Optimal Strategy Analysis of an N-policy M/E_k/1 Queueing System with Server Breakdowns and Multiple Vacations

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Abstract- This paper studies the optimal control of an N-policy M/E_k/1 queueing system with server breakdowns and multiple vacations. The server is turned on when N units are accumulated in the system. The server is turned off and takes a vacation with exponential random length whenever the system is empty. If the number of units waiting in the system at any vacation completion is less than N, the server will take another vacation. If the server returns from a vacation and finds at least N units in the system, it immediately starts to serve the waiting units. It is assumed that the server breakdown is according to Poisson process and the repair time has an exponential distribution. We derive the distribution of the system size and employ the probability generating function to obtain the mean queue length. It is proved that the service station is busy in the steady state is equal to the traffic intensity. The total expected cost function per unit time is developed to determine the optimal operating policy at minimum cost. This paper provides the minimum expected cost and the optimal operating policy based on numerical values of the system parameters. Sensitivity analysis is also provided.

***Index Terms*- :** M/E_k/1 queueing system, multiple vacations, N-policy, probability generating function, Server breakdowns.

I. INTRODUCTION

This paper considers the modeling of a production system at which arrivals of production order follows a Poisson process at a rate λ. The production times of the orders are made up of k independent and identically distributed exponential random variables with mean $1/k\mu$ which yields an Erlang type k distribution. The system operation starts (turned on) only when N orders have accumulated and is shut down (turned off) when no orders are present. When the server is working he may meet unpredictable breakdown but it is immediately repaired. When the system is turned off, the server leaves the system for a random period of time called vacation. On returning to the system if the server finds less than N units in the system immediately he takes another vacation. This production system can be modeled by an M/E_k/1 queueing system with server breakdowns and multiple vacations under N-policy. The concept of *N* policy was first introduced by Yadin and Naor [6]. Past work regarding queueing systems under the *N* policy may be divided into two categories: (i) cases with server's vacations and (ii) cases with server's breakdowns. For cases with server breakdowns, Wang [2] first proposed a management policy for Markovian queueing systems under the *N* policy with server breakdowns. Wang [4] and Wang *et al.* [5] extended the model proposed by Wang [2] to M/Ek/1 and M/H2/1 queueing systems respectively. Vasantha Kumar and Chandan [7] presented the optimal strategy analysis of two phase M/Ek/1 queueing system with server breakdowns and gating. Also they obtained the total expected cost function for the system and determine the optimal value of the control parameter N.

Existing research works, including those mentioned above, have never covered cases involving both server breakdowns and vacations. Queueing models with server breakdowns and vacations accommodate the real-world situations more closely. The purpose of this paper is threefold.

1. The steady state equations are established to get the steady state probability distribution and to show that it generalizes the previous results.
2. We formulate the system total expected cost in order to determine the optimal operating N-policy numerically at the minimum cost for various values of system parameters while maintaining the minimal service quantity.
3. We perform a sensitivity analysis

II. MODEL DESCRIPTION

For the purpose of analytical investigation, we consider the model with the following assumptions:

1. The arrival is poisson process with parameter λ and with service times according to an Erlang distribution with mean $1/\mu$ and stage parameter k. The Erlang type k distribution is made up of k independent and identical exponential stages, each with mean $1/k\mu$. A customer goes into the first stage of the service (say stage k) then progresses through the remaining stages and must

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complete the last stage (say stage 1) before the next customer enters the last stage. We assume that customers arriving at the service station form a single waiting line and are served in the order of their arrivals.

2. When the system is turned off, the server leaves the system for a random period of time called vacation which is exponentially distributed with parameter θ.
3. When the server is working, the server may breakdown at any time with a Poisson breakdown rate α.
4. When the server fails, it is immediately repaired at a repair rate β, where the repair times are exponentially distributed.

III. STEADY STATE RESULTS

In steady state the following notations are used.

$P_{0,0,0}$ = Probability that there are no customers in the system when the server is on vacation

$P_{n,i,0}$ = Probability that there are n customers in the system and the customers in service is in stage i while the server is on vacation.

$P_{n,i,1}$ = Probability that there are n customers in the system and the customers is in the stage i while the server is busy.

$P_{n,i,2}$ = Probability that there are n customers in the system and the customer in service is in stage i when the server is in operation but found to be broken down.

The steady state equations are given as follows

$$\lambda P_{1,k,0} = \lambda P_{0,0,0} \quad (1)$$

$$\lambda P_{n,k,0} = \lambda P_{n-1,k,0} \qquad (2 \le n \le N-1) \quad (2)$$

$$\lambda P_{0,0,0} = k\mu P_{1,1,1} \quad (3)$$

$$(\lambda+\theta)P_{n,k,o} = \lambda P_{n-1,k,0} \qquad (n \ge N) \quad (4)$$

$$(\lambda+k\mu+\alpha)P_{1,i,1} = k\mu P_{1,i+1,1} + \beta P_{1,i,2} \qquad (1 \le i \le k-1) \quad (5)$$

$$(\lambda+k\mu+\alpha)P_{1,k,1} = k\mu P_{2,1,1} + \beta P_{1,k,2} \quad (6)$$

$$(\lambda+k\mu+\alpha)P_{n,i,1} = \lambda P_{n-1,i,1} + k\mu P_{n,i+1,1} + \beta P_{n,i,2} \qquad (2 \le n \le N-1 \;\; 1 \le i \le k-1) \quad (7)$$

$$(\lambda+k\mu+\alpha)P_{n,k,1} = \lambda P_{n-1,k,1} + k\mu P_{n+1,1,1} + \beta P_{n,k,2} \qquad (2 \le n \le N-1) \quad (8)$$

$$(\lambda+k\mu+\alpha)P_{n,k,1} = \lambda P_{n-1,k,1} + k\mu P_{n+1,1,1} + \theta P_{n,k,0} + \beta P_{n,k,2} \qquad (n \ge N) \quad (9)$$

$$(\lambda+k\mu+\alpha)P_{n,i,1} = \lambda P_{n-1,i,1} + k\mu P_{n,i+1,1} + \beta P_{n,i,2} \; (n \ge N, 1 \le i \le k-1) \quad (10)$$

$$(\lambda+\beta)P_{1,k,2} = \alpha P_{1,k,1} \quad (11)$$

$$(\lambda+\beta)P_{n,i,2} = \alpha P_{n,i,1} + \lambda P_{n-1,i,2} \qquad (1 \le i \le k-1) \qquad (n \ge 2) \quad (12)$$

$$(\lambda+\beta)P_{1,i,2} = \alpha P_{1,i,1} \; (1 \le i \le k-1) \quad (13)$$

$$(\lambda+\beta)P_{n,k,2} = \lambda P_{n-1,k,2} + \alpha P_{n,k,1} \qquad (n \ge 2) \quad (14)$$

Solving equations (1),(2) and (4) recursively, we finally get

$$P_{n,k,0} = \begin{cases} P_{0,0,0}, & 1 \le n \le N-1 \\ R^{n-(N-1)} P_{0,0,0}, & n \ge N \end{cases} \quad (15)$$

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where $R = \dfrac{\lambda}{\lambda+\theta}$

IV. PROBABILITY GENERATING FUNCTION

The technique of using the probability generating function may be applied in a recursive manner from equations (1) to (14) to obtain the analytical solution of $P_{0,0,0}$ in a neat closed form of expression. Define the probability generating function of $G_0(z)$, $G_1(z)$ and $G_2(z)$respectively as follows:

$$G_0(z) = P_{0,0,0} + \sum_{n=1}^{\infty} z^n P_{n,k,0} \tag{16}$$

$$H_i(z) = \sum_{n=1}^{\infty} z^n P_{n,i,1} \quad 1 \le i \le k-1 \tag{17}$$

$$G_1(z) = \sum_{i=1}^{K} H_i(z) \tag{18}$$

$$G_i(z) = \sum_{n=1}^{\infty} z^n P_{n,i,2} \qquad 1 \le i \le k-1 \tag{19}$$

$$G_2(z) = \sum_{i=1}^{K} G_i(z) \tag{20}$$

where $|z| \le 1$

Applying algebraic manipulation technique to equations (2) and (3), we get the following

$$G_0(z) = \left[\frac{1-z^N}{1-z} + \frac{Rz^N}{1-Rz}\right] P_{0,0,0} \tag{21}$$

Multiplying equation (5) by z and (7)and (10) by z^n and summary over n we get

$$H_{i+1}(z) = \left(r+1-rz+s\right) H_i(z) - tG_i(z) \tag{22}$$

Where $r = \dfrac{\lambda}{k\mu}$ $s = \dfrac{\alpha}{k\mu}$ $t = \dfrac{\beta}{k\mu}$

Multiplying equation (6) by z (8) and (9) by z^n and summing over n we get

$$\left(r+1+s\right) H_K(z) = rzH_K(z) + \frac{1}{z} H_1(z) + \lambda \left[\frac{\theta R z^N}{1-Rz} - 1\right] P_{0,0,0} + tG_K(z) \tag{23}$$

Again multiplying (11) and (13),(12) and (14) respectively by appropriate powers of z and summary over n we find

$$G_i(z) = \frac{\alpha}{\lambda+\beta-\lambda z} H_i(z) \tag{24}$$

$$G_K(z) = \frac{\alpha}{\lambda+\beta-\lambda z} H_K(z) \tag{25}$$

Substituting $G_i(z)$ in equation (23) we get

$$H_{i+1}(z)=\left[\left(r+1-rz+s\right)-t\frac{\alpha}{\lambda+\beta-\lambda z}\right]H_i(z)$$

Therefore $H_i(z)=\left[\left(r+1-rz+s\right)-t\left(\frac{\alpha}{\lambda+\beta-\lambda z}\right)\right]^{i-1}H_1(z)$ (26)

Solving (21) and (22) in equation (19) we obtain

$$H_1(z)=\frac{rz\left[1-\frac{\theta Rz^N}{\lambda\left(1-Rz\right)}\right]}{1-z\left[\left(r+1-rz+s\right)-t\left(\frac{\alpha}{\lambda+\beta-\lambda z}\right)\right]^K}P_{0,0,0} \tag{27}$$

We solve equations (26) and (27) for $G_1(z)$ and $G_2(z)$ to obtain the following

$$G_1(z)=\frac{z\left[\frac{\theta Rz^N}{\lambda\left(1-Rz\right)}-1\right]}{1-z+s-t\left(\frac{\alpha}{\lambda+\beta-\lambda z}\right)}\qquad\frac{1-\left[\left(r+1-rz+s\right)-t\left(\frac{\alpha}{\lambda+\beta-\lambda z}\right)\right]^k}{1-z\left[\left(r+1-rz+s\right)-t\left(\frac{\alpha}{\lambda+\beta-\lambda z}\right)\right]^K}P_{0,0,0} \tag{28}$$

$$G_2(z)=\frac{\alpha}{\lambda+\beta-\lambda z}G_1(z) \tag{29}$$

We evaluate the probability $P_{0,0,0}$ using normalizing condition. For this purpose we evaluate $G_0(1),G_1(1)$ and $G_2(1)$ from equations (21),(28) and (29) respectively as

$$G_0(1)=\left(N+\frac{\lambda}{\theta}\right)P_{0,0,0} \tag{30}$$

$$G_1(1)=\left(N+\frac{\lambda}{\theta}\right)\left[\frac{\rho}{1-\rho\left[1+\frac{\alpha}{\beta}\right]}\right]P_{0,0,0} \tag{31}$$

$$G_2(1)=\frac{\alpha}{\beta}G_1(1) \tag{32}$$

Now using the normalizing condition given by

$G_1(1)=G_0(1)+G_1(1)+G_2(1)=1$, we obtain the value of probability that the system empty is

$$P_{0,0,0}=\frac{1-\rho\left(1+\frac{\alpha}{\beta}\right)}{N+\frac{\lambda}{\theta}} \tag{33}$$

V. SOME OF THE PERFORMANCE MEASURES

Denote the long run fraction time for which the server is on vacation, busy and broken down by P_0, P_1 and P_2 respectively. Thus

$$P_0 = G_0(1) = 1-\rho\left(1+\frac{\alpha}{\beta}\right) \tag{34}$$

$$P_1 = G_1(1) = \rho \tag{35}$$

$$P_2 = G_2(1) = \frac{\alpha}{\beta}\rho \tag{36}$$

We define the expected number of customers in the system as follows.

L_v = the expected number of units in the system when the server is on vacation.

L_b = the expected number of units in the system when the server is working.

L_d = the expected number of units in the system when the server is broken down.

L_s = Expected number of customers in the system.

To find L_v, we compute $G_0^{'}(1)$ in equation (21) by applying L'Hospital rule twice to obtain.

$$L_v = \left\{\frac{N(N-1)}{2\left[N+\frac{\lambda}{\theta}\right]}+\frac{\lambda}{\theta}\right\}\left[1-\rho\left(1+\frac{\alpha}{\beta}\right)\right] \tag{37}$$

Similarly, we compute $G_1^{'}(1)$ and $G_2^{'}(1)$ in equations (28) and (29) respectively, by applying L'hospital rule twice to obtain

$$L_b = G_1^{'}(1)$$

$$L_b = -\frac{1}{2\left(1+\frac{\alpha}{\beta}\right)}\left\{\left[\frac{2sk\frac{\lambda^2}{\beta^2}+\left(1+\frac{\alpha}{\beta}\right)^2\left(\rho^2+\rho r\right)}{\left[\rho\left(1+\frac{\alpha}{\beta}\right)-1\right]}\right]+\rho\left\{\left[\frac{N(N-1)}{\left(N+\frac{\lambda}{\theta}\right)}+2\left(\frac{\lambda}{\theta}+1\right)\right]\left[-\left(1+\frac{\alpha}{\beta}\right)\right]+2\alpha\frac{\lambda}{\beta^2}\right\}\right\} \tag{38}$$

$$L_d = G_2^{'}(1)$$

$$L_d = \frac{\alpha}{\beta}G_1^{'}(1)+G_1(1)\left[-\frac{\alpha\lambda}{\beta^2}\right] \tag{39}$$

$$L_s = G_0^{'}(1)+G_1^{'}(1)+G_2^{'}(1)$$

$$\mathrm{L}_s = \left[\frac{N(N-1)}{2\left(N+\frac{\lambda}{\theta}\right)}+\frac{\lambda}{\theta}\right]\left[1-\rho\left(1+\frac{\alpha}{\beta}\right)\right]-\frac{1}{2}\left\{\frac{2sk\frac{\lambda^2}{\beta^2}+\left(1+\frac{\alpha}{\beta}\right)^2\left(\rho^2+\rho r\right)}{\rho\left(1+\frac{\alpha}{\beta}\right)-1}+\rho\left[\frac{N(N-1)}{N+\frac{\lambda}{\theta}}+2\left(\frac{\lambda}{\theta}+1\right)\right]\left[-\left(1+\frac{\alpha}{\beta}\right)\right]\right\}-2\rho\frac{\alpha\lambda}{\beta^2} \tag{40}$$

VI. SPECIAL CASES

In this section we present some existing results in the literature which are special cases of our model.

Case (i):

If Θ=∞,α=0 and β=∞, expressions (37),(38) and (40) reduces to a special cases of L_{off}, L_{on} and L_N respectively of Wang and Huang[3](p.1019)

Case (ii):

If N=1, Θ=∞,α=0 and β=∞, the expression (33) reduces to a special case of expression (2.8) of Gross and Harris[1]

VII. OTHER SYSTEM CHARACTERISTICS

A grand vacation and the grand vacation process are defined to investigate the operating characteristics of our model. The first grand vacation (G_1) starts from the point the system becomes empty and the server leaves for the first vacation (V_1) and lasts until the server finds one (or) more customers after returning from a vacation. At the end of the first grand vacation, if the number of customers in the queue is less than N, the server leaves for another vacation and it becomes the new starting pint of the second grand vacation (G_2). This second grand vacation continues until a different system state is observed after a vacation. Grand vacations (G_1,G_2….) continue in this manner until the number of units observed after a grand vacation is found to be greater than or equal to N.

The grand vacation process is a process imbedded in the idle period in which the imbedded states are the number of units in the queue just after the server leaving for grand vacations. One cycle begins when the system is empty and the server takes a vacation. The server remains on vacation until there are at least N units in the system when it returns from a vacation. We call this, the idle period. The busy period is initiated when the server starts serving the waiting units and terminates when there are no units in the system. While providing service the server may breakdown and sent for repair immediately. This is called breakdown period.

The idle period I is developed by means of the grand vacation process. It is defined that

ϕ_k = the probability that the grand vacation process passes through state k.

π_k = the probability that k customers arrive during a vacations

Then using the concept of grand vacation process, it is determined

$$\phi_n = \sum_{k=1}^{n} \frac{\pi_k}{1-\pi_0}\phi_{n-k} \qquad k = 1,2\ldots\ldots\ldots..$$

$$\phi_0 = 1$$

Calculating ϕ_n recursively it is determined

$$\phi_n = \frac{\theta}{\lambda+\theta}$$

Since the expected length of a grand vacation is $\frac{E(V)}{1-\pi_0} = \frac{1}{\theta(1-\pi_0)}$ then $\frac{\phi_k}{\theta(1-\pi_0)}$ is the expected length of the grand vacation which starts with k customers. Hence we have the expected length of the idle period is given by

$$E(I) = \frac{1}{\theta(1-\pi_0)}\sum_{k=0}^{N-1}\phi_k \tag{41}$$

Substituting for π_0 and ϕ_k we have

$$E(I) = \frac{N}{\lambda} \tag{42}$$

If E(B),E(D) and E(C) denote the expected busy period, broken down period and busy cycle respectively, we have

E(C) = E(I) + E(B)+E(D)

From equations (34) to (36) we obtain the long run fraction of time for the server is idle, busy and broken down respectively:

$$\frac{E(I)}{E(C)} = P_0 = 1-\rho\left(1+\frac{\alpha}{\beta}\right) \tag{43}$$

$$\frac{E(B)}{E(C)} = P_1 = \rho \tag{44}$$

$$\frac{E(D)}{E(C)} = P_2 = \frac{\alpha}{\beta}\rho \tag{45}$$

Thus we have the number of cycles per unit time

$$E(C) = \frac{N}{\lambda\left[1-\rho\left(1+\frac{\alpha}{\beta}\right)\right]} \tag{46}$$

VIII. OPTIMAL N-POLICY

We develop a steady state total expected cost function per unit time for the N-policy $M/E_K/1$ Queueing system with server vacations and breakdowns, in which N is a decision variable. Following cost structure is constructed, our objective is to determine the optimal operating N policy so as to minimize this cost function. Let

C_h = holding cost per unit time for each customer present in the system.

C_o = Cost per unit time for the operating service station.

C_s =Set up cost per cycle.

C_d =breakdown cost per unit time

C_r =removable cost per unit time for removing the service station.

C_γ =removable per unit time for the server being on vacation.

Using the definition of each cost elements and its corresponding characteristics, the total expected cost function per unit time is given by

$$T(N) = C_h L_s + C_o\frac{E(B)}{E(C)} + (C_s + C_r)\frac{1}{E(C)} + C_d\frac{E(D)}{E(C)} - C_\gamma\frac{E(I)}{E(C)} \tag{47}$$

We obtain the optimal value N*,which minimizes cost function by differentiating it with respect to N and setting the result to be zero. i.e., $\frac{\partial}{\partial N}(T(N)) = 0$ .The solution N to (47) may not be an integer and the optimal positive integer value of N is one of the integers surrounding N* which gives a smaller cost T. Here we should be pointed out explicitly that the solution really gives the minimum value and $\frac{\partial^2}{\partial N^2}(T(N)) = 0$ at N=N* is greater than zero when the values of system parameters satisfy suitable conditions. However, it is quite tedious to present the explicit expression. Therefore we will perform the numerical experiments to demonstrate that the function is really convex and the solution gives a minimum.

IX. SENSITIVITY ANALYSIS

In the course of analysis, sensitivity analysis has been carried out to find the optimum value of N (ie.,N*), expected system length and minimum cost based on changes in the system parameters by using MATLAB.

In order to arrive at the conclusions, the following arbitrary values of the system parameters are considered.
$\lambda = 2$, $\alpha = 3$, $\mu = 5$, $\beta = 5$, $C_h=5$, $C_0=200$, $C_s=500$, $C_d=100$, $C_\gamma=25$, $C_r=25$, $\Theta=3$, $k=2$

It is noticed from the results in table 1 that as λ increases, the value of N*, L (N*) and expected cost T(N*) increases.

Table 1:

λ	0.5	0.75	1	1.25	1.5
N*	9	10	11	12	13
L(N*)	17.43688	26.5325	43.4053	80.8181	191.9188
T(N*)	179.3854	218.50	295.5919	474.3281	1020.81

Computed values in table 2 shows that as μ increases N* increases, the expected cost and queue length decreases as μ increases.

Table 2:

μ	4	6	8	10	12
N*	9	14	16	17	17
L(N*)	69.3989	46.9767	28.6198	14.0804	2.9796
T(N*)	436.0279	326.0808	237.4133	169.5034	120.6893

It can be observed from table3 that N* increases for smaller values of β and does not change for larger values β, the expected cost and queue length increases with increase in β.

Table 3:

β	4	6	8	10	12
N*	10	12	13	14	14
L(N*)	5.14644	11.11154	17.91131	26.11784	35.87028
T(N*)	119.3701	163.658	204.6549	249.8467	301.3808

From table 4, it is observed that N* remained unchanged when α increased from 1.1 to 1.9. Thus N* is insensitive to the changes in α. L(N*) and T(N*) decreased as α increased.

Table 4:

α	1.1	1.3	1.5	1.7	1.9
N*	14	14	14	14	14
L(N*)	46.7689	34.7618	27.0737	21.8137	18.0288
T(N*)	357.4992	295.6264	254.6264	226.1125	204.9639

Computed values in table 5 shows that N* is insensitive to the changes in ϴ. L (N*) and T (N*) decreases as ϴ increases.

Table 5:

ϴ	1	3	5	7	9
N*	12	12	12	12	12
L(N*)	10.5117	8.1157	7.6515	7.4540	7.3451
T(N*)	154.5856	142.9433	140.6922	139.7319	139.1949

X. CONCLUSION

Optimal strategy analysis of N-policy $M/E_k/1$ queueing system with server breakdowns and vacations has been studied. Some of the system performance measures have been derived. A cost function is formulated to determine the optimal value of N. Sensitivity analysis is carried out through numerical illustrations. These numerical values will be useful in analyzing practical queueing system and make decision to improve the grade of service by selecting appropriate system descriptors. The present study can be extended by working vacation. The future scope of the study is the cost and profit analysis of this model.

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