right) * \left({}^{max.D}{}_f[3q] + {}^{max.D}{}_f[2]\right)$
$= {}^{max.D}{}_f[3p] * {}^{max.D}{}_f[3q] + {}^{max.D}{}_f[3p] * {}^{max.D}{}_f[2] + {}^{max.D}{}_f[3q] * {}^{max.D}{}_f[2] + {}^{max.D}{}_f[2] * {}^{max.D}{}_f[2]$
$= pq + 0,(p) + 0,(q) + (0,1) * (0,1)$
$= pq, (pq+p), (pq+q), (pq+p+q+1) = p,(p+1) * q,(q+1)$

Hence, we can see:

$A,(A+1) * B(B+1) = AB + 0,(A) + 0,(B) + (0,1)^2 \ldots\ldots (\mathbf{3.12})$

Coming further, we see that if we are given functions like ${}^{max.D}{}_f[A,(A+1),(A+2)]$, *we can break them into* ${}^{max.D}{}_f[A], {}^{max.D}{}_f[A+1], {}^{max.D}{}_f[A+2]$.

➢ **InverseRelation:**

But at the same time, we can be given ${}^{max.D^{-1}}{}_f[\boldsymbol{A},(\boldsymbol{A}+\mathbf{1}),(\boldsymbol{A}+\mathbf{2})]$**, i.e. the inverse of A, (A+1), (A+2). Max.D-f inverse is just the opposite of Max.D-f.**

$E.g.\ Find\ {}^{max.D^{-1}}{}_f[a_1].$
$Ans: 3a_1, 3a_1+1$
$Or, {}^{max.D^{-1}}{}_f[a_1,(a_1+1)] = 3a_1+2$

But then what is ${}^{max.D^{-1}}{}_f[A,(A+1),(A+2)]?$

We have to understand that Max.D-f functions can be at most yielding a result like a, (a+1). So, A, (A+1), (A+2) is clearly [A, (A+1)], [(A+1), (A+2)]. The extra A+1 comes from the fact that only unique values are kept.

Hence the expression simplifies to:

${}^{max.D^{-1}}{}_f[A,(A+1)], {}^{max.D^{-1}}{}_f[(A+1),(A+2)]$
$= (3A+2), (3A+5)$

Or,

${}^{max.D^{-1}}{}_f[A], {}^{max.D^{-1}}{}_f[(A+1)], {}^{max.D^{-1}}{}_f[(A+2)]$
$= 3A, 3A+1\ , \quad 3A+3, 3A+4, \quad 3A+6, 3A+7$

At the same time, we cannot solve ${}^{max.D^{-1}}{}_f[A,B]$ in the same way if "A" and "B" is not same or adjacent non-negative integers. So, then we have to divide it into two parts:

${}^{max.D^{-1}}{}_f[A], {}^{max.D^{-1}}{}_f[B] = 3A, 3B$

Clearly, it can be understood that the inverse relation yields many results. There is no clear/distinct pre-image of the relation. This accounts for the relation to be more forwardly than backwardly explicit.

➢ **Summation of like Max.D-f relations:**

Let us add two Max.D-f relations of dimensions of form M (3) +2.

$$^{max.D}{}_{f}[3p+2] + {}^{max.D}{}_{f}[3q+2] = p,(p+1)+q,(q+1)$$
$$= p,(p+q+1),(p+q+1),(p+q+2) = p,(p+q+1),(p+q+2)$$
$$Now, \quad (p+q),(p+q+1),(p+q+2) + {}^{max.D}{}_{f}[3r+2]$$
$$= (p+q+r),(p+q+r+1),(p+q+r+2),(p+q+r+3)$$

In general,

$$^{max.D}{}_{f}[3a_1+2] + \cdots + {}^{max.D}{}_{f}[3a_n+2] = \Psi_{j=1}^{n}\left(\sum_{i=1}^{n} a_i + j\right)$$

Let us introduce a new relation here. Let it be called the Psi(Ψ) relation.

To Note: $\Psi_{i=1}^{n} = 1,2,3, \ldots\ldots\ldots, (n-1), n$

$E.g. \ \Psi_{i=p}^{p+4} = p,(p+1),(p+2),(p+3),(p+4)$

Hence,

$$\sum_{i=1}^{n} {}^{max.D}{}_{f}[3a_i+2] = \Psi_{j=1}^{n}\left(\sum_{i=1}^{n} a_i + j\right)$$

But can we write the Max.D-f relations as a single relation? Yes we can.

We know that: $^{max.D^{-1}}{}_{f}[A,(A+1),(A+2)]$

$$= {}^{max.D^{-1}}{}_{f}[A,(A+1)], {}^{max.D^{-1}}{}_{f}[(A+1),(A+2)]$$
$$= (3A+2),(3A+5)$$

Or, $^{max.D}{}_{f}[(3A+2),(3A+5)] = A,(A+1),(A+2)$

Also, 3A+5 = 3A +{3*2 - 1}

Hence;

$$\sum_{i=1}^{n} {}^{max.D}{}_{f}[3a_i+2] = {}^{max.D}{}_{f}\left[\Psi_{j=1}^{n}\left(3\sum_{i=1}^{n} a_i + \{3j-1\}\right)\right] = \Psi_{j=1}^{n}\left(\sum_{i=1}^{n} a_i + j\right)$$

As for dimensions of form 3k or 3k+1, it is easy to denote their summation.

Hence, the equations for the dimensions are:

$$\sum_{i=1}^{n}[3a_i] = {}^{max.D}{}_{f}\left[3\sum_{i=1}^{n} a_i\right] = \sum_{i=1}^{n} a_i \qquad (\boldsymbol{a})$$

$$\sum_{i=1}^{n}[3a_i+1] = {}^{max.D}{}_{f}\left[3\sum_{i=1}^{n} a_i + 1\right] = \sum_{i=1}^{n} a_i \ (\boldsymbol{b})$$

$$\sum_{i=1}^{n} {}^{max.D}{}_{f}[3a_i+2] = {}^{max.D}_{f}\left[\Psi_{j=1}^{n}\left(3\sum_{i=1}^{n} a_i + \{3j-1\}\right)\right] = \Psi_{j=1}^{n}\left(\sum_{i=1}^{n} a_i + j\right)(\boldsymbol{c})$$
$\ldots\ldots(\mathbf{3.13\ eqns.})$

$$\therefore \sum_{j=3a}^{3(a+n)+2} {}^{max.D}{}_{f}[j] = 2\sum_{i=0}^{n}(a+i) + \left(\sum_{i=0}^{n}(a+i)\right), \ldots, \left(\sum_{i=0}^{n}(a+i)+(n+1)\right)$$
$$= 2\sum_{i=0}^{n}(a+i) + \Psi_{k=0}^{(n+1)}\left(\sum_{i=0}^{n}(a+i)+k\right)$$
$$= (n+1)(2a+n) + \Psi_{k=0}^{(n+1)}\left(\frac{(n+1)(2a+n)}{2}+k\right)\ldots\ldots(\mathbf{3.14})$$

- **Simplified term of a factorial:**

$${}^{max.D}{}^{t}_{f}[M!] = \left[\frac{M!}{3^t}\right]$$

Can we find p such that $3 \nmid {}^{max.D}{}^{p}_{f}[M!]$?
Yes, we can.
We know that the total number of "n's" present in "M!" is given by:

$$\left[\frac{M}{n}\right]+\left[\frac{M}{n^2}\right]+\left[\frac{M}{n^3}\right]+\cdots+\left[\frac{M}{n^p}\right] < DE\ PROLIGNAC'S\ FORMULA >$$

Such that "n^p" is the largest power of "n" in "M!"
By replacing n with 3, we get:

$$\left[\frac{M}{3}\right]+\left[\frac{M}{3^2}\right]+\left[\frac{M}{3^3}\right]+\cdots+\left[\frac{M}{3^p}\right]$$

Hence, $3 \nmid {}^{max.D}{}_{f}^{\left[\frac{M}{3}\right]+\left[\frac{M}{3^2}\right]+\left[\frac{M}{3^3}\right]+\cdots+\left[\frac{M}{3^p}\right]}[M!]\ldots\ldots(\mathbf{3.15})$

Short formula:
i) Finding Max.D-f for given coefficients:

$$Given,\ {}^{max.D}{}_{f}[2n] = q,(q+1),\quad and\ r = 3t+p\quad \{0\le p<3\,;\ p\,\epsilon\,|I|\},$$
$$then\quad 2(n-r) = 3(q+1-3t-p)+(3t+p-1)$$

$$i)Hence\ {}^{max.D}{}_{f}[2(n-r)] = (2q-2t)\ \{p=0\}$$

$$ii)Or,\ {}^{max.D}{}_{f}[2(n-r)] = (2q-2t),(2q-2t+1)\ \{p=1\}$$

$$iii)Or,\ {}^{max.D}{}_{f}[2(n-r)] = (2q-2t)\ \{p=2\}\ldots\ldots(\mathbf{3.16})$$

- **Division of Max.D-f relations (*Decimal estimated):***

We can show decimal estimation by third brackets "[ ]".

We can put forward some general division rules:

To Note: $p \neq 0$

$$Case\ (i) \qquad \left[\frac{^{max.D}_{f}[3p+1]}{^{max.D}_{f}[3p]}\right] = \left[\frac{p}{p}\right] = 1$$

$$Hence, \left[\frac{^{max.D}_{f}[3p+1]}{^{max.D}_{f}[3p]}\right] = \left[\ \left[\frac{3p+1}{3p}\right]\right] = \left[\ \left[\frac{3p}{3p} + \frac{1}{3p}\right]\right]$$

$$= \left[\ \left[1 + \frac{0.3333\ldots\ldots\infty}{p}\right]\right] = [\ \ 1] = 1$$

$$Case\ (ii) \left[\frac{^{max.D}_{f}[3p+2]}{^{max.D}_{f}[3p]}\right] = \left[\frac{p,(p+1)}{p}\right] = 1, \left[1 + \frac{1}{p}\right]$$

Here, if $p > 2,\ p \in |I|\ then\ E_p = 1 \quad but\ if\ 0 < p \leq 2, then\ E_p = 1,2$

$$Case(iii) \quad Finally, \left[\frac{^{max.D}_{f}[3p+2]}{^{max.D}_{f}[3p+1]}\right] = \left[\frac{p,(p+1)}{p}\right] = 1, \left[1 + \frac{1}{p}\right]$$

If $p > 2,\ p \in |I|\ then\ E_p = 1 \quad but\ if\ 0 < p \leq 2, then\ E_p = 1,2$

$$\therefore\ E_p = 1\ or\ 1,2 \qquad \ldots\ldots(\mathbf{3.17\ eqns.})$$

Thus, several theories on hypercubes can actually be clubbed into an overall expression of faces in Hypercubes derived from Euler's Polyhedron (Polytope) Formula.

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