A Mixed Type Approximation Method for Radiation and Scattering Problems

J. Sriranganesan*, H. M. Nasir**

[*] Department of Mathematics, Faculty of Science, Eastern University, Sri Lanka
[**]Department of Mathematics and Statistics, College of Science, SulatanQaboos University, Oman

Abstract- We consider the scattering problem given by the Exterior Helmholtz equation in two dimension by the finite element method. We use a finite element approximation method for the matrix element which correspond to the exact non-local radiation boundary condition on the artificial boundary. We propose a mixed type approximation method to implement the exact Dirichlet to Neumann (DtN) boundary condition that directly gives an approximation matrix for the Helmholtz problem. We use non- uniform partitioning of the artificial boundary by considering some example for which analytical solution was computed. We prove the convergence of the solution of our method to the exact solution when the exact DtN boundary condition is applied on a circular boundary.

Index Terms- Helmholtz equation, Non-local operator, DtN mapping, Artificial boundary condition, Mixed-type method.

I. Introduction

We consider the following exterior Helmholtz problem with boundary and Sommerfeld radiation conditions in two dimensions:

$$-\Delta u - k^2 u = 0 \qquad \text{in } \Omega \qquad (1)$$

$$\frac{\partial u}{\partial n} = -\frac{\partial u^{inc}}{\partial n} \text{ on } \partial\Omega \qquad (2)$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial n} - iku\right) = 0 \qquad (3)$$

where Ω is the interior of the complement of a bounded obstacle O in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ of the scatterer$O$. We assume that $O = \mathbb{R}^2 \backslash \overline{\Omega}$ is a bounded open set. Δ is the Laplace operator and **n** is the outward unit normal vector on the scatterer. u^{inc} is the time harmonic incident wave, which is the plane wave, given by $u^{inc} = e^{ik.x}$ whose direction of propagation is given by the vector k. The boundary condition (3) at infinity is the Sommerfeld radiation condition in two dimensions, which allows only the unique outgoing wave in the solution of the Helmholtz equation (1) [11].

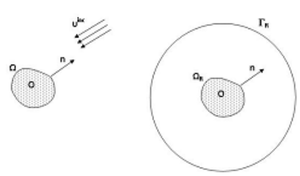


Figure 1: Obstacle and artificial boundary

To solve the exterior Helmholtz problem by numerical methods in an unbounded domain, it is usual to introduce an artificial boundary to make the computational domain as a bounded domain. Then some boundary conditions must be imposed on this boundary. An artificial boundary condition, which is called a Radiation Boundary Condition, is imposed on this boundary such that the solution in the bounded domain is the same as the behavior of the solution of the original exterior Helmholtz problem. The radiation boundary condition satisfies the exact Dirichlet to Neumann boundary condition which is defined by Dirichlet to Neumann mapping (DtN mapping). The exact DtN mapping is defined as the mapping from the Dirichlet data on the artificial boundary to its normal derivative on the same boundary of the solution of the Helmholtz problem. The DtN mapping is non-local and contains infinite sum whose terms involve the Hankel functions and their derivatives. In numerical computation, this infinite series is truncated into a finite number of terms.

The non-local radiation boundary conditions expressed as a pseudo-differential operator and considered its numerical approximation. Also, we introduce a mixed type approximation method for the sesqui-linear form corresponding to the pseudo-differential operator. This gives a simple finite element matrix for these forms when non-uniform mesh is used. We represent the non-local DtN mapping on the artificial boundary as a pseudo-differential operator, which is defined as a function of the Laplace-Beltrami operator $D^2 := -\frac{\partial^2}{\partial\theta^2}$ on the circular artificial boundary. Specifically, the pseudo-differential operator $M(D^2)$ gives the sesque-linear form defined by $(M(D^2).\,,\,.)_{\Gamma_R}$ with respect to the basic inner product $(.\,,\,.)_{\Gamma_R}$ The Finite element method matrix is then given by [B]RM([B]$^{-1}$[A]), where [A] and [B] are finite element matrices corresponding to the sesqui-linear forms $(D^2\,.\,,\,.)_{L^2(0,\ 2\pi)}$ and $(.\,,\,.)_{L^2(0,\ 2\pi)}$ respectively. This matrix is used in the finite element approximation of the Helmholtz problem.

In this paper, we implement a mixed type approximation method to obtain a finite element matrix corresponding to the exact Dirichlet to Neumann boundary condition using non-uniform partition. Then apply this approximation method to the scattering problems.

II. METHODOLOGY

A. Radiation boundary condition

When solving the exterior Helmholtz problem numerically, the unbounded domain Ω is truncated into a bounded domain by introducing an artificial boundary Γ_R and an artificial boundary condition is imposed on the Γ_R (Figure 1). Then artificial boundary condition is imposed on Γ_R such that the outgoing scattering waves pass through the boundary without any reflection. This new boundary condition is referred as radiation boundary condition [9].
We consider the two dimensional polar coordinate system $(r,\ \theta)$ in the plane to represent the Helmholtz problem (1) and choose the artificial boundary as a circle of large enough radius R. Let B_R be the circular domain bounded by Γ_R which includes O in its interior. The computational domain is then $\Omega_R = B_R \backslash \bar{O}$.
On Γ_R , we impose the exact radiation boundary condition

$$\frac{\partial u}{\partial r} = -Mu, \qquad |x| = R,$$

where$M$ is the Dirichlet- to -Neumann (DtN) mapping because it maps the Dirichlet data of a solution to the exterior Helmholtz problem with respect to the exterior of Γ_R to the corresponding Neumann data. The exact solution $u(r,\ \theta)$for (1) outside the$B_R$has the following representation:

$$u(r,\theta) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\frac{H^{(1)}(kr;\ m)}{H^{(1)}(kR;m)}\int_0^{2\pi} u(R,\phi)e^{im(\theta-\phi)}\,d\phi\,,$$

where$H^{(1)}(x,m) = H_m^{(1)}(x)$ is the Hankel function of the first kind of order m. And

$$\frac{\partial u}{\partial r}\Big|_{r=R} = \frac{k}{2\pi}\sum_{m=-\infty}^{\infty}\frac{H^{(1)'}(kR;\ m)}{H^{(1)}(kR;m)}\int_0^{2\pi} u(R,\phi)e^{im(\theta-\phi)}\,d\phi\,.$$

The DtN operator M can be analytically represented as follows:

$$(M\omega)(\theta) = -\frac{k}{2\pi}\sum_{m=-\infty}^{\infty}\frac{H^{(1)'}(kR;\ m)}{H^{(1)}(kR;m)}\int_0^{2\pi} \omega(\ \phi)e^{im(\theta-\phi)}\,d\phi\,.$$

The reformulated problem in the bounded domain $\Omega_R$is then given as

$$\Delta u - k^2 u = 0 \qquad \text{in} \ \ \Omega_R (4)$$

$$\frac{\partial u}{\partial n} = -\frac{\partial u^{inc}}{\partial n} \text{on} \ \ \partial\Omega \qquad (5)$$

$$\frac{\partial u}{\partial n} = -Mu \qquad \text{on} \ \ \Gamma_R (6)$$

B. Variational formulation

Variational formulations are derived by a given finite exterior domainΩ_R. The Helmholtz equation (1) is multiplied by a test function v , where $v \in V \equiv H^1(\Omega_R)$, $H^s(\Omega_R)$is the Sobolev space of order $s \in \mathbb{R}$ in Ω_R and then integrated over Ω_R. This gives a sesqui-linear form after integration by parts:

$$a(u,v) + < \gamma u, \gamma v >_M + \left(\frac{\partial u^{inc}}{\partial n},\ v\right)_{\partial\Omega} = 0$$

where$\gamma :\ H^1(\Omega_R) \rightarrow\ H^{1/2}(\Gamma_R)$is the trace operator and the sesqui-linear forms $a(.\ ,\ .), <.\ ,\ .>_M$ and $(.\ ,\ .)_{\partial\Omega}$ are respectively.

$$a(u,v) = \int_{\Omega_R} \left(\frac{\partial u}{\partial r}\frac{\partial \bar{v}}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\frac{\partial \bar{v}}{\partial \theta} - k^2 u\bar{v}\right) r\, dr\, d\theta, \qquad u, v \in H^1(\Omega_R)$$

$$< p, q >_M = \int_0^{2\pi} (Mp)(\theta)\bar{q}(\theta)\, R\, d\theta\,, \qquad p,\ q \in H^{1/2}(\Gamma_R) \qquad (7)$$

and $(f,\ g)_{\partial\Omega} = \int_{\partial\Omega} fg\, d\sigma\,,$ $\qquad f,\ g \in L^2(\partial\Omega)$.

Now, based on the element partitioning of the computational domain, we introduce a finite dimensional subspace $V_h$of V. Then the finite element approximation is to find $u_h \in V_h$such that

$$a(u_h, v_h) + < \gamma u_h, \gamma v_h >_M + \left(\frac{\partial u^{inc}}{\partial n},\ v_h\right)_{\partial\Omega} = 0\,, \qquad \forall\, v_h \in V_h\,.$$

C. Finite Element Method

The computational domain is discretized with the first order triangular element. Triangular mesh was created using the pdetool method in Matlab (version 7.6.0.324 (R2008a)). From the mesh the triangle node numbers, Co-ordinates of the triangle node and boundary node number of the computational domain are retrieved. Global stiffness matrix and general force vector are assembled using the mesh data. Then the element stiffness matrices are assembled to the boundary of the computational domain. After that the mixed type approximation matrix is calculated using the equation $[B]RM([B]^{-1}[A])$. This matrix is also assembled with the global stiffness matrix. The incident wave is applied at the obstacle. The mixed type approximation method solution is computed.

The element stiffness matrix is evaluated for each element and is assembled over the entire region according to the global node numbering to obtain a global matrix equation. The global stiffness matrix is assembled by adding the coefficients of the element matrices with an entries corresponding to their global nodal numbers. Then the global stiffness matrix is assembled with the mixed type approximation matrix M^{Mixed} .

D. Mixed type approximation matrix

Commonly, the non-local operator in the radiation boundary condition is expressed in the form of Fourier series. Hence the operator itself, the corresponding sesqui-linear form and the finite element matrix with respect to a partition of the artificial boundary are all expressed in infinite series. In numerical computation, the infinite series is reduced to a finite sum, which introduces some additional truncation error.

We observe that the non-local operator is a pseudo differential operator expressed as a function of the Laplace-Beltrami operator $D^2 = -\frac{\partial^2}{\partial\theta^2}$on the artificial boundary and define a Finite Element approximation matrix which we call a mixed type approximation matrix corresponding to the sesqui-linear form of the operator with respect to a given partition.

Specifically, the pseudo-differential operator $M(D^2)$, as a function of a differential operator $D^2$is defined as follows:

$$M(D^2) = -k\frac{H^{(1)'}\left(kR; \sqrt{D^2}\right)}{H^{(1)}\left(kR; \sqrt{D^2}\right)}.$$

With the same partition and basis-functions, the finite element matrices corresponding to the sesqui-linear forms $(p', q')_{L^2(0,2\pi)}$ and $(p,q)_{L^2(0,2\pi)}$respectively are given by $[A]$and $[B]$, where $h_{i\,,\ i=1,\ 2,\ \cdots,\ N-1}$ are the length of the boundary element.

$$[A] = [A_{ij}] = \begin{cases} \frac{1}{h_{i-1}} + \frac{1}{h_i} & \text{if} \quad i = j \\ -\frac{1}{h_{i-1}} & \text{if} \quad i = j+1 \\ -\frac{1}{h_{i-1}} & \text{if} \quad i = j-1 \\ -\frac{1}{h_{N-1}} & \text{if} \quad i = N-1, j = 1 \text{ and } i = 1, j = N-1 \\ 0 & \text{elsewhere}, \end{cases}$$

where index 0 is regarded $N - 1 \equiv 0$.

$$[B] = [B_{ij}] = \begin{cases} \frac{2}{6}(h_{i-1} + h_i) & \text{if} \quad i = j \\ \frac{1}{6}h_{i-1} & \text{if} \quad i = j+1 \\ \frac{1}{6}h_{i-1} & \text{if} \quad i = j-1 \\ \frac{1}{6}h_{N-1} & \text{if} \quad i = N-1, j = 1 \text{ and } i = 1, j = N-1 \\ 0 & \text{elsewhere}, \end{cases}$$

where index 0 is regarded $N - 1 \equiv 0$.

The mixed type approximation matrix is defined as follow:

Definition: A mixed type approximation matrix corresponding to the sesqui-linear form $<.\,,\,.>_M$ defined in the equation (7) for the pseudo-differential operator $M(D^2)$ with respect to a partition of Γ_R is defined by

$$M^{\text{Mixed}} = [B]RM([B]^{-1}[A]),$$

where the matrices $[A]$ and $[B]$ are given above and R is the radius of the circular artificial boundary.

The matrix M^{Mixed} is a function of the matrices $[A]$ and $[B]$ and also its eigenvalues can be expressed as a function of the eigenvalues of the matrices $[A]$and $[B]$. Therefore the j^{th} eigenvalue λ_j^{Mixed} of M^{Mixed} is given by

$$\lambda_j^{\text{Mixed}} = RM(v_j^2)\lambda_j^B$$

where $v_j^2 = \frac{\lambda_j^A}{\lambda_j^B}$ is the eigenvalue of $B^{-1}A$ and $\lambda_j^A$and $\lambda_j^B$are the eigenvalues of the matrices $[A]$ and $[B]$ respectively.

The eigenvalues λ_j^{Mixed} in above of the mixed type approximation matrix are expressed in one term. The function $M(v^2)$is the logarithmic derivative of the Hankel function of the first kind of fractional order v. That is

$$M(v^2) = -k\frac{H^{(1)'}(kR;\, v)}{H^{(1)}(kR; v)}.$$

The computation of this function requires the computation of the Hankel function and its derivative.

III. Numerical Results

We present some of the results of numerical testing of our application of the mixed type method. In order to test the convergence of the computed solutions, we compared the computed solutions with the exact solution as the non-uniform mesh size decreases.

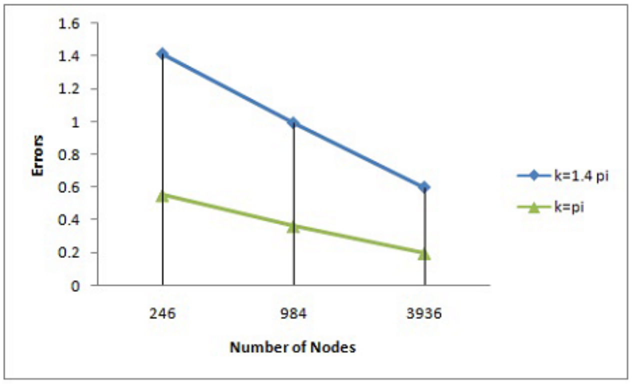


Figure 2: Convergence of mixed and exact solutions

Table 1: Maximum difference between the computed solution and the exact solution

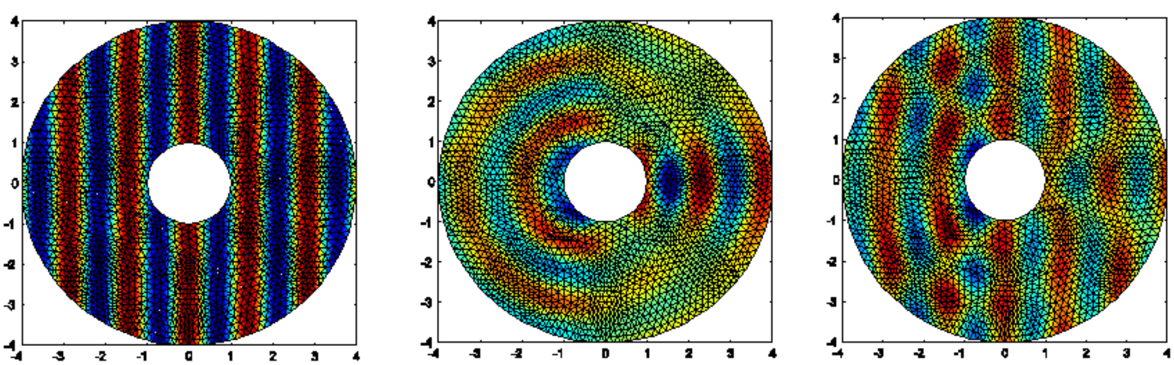
Wave Number	Number of Nodes	Errors
π	246	0.5481
	984	0.3400
	3936	0.1912
1.4π	246	1.4320
	984	0.9902
	3936	0.5750

The figure shows that the difference between the computed solutions of exact and mixed type finite element method converge.
To see the difference between the computed solutions and the exact solution, we computed the maximum difference

$$\| u^{\text{exact}} - u^{\text{Mixed}} \|_{\infty,\Omega_{R,h}} = \max_{x\in \Omega_{R,h}} \left| u^{\text{exact}}(x) - u^{\text{Mixed}}(x) \right|$$

and plotted against the non-uniform mesh size, where $\Omega_{R,h}$ is the discretized computational domain. The results are shown in Table 1. For the finite element mesh, we choose the mesh size from 246 to 3936. Figure 2 shows that the number of nodes increases, that is mesh size decreases, the error decrease. So our mixed type method is convergent.

Here we consider some examples of circular, elliptical and square obstacle with various wave numbers. For the first test, we considered an example of a circular obstacle of radius $a = 1$ with artificial boundary radius $R = 4$. The incident wave is a plane wave in the x-



axisdirection $\phi = 0$. We choose the wave number $k = \pi,\ 1.4\pi$.

Figure 3: Incident, Scattered & Total waves respectively for the circular obstacle, k=1.4π, ϕ=0

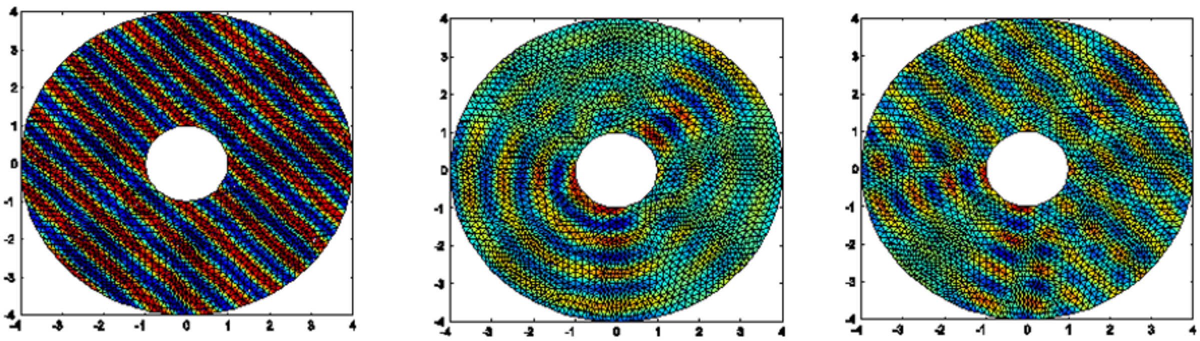


Figure 4:Incident, Scattered & Total waves respectively for the circular obstacle, k=π, φ=π/4

For the elliptical obstacle, the major axis and minor axis are 2a = 4 and 2b = 2 respectively. The artificial boundary is a circle of radius R = 3. Figure 5 shows that the incident waves inthe direction of x-axis from the right, scattering waves and total waves with the wave number $k = \frac{3\pi}{2}$.

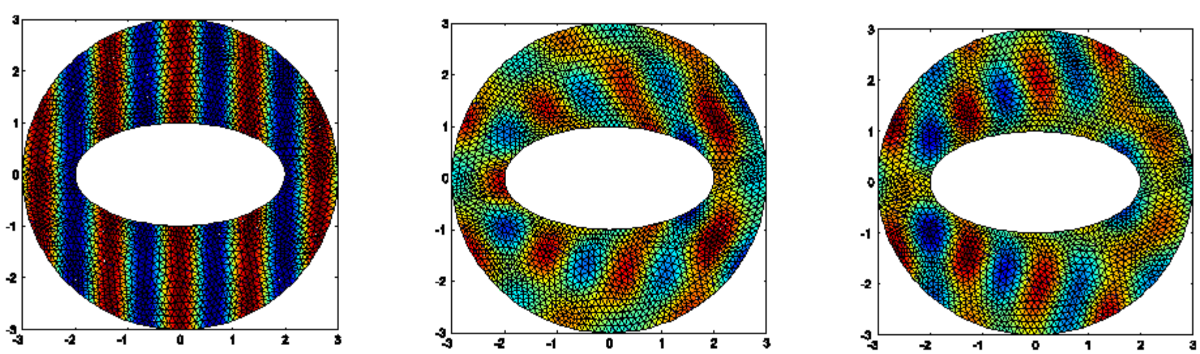


Figure 5: Incident, Scattered & Total waves respectively for the elliptical obstacle, k = 3π/2, ϕ = 0

For the square obstacle, the length of square is 2. The artificial boundary is a circle of radius R = 4. Figure 6 shows that the scattering waves corresponding to the incident wave with the wave number $k = 1.5\pi$ and wave direction $\phi = \frac{\pi}{3}$.

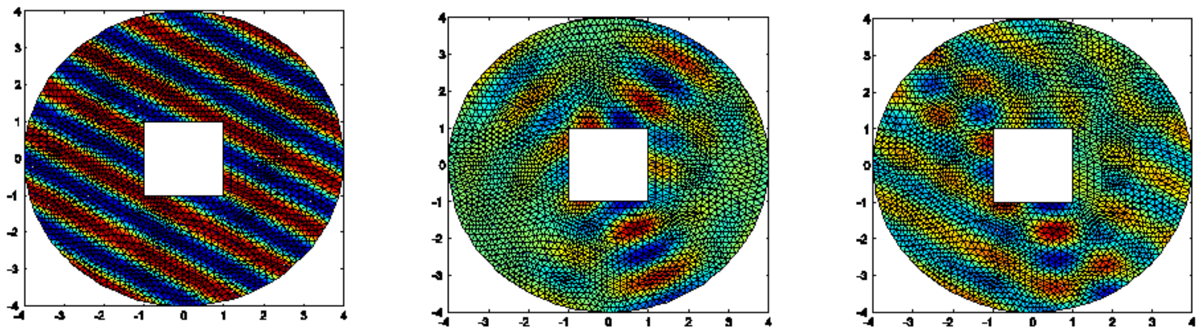


Figure 6: Incident, Scattered & Total waves respectively for the circular obstacle, $k = 1.5\pi$, $\phi = \pi/3$

IV. CONCLUSION

We propose an approximation method to implement the exact Dirichlet to Neumann boundary condition that directly gives an approximation matrix for the Helmholtz problem. We define a mixed type approximation matrix to implement the sesqui-linear forms corresponding to the DtN operator for non-uniform partitions.

Numerically, we confirm the convergence of the solution by considering an example of a circular obstacle for which analytical solution can be derived and computed. The numerical results show that our mixed type method converges to the exact solution.

REFERENCES

[1] B. Engquist, A. Madja, *Absorbing boundary conditions for the numerical simulation of waves*, Mathematics of Computation 1977, 31 139, 629-651.

[2] O.G. Ernst, *A Finite element capacitance matrix method for exterior Helmholtz problems*, NumerischeMathematik, 1996, 75, 2, 175-204.

[3] E. Heikkola, Y. A. Kusnetsov, P. Neittaanmaki, J. Toivanen, *Fictitious domain methods for the numerical methods for the numerical solution of two dimensional scattering problems*, Journal Computational Physics, 1998, 145, 89-109.

[4] T. Kako, K. Touda, *Numerical approximation of Dirichler-to-Neumann mapping and its application to voice generation problem*, Lecture Notes in Computational Science and Engineering, 2005, 40, 51-65 .

[5] O.G. Ernst, *A Finite element capacitance matrix method for exterior Helmholtz problems*, Numer.Math, 1996, 75, 2, 175-204.

J. B. Keller, D. Givoli, *Exact non –reflecting boundary conditions*, Journal of Computational Physics, 1989, 82, 172-192.

[6] M. Masmoudi, *Numerical solution for exterior problems*, NumerischeMathematik, 1987, 51, 87-101.

[7] H. M. Nasir, T. Kako, *A Numerical approximation method for a non-local operator applied to radiation problem*, Record Kokyu Rims., 2002, 1265, 173-183.

[8] H. M. Nasir, T. Kako, *Fictitious domain method for structural acoustic coupling problem in unbounded region*, Theoretical and Applied Mechanics, 2001, 50, 391-401.

[9] H. M. Nasir, *Ph.D Thesis, A mixed type finite element method for radiation and scattering problems with applications to structural acoustic coupling problem in unbounded domain*, University of Electro Communications, 2003.

[10] L. J. Segerlind, *Applied finite element analysis*, New York, John Welly and sons, 1984.

[11] C. H. Wilcox, *Scattering theory for the d'Alembert equation in exterior domains*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1975, 442, 184.

AUTHORS

First Author – Mrs.JeyanthiniSriranganesan, Senior Lecturer , Department of Mathematics, Faculty of Science, Eastern University, Sri Lanka and sriranganesanj@esn.ac.lk

Second Author – Dr.H.M.Nasir, Senior Lecturer,Department of Mathematics and Statistics, College of Science, SulatanQaboos University, P. O. Box: 36, Al-Khoud, Muscat 123, Sultanate of OMANand nasirhm11@yahoo.com

Correspondence Author – Mrs.JeyanthiniSriranganesan, Senior Lecturer , Department of Mathematics, Faculty of Science, Eastern University, Sri Lanka and sriranganesanj@esn.ac.lk