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Some Fixed Point and Common Fixed Point Theorems of Integral Expression on 2-Banach Spaces

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Abstract- In the present paper we prove some fixed point and common fixed point theorems in 2-Banach spaces for rational expression. Which generalize the well-known results.

Index Terms- Banach Space, 2-Banach Spaces, Fixed point, Common Fixed point **.**

I. Introduction

Fixed point theory plays basic role in application of various branches of mathematics from elementary calculus and linear algebra to topology and analysis.

The study of non-contraction mapping concerning the existence of fixed points draws attention of various authors in non- linear analysis. It is well know that the differential and integral equations that arise in physical problem are generally non-linear, therefore the fixed point methods especially Banach's contraction principle provides a powerful tooi for obtaining the solutions of these equations which were very difficult to solve by any other methods .Resently Verma[14] described about the application of Banach's contraction principal [4]. Gahler [8]introduced the concept of 2-Banach spaces. Recently Badshah and Gupta[5],Yadava, Rajput and Bhardwaj[15] also worked for Banach and 2-Banach spaces for non-contraction mappings. In present paper we prove some fixed point and common fixed point theorems for non-contraction mapping, in 2-Banach spaces motivated by above, before starting the main result first we write some definitions.

Definition: (1.1), 2-Banach Spaces:

In a paper Gahler[8] define a linear 2-normed space to be pair (L, ‖. , ‖) where L is a linear space and ‖. , ‖ is non-negative, real valued function defined on L such that a, b, c $\in$ L

(i) ‖a, b‖ = 0 if and only if a and b are Linearly dependent
(ii) ‖a, b‖ = ‖b, a‖
(iii) ‖a, βb‖ = |β| ‖a, b‖, β is real
(iv) ‖ a, b + c‖ ≤ ‖a, b‖ + ‖a, c‖
Hence ‖ . , . ‖ is called a 2-norm.

Definition: (1.2), A sequence $\{x_n\}$ in a linear 2-normed space L, is called a convergent sequence if there is, x $\in$ L, such that $\lim_{n\to\infty} \|x_n - x, y\| = 0$ for all y $\in$ L

Definition: (1.3), A sequence $\{x_n\}$ in a linear 2-normed space L, is called a Cauchy sequence if there exists y, z $\in$ L, such that y and z are linearly independent and
$\lim_{m,n\to\infty} \|x_m - x_n, y\| = 0$

Definition: (1.4), A linear 2-normed space in which every Cauchy sequence is convergent is called 2-Banach spaces.

Theorem (1.1) Let (X, d) be a complete metric space, c $\in$ (0, 1) and f: X → X be a mapping such that for each x, y $\in$ X, d(fx,fy) ≤ cd(x, y) Then f has a unique fixed point a $\in$ X, such that for each
x $\in$ X, $\lim_{n\to\infty} f^n(x) = a.$

After the classical result, Kannan[12] gave a subsequently new contractive mapping to prove the fixed point theorem. Since then a number of mathematicians have been worked on fixed point theory dealing with mappings satisfying various type of contractive conditions.

In 2002, A. Branciari[3] analysed the existence of fixed point for mapping f defined on a complete metric space(X, d) satisfying a general contractive condition of integral type.

Theorem(1.2): (Branciari) Let (X, d) be a complete metric space, c $\in$ (0, 1) and let f : X → X be a mapping such that for each x, y $\in$ X,

$$\int_0^{d(fx,fy)} \emptyset(t)dt \leq c\int_0^{d(x,y)} \emptyset(t)\,dt \text{ where } \emptyset:[0,+\infty)\rightarrow[0,+\infty)$$

is a Lebesgue integrable mapping which is summmable on each compact subset of $[0,+\infty)$, non-negative, and such that for each $\varepsilon > 0, \int_0^{\varepsilon} \emptyset(t)dt$,then f has a unique fixed point a $\in$ X, such that for each x $\in$ X

$\lim_{n\to\infty} f^n(x) = a$.

After the paper of Branciari, a lot of research works have been carried out on generalizing contractive condition of integral type for different contractive mappings satisfying various

Known properties. A fine work has been done by Rhoades [5] extending the result of Branciari by replacing the condition [1.2] by the following

$$\int_0^{d(fx,fy)} \emptyset(t)dt \leq c\int_0^{\max\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(y,fx)}{2}\}} \emptyset(t)\,dt .$$

Theorem(1.3): Let T be a mapping of a 2-Banach spaces into itself. If T satisfies the following conditions:

(1) $T^2 = I$, where I is identity mapping

(2) $\|Tx - Ty, a\|$

$$\geq \alpha\,\frac{\|x-Tx,a\| + \|y-Ty,a\|}{\|x-y,a\|} + \beta\,\frac{\|y-Ty,a\|\,\|y-Tx,a\|\,\|x-Ty,a\| + \|x-y,a\|^3}{\|x-y,a\|^2} + \gamma\left[\frac{\|x-Tx,a\| + \|y-Ty,a\|}{2}\right] + \delta\left[\frac{\|x-Ty,a\| + \|y-Tx,a\|}{2}\right] + \eta\,\|x-y,a\|$$

Where $x \neq y$, $a > 0$ is real with $8\alpha + 10\beta + 4\gamma + 2\delta + 3\eta > 4$. Then T has unique fixed point.

Our main result is modified the above result in integral type mapping. A. S. Saluja[1].

II. Main Result

Theorem (2 . 1)

Let T be a mappings of a 2-Banach space X into itself. T satisfy the following conditions:

(1) $T^2 = I$, where I is identity mapping,

(2) $$\int_0^{\|Tx-Ty,a\|} \emptyset(t)dt \geq \alpha\int_0^{\frac{\|x-Tx,a\|\|y-Ty,a\|}{\|x-y,a\|}} \emptyset(t)dt + \beta\int_0^{\frac{\|y-Ty,a\|\,\|y-Tx,a\|\|x-Ty,a\|+\|x-y,a\|^3}{\|x-y,a\|^2}} \emptyset(t)dt + \gamma\int_0^{\frac{\|x-Tx,a\|+\|y-Ty,a\|}{2}} \emptyset(t)dt + \delta\int_0^{\frac{\|x-Ty,a\|+\|y-Tx,,a\|}{2}} \emptyset(t)dt + \eta\int_0^{\|x-y,a\|} \emptyset(t)dt$$

For every x, y $\in X, \alpha, \beta, \gamma, \delta, \eta \in [0,1]$ with $x \neq y$ and

$8\alpha + 10\beta + 4\gamma + 3\delta + 2\eta > 4$. Also $\emptyset:[0,+\infty)\rightarrow[0,+\infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0,+\infty)$, non- negative and such that for each $\varepsilon > 0, \int_0^{\varepsilon} \emptyset(t)dt,$ Then T has unique fixed point.

Proof : Suppose x is any point in 2-Banach space X.

Taking y = $\frac{1}{2}(T+I)$ x, z = T(y)

$$\int_0^{\|z-x,a\|} \emptyset(t)dt = \int_0^{\|Ty-T^2x,a\|} \emptyset(t)dt = \int_0^{\|Ty-T(Tx),a\|} \emptyset(t)dt$$

$$\geq \alpha \int_0^{\frac{\|y-Ty,a\|\|Tx-T(Tx),a\|}{\|y-Tx,a\|}} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{\|Tx-T(Tx),a\|\|Tx-Ty,a\|\|y-T(Tx),a\|+\|y-Tx,a\|^3}{\|y-Tx,a\|^2}} \emptyset(t)dt$$

$$+ \gamma \int_0^{\frac{\|y-Ty,a\|+\|Tx-T(Tx),a\|}{2}} \emptyset(t)dt + \delta \int_0^{\frac{\|y-T(Tx),a\|+\|Tx-Ty,a\|}{2}} \emptyset(t)dt$$

$$+\eta \int_0^{\|y-Tx,a\|} \emptyset(t)dt$$

$$\geq \alpha \int_0^{\frac{\|y-Ty,a\|\|Tx-x,a\|}{\frac{1}{2}\|x-Tx,a\|}} \emptyset(t)dt$$

$$+ \beta \int_0^{\frac{\|Tx-x,a\|[\|Tx-y,a\|+\|y-Ty,a\|]\|y-x,a\|+\|y-Tx,a\|^3}{\frac{1}{4}\|x-Tx,a\|^2}} \emptyset(t)dt$$

$$+ \gamma \int_0^{\frac{\|y-Ty,a\|+\|Tx-x,a\|}{2}} \emptyset(t)dt + \delta \int_0^{\frac{\|y-x,a\|+\|Tx-y,a\|+\|y-Ty,a\|}{2}} \emptyset(t)dt$$

$$+\eta \int_0^{\|y-Tx,a\|} \emptyset(t)dt$$

$$\geq 2\alpha \int_0^{\|y-Ty,a\|} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{\|Tx-x,a\|\left[\frac{1}{2}\|x-Tx,a\|+\|y-Ty,a\|\right]\frac{1}{2}\|x-Tx,a\|+\frac{1}{8}\|x-Tx,a\|^3}{\frac{1}{4}\|x-Tx,a\|^2}} \emptyset(t)dt$$

$$+ \gamma \int_0^{\frac{\|y-Ty,a\|+\|Tx-x,a\|}{2}} \emptyset(t)dt + \delta \int_0^{\frac{\frac{1}{2}\|x-Tx,a\|+\frac{1}{2}\|x-Tx,a\|+\|y-Ty,a\|}{2}} \emptyset(t)dt$$

$$+\eta \int_0^{\frac{1}{2}\|x-Tx,a\|} \emptyset(t)dt$$

$\geq$
$2\alpha \int_0^{\|y-Ty,a\|} \emptyset(t)dt +$
$\frac{\beta}{2} \int_0^{4\left[\frac{1}{2}\|x-Tx,a\|+\|y-Ty,a\|\right]+\frac{\|x-Tx,a\|^3}{\|x-Tx,a\|^2}} \emptyset(t)dt + \gamma$

$$\int_0^{\frac{\|y-Ty,a\|+\|Tx-x,a\|}{2}} \emptyset(t)dt + \delta \int_0^{\frac{\|x-Tx,a\|+\|y-Ty,a\|}{2}} \emptyset(t)dt + \frac{n}{2} \int_0^{\|x-Tx,a\|} \emptyset(t)dt$$

$\geq$
$2\alpha \int_0^{\|y-Ty,a\|} \emptyset(t)dt +$
$\frac{\beta}{2} \int_0^{[3\|x-Tx,a\|+4\|y-Ty,a\|]} \emptyset(t)dt + \gamma$

$$\int_0^{\frac{\|y-Ty,a\|+\|Tx-x,a\|}{2}} \emptyset(t)dt + \delta \int_0^{\frac{\|x-Tx,a\|+\|y-Ty,a\|}{2}} \emptyset(t)dt + \frac{n}{2} \int_0^{\|x-Tx,a\|} \emptyset(t)dt$$

$\geq (\frac{3\beta}{2} + \frac{\gamma}{2}$

$+ \frac{\delta}{2} + \frac{\eta}{2}) \int_0^{\|x-Tx,a\|} \emptyset(t)dt + (2\alpha + 2\beta + \frac{\gamma}{2} +$
$\frac{\delta}{2}) \int_0^{\|y-Ty,a\|} \emptyset(t)dt$

$$\geq \frac{1}{2}(3\beta + \gamma + \delta + \eta) \int_0^{\|x-Tx,a\|} \emptyset(t)dt + \frac{1}{2}(4\alpha + 4\beta + \gamma + \delta) \int_0^{\|y-Ty,a\|} \emptyset(t)dt \quad \ldots(2.1.1)$$

Now for

$$\int_0^{\|u-x,a\|} \emptyset(t)dt = \int_0^{\|2y-z-x,a\|} \emptyset(t)dt = \int_0^{\|Ty-Ty,a\|} \emptyset(t)dt$$

$$\geq \alpha \int_0^{\frac{\|x-Tx,a\|\|y-Ty,a\|}{\|x-y,a\|}} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{\|y-Ty,a\|\|y-Tx,a\|\|x-Ty,a\|+\|x-y,a\|^3}{\|x-y,a\|^2}} \emptyset(t)dt$$
$$+\gamma \int_0^{\frac{\|x-Tx,a\|+\|y-Ty,a\|}{2}} \emptyset(t)dt + \delta \int_0^{\frac{\|x-Ty,a\|+\|y-Tx,a\|}{2}} \emptyset(t)dt$$
$$+\eta \int_0^{\|x-y,a\|} \emptyset(t)dt$$

$$\geq \alpha \int_0^{\frac{\|x-Tx,a\|\|y-Ty,a\|}{\frac{1}{2}\|x-Tx,a\|}} \emptyset(t)dt$$
$$+\beta \int_0^{\frac{\|y-Ty,a\|\frac{1}{2}\|y-Tx,a\|\frac{1}{2}\|x-Ty,a\|+\frac{1}{8}\|x-Tx,a\|^3}{\frac{1}{4}\|x-Tx,a\|^2}} \emptyset(t)dt$$

$+$
$\gamma \int_0^{\frac{\|x-Tx,a\|+\|y-Ty,a\|}{2}} \emptyset(t)dt +$
$\delta \int_0^{\frac{\frac{1}{2}\|x-Tx,a\|+\frac{1}{2}\|x-Tx,a\|}{2}} \emptyset(t)dt$

$+\eta \int_0^{\frac{1}{2}\|x-Tx,a\|} \emptyset(t)dt$

$\geq$
$2\alpha \int_0^{\|y-Ty,a\|} \emptyset(t)dt +$
$\beta \int_0^{[\|y-Ty,a\|+\frac{1}{2}\|x-Tx,a\|]} \emptyset(t)dt + \gamma$

$$\int_0^{\frac{\|x-Tx,a\|+\|y-Ty,a\|}{2}} \emptyset(t)dt + \delta \int_0^{\frac{\|x-Tx,a\|}{2}} \emptyset(t)dt + \eta \int_0^{\frac{1}{2}\|x-Tx,a\|} \emptyset(t)dt$$

$\geq (\frac{\beta}{2} + \frac{\gamma}{2}$
$+ \frac{\delta}{2} + \frac{\eta}{2}) \int_0^{\|x-Tx,a\|} \emptyset(t)dt + (2\alpha + \beta +$
$\frac{\gamma}{2}) \int_0^{\|y-Ty,a\|} \emptyset(t)dt$

$$\geq \frac{1}{2}(\beta + \gamma + \delta + \eta) \int_0^{\|x-Tx,a\|} \emptyset(t)dt + \frac{1}{2}(4\alpha + 2\beta + \gamma) \int_0^{\|y-Ty,a\|} \emptyset(t)dt \quad \ldots(2.1.2)$$

Now
$$\int_0^{\|z-u,a\|} \emptyset(t)dt = \int_0^{\|z-x,a\|} \emptyset(t)dt + \int_0^{\|x-u,a\|} \emptyset(t)dt$$

$\geq \frac{1}{2}(3\beta + \gamma + \delta + \eta) \int_0^{\|x-Tx,a\|} \emptyset(t)dt + \frac{1}{2}(4\alpha + 4\beta + \gamma + \eta + \delta)$

$\int_0^{\|y-Ty,a\|} \emptyset(t)dt \geq$
$\frac{1}{2}(\beta + \gamma + \delta + \eta) \int_0^{\|x-Tx,a\|} \emptyset(t)dt$

$+\frac{1}{2}(4\alpha + 2\beta + \gamma) \int_0^{\|y-Ty,a\|} \emptyset(t)dt$

$$\geq \frac{1}{2}(3\beta + \gamma + \delta + \eta + \beta + \gamma + \delta + \eta) \int_0^{\|x-Tx,a\|} \emptyset(t)dt$$

$$+\frac{1}{2}(4\alpha + 4\beta + \gamma + \delta + 4\alpha + 2\beta + \gamma) \int_0^{\|y-Ty,a\|} \emptyset(t)dt$$

$$\geq \frac{1}{2}(4\beta + 2\gamma + 2\delta + 2\eta) \int_0^{\|x-Tx,a\|} \emptyset(t)dt$$

$$+\frac{1}{2}(8\alpha + 6\beta + 2\gamma + \delta) \int_0^{\|y-Ty,a\|} \emptyset(t)dt$$

On the other hand

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$$\int_0^{\|z-u,a\|} \emptyset(t)dt = \int_0^{\|T(y)-(2y-z),a\|} \emptyset(t)dt$$

$$= \int_0^{\|T(y)-2y+T(y),a\|} \emptyset(t)dt$$

$$= 2\int_0^{\|Ty-y,a\|} \emptyset(t)dt$$

So

$$2\int_0^{\|Ty-y,a\|} \emptyset(t)dt \geq \frac{1}{2}(4\beta+2\gamma+2\delta+2\eta)\int_0^{\|x-Tx,a\|} \emptyset(t)dt$$

$$+\frac{1}{2}(8\alpha+6\beta+2\gamma+\delta)\int_0^{\|y-Ty,a\|} \emptyset(t)dt$$

$$[4-(8\alpha+6\beta+2\gamma+\delta)\int_0^{\|Ty-y,a\|} \emptyset(t)dt \geq$$

$$(4\beta+2\gamma+2\delta+2\eta)\int_0^{\|x-Tx,a\|} \emptyset(t)dt$$

$$\int_0^{\|x-Tx,a\|} \emptyset(t)dt \leq \frac{4-(8\alpha+6\beta+2\gamma+\delta)}{(4\beta+2\gamma+2\delta+2\eta)}\int_0^{\|Ty-y,a\|} \emptyset(t)dt$$

$$\int_0^{\|x-Tx,a\|} \emptyset(t)dt \leq k\int_0^{\|Ty-y,a\|} \emptyset(t)dt$$

as $(8\alpha+10\beta+4\gamma+3\delta+2\eta>4)$

Where k = $\frac{4-(8\alpha+6\beta+2\gamma+\delta)}{(4\beta+2\gamma+2\delta+2\eta)} < 1$

Let R = $\frac{1}{2}(T+1)$, then

$$\int_0^{\|R^2(x)-R(x),a\|} \emptyset(t)dt = \int_0^{\|R(R(x))-R(x),a\|} \emptyset(t)dt$$

$$\int_0^{\|R(y)-y,a\|} \emptyset(t)dt = \frac{1}{2}\int_0^{\|y-Ty,a\|} \emptyset(t)dt$$

$$< \frac{k}{2}\int_0^{\|x-Tx,a\|} \emptyset(t)dt$$

By the definition of R we claim that $\{R^n(x)\}$ is a Cauchy sequence in X, $\{R^n(x)\}$ is converges to so element x_0 in X .So $\lim_{n\to\infty}\{R^n(x)\} = x_0$. Hence $T(x_0) = x_0$
S0 x_0 is a fixed point of T.

Uniqueness:

If possible $y_0 \neq x_0$ is another fixed point of T. Then

$$\int_0^{\|x_0-y_0,a\|} \emptyset(t)dt = \int_0^{\|Tx_0-Ty_0,a\|} \emptyset(t)dt$$

$$\geq \alpha \int_0^{\frac{\|x_0-Tx_0,a\|\|y_0-Ty_0,a\|}{\|x_0-y_0,a\|}} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{\|y_0-Ty_0,a\|\|y_0-Tx_0,a\|\|x_0-Ty_0,a\|+\|x_0-y_0,a\|^3}{\|x_0-y_0,a\|^2}} \emptyset(t)dt$$

$$+ \gamma \int_0^{\frac{\|x_0-Tx_0,a\|+\|y_0-Ty_0,a\|}{2}} \emptyset(t)dt + \delta \int_0^{\frac{\|x_0-Ty_0,a\|+\|y_0-Tx_0,a\|}{2}} \emptyset(t)dt$$

$$+\eta \int_0^{\|x_0-y_0,a\|} \emptyset(t)dt$$

$$\geq \beta \int_0^{\|x_0-y_0,a\|} \emptyset(t)dt + \delta \int_0^{\|x_0-y_0,a\|} \emptyset(t)dt + \eta \int_0^{\|x_0-y_0,a\|} \emptyset(t)dt$$

$$\geq (\beta+\delta+\eta) \int_0^{\|x_0-y_0,a\|} \emptyset(t)dt$$

Which is contradiction as $(8\alpha + 10\beta + 4\gamma + 3\delta + 2\eta > 4)$

So $x_0 = y_0$. Hence fixed point in unique.

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