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Determination of Compromise Solutions for Linear Steady State Regulator with Vector Valued Performance Index

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Abstract- The use of vector valued performance index as a performance measure in the design of optimal control systems has invited much attention from researchers in the field of control engineering. The problem is of practical significance as the system presents an optimal solution for multiple criteria optimal control problems. A general performance index is defined in the form of a norm which can be expressed as the sum of p^{th} power of deviations of the different performance index elements from their optimal values considering individual performance measures separately. The norm is minimized in k dimensional performance index space to obtain a set of non inferior systems corresponding to a class of compromise solutions. The paper presents a solution method for determining a class of compromise solutions for the standard linear quadratic steady state regulator problem optimizing k performance indices simultaneously. Selection of a single system out of the class of systems obtained is made by an appropriate choice of the parameter p based on the specified application.

Index Terms- Optimal Control, Compromise Solutions, Multi Objective Optimal Control, Vector Valued Performance Index.

I. Introduction

One of the most important directions in research in the field of control theory is the development of methods for analyzing the quality of control processes. The fact is that contemporary progress in science and technology very sharply presents scientists and engineers with the problem of creating even more perfect automatic control systems. In addition to ensuring stability the requirement is to create higher quality systems ranked based on the various performance indices of quality.

Optimal control theory involves the design of controllers that can satisfy some control objective while simultaneously minimizing certain performance measure. A sufficient condition to solve an optimal control problem is to solve the Hamilton-Jacobi-Bellman (HJB) equation. For the special case of linear time-invariant systems, the HJB equation reduces to an algebraic Riccati equation (ARE) [1, 2]. Optimal control design using single or scalar performance index is an established area in control engineering research. However, current requirement for control systems can no longer be satisfied by using single objective function and therefore multi-objective optimal control appears as the logical future trend.

The always increasing quality requirements for new products and consequently the natural advance in control system design have led to the introduction of more than one design criterion which requires more sophisticated techniques like multiple objective optimizations [3]. A multi objective optimization problem requiring simultaneous optimization of a collection of functions each of which measures a definite aspect of the system performance is addressed. The objective functions normally define mathematical description of design specifications which are in conflict with each other. Recently the problem of vector valued criteria has become a central part of control theory. Great attention is given to the design and construction of modern control systems by taking into account different aspects of system performance to be optimized by employing appropriate control measures [4].

The paper is organized as follows. Section II gives a brief overview of the research in this field and the common methods followed for solving multi criteria optimal control problems. Mathematical preliminaries for the optimal controller design for a linear steady state regulator problem with *k* performance indices is provided in Section III. A numerical example of the problem is presented and solved using Matlab in Section IV. Section V is devoted to draw conclusions from this work.

II. Literature Review and Solution Methods

The problem of vector valued optimal control was first presented by Lotfi Zadeh in 1963 where he addressed the problem of designing control systems which are optimal relative to several performance indices. The paper though not providing a rigorous mathematical solution to multi objective optimization, paves a founding stone for the concept of optimality and non-scalar valued performance criteria. The paper proposes definitions for optimality and non-inferiority in the context of multi criteria optimal control system design. According to Zadeh, a system S_0 in the constraint set C is non-inferior in C if the intersection of C and a subclass of S consisting of all systems which are superior to or better than S_0 is an empty set. Also a system S_0 in the constraint set C is optimal in C if C is contained in a subclass of S consisting of all systems which are inferior to or equal to S_0 [5].

The largest collection of works in vector valued optimization is devoted to the method of determination of set of un-improvable points which are also called Pareto Optimal solutions in group decision problems. The methods adopted so far for solving optimal control problems with vector valued performance index can be summarized as follows [6-8].

A. Optimization of a hierarchical sequence of performance indices:

The method is based on addressing the performance measures specified for the optimal control problem in the order of preference for a particular application. If the performance indices arranged in hierarchical order are represented as $J_1(x)$, $J_2(x)$... $J_l(x)$ and the solution is determined starting with the scalar performance measure of highest preference. The optimization of the performance measure listed as second in the hierarchical order is then done relative to the solution of the first optimization problem.

A major drawback of this method is that optimization with respect to first performance measure which is of highest preference leads to a unique optimal solution and the problem then reduces to optimization relative to the first performance measure only. The solution hence will be a subclass of the unique solution obtained in the first step.

B. Determination of a set of unimprovable points:

A point x^0 ε x is called unimprovable in x relative to J(x), if among all x ε x there does not exist a point x* such that $J_\alpha(x^*) \le J_\alpha(x^0)$, α= 1, 2, k with one of the inequalities being strict. [5] The problem of determination of unimprovable points is proposed as minimization of a linear form of the components of the vector J(x) with constant weighing coefficients. It can be shown that the set of unimprovable points are obtained by minimizing the expression $J = \sum_{i=1}^{k} C_i \; J_i(u)$ where $C_i>0$ is the positive weighing constant and u is the control vector and u ε U is the set of admissible controls.

C. Methods for determining solution based on one form of compromise or another:

This method is based on the assumption that the choice of a solution in the multi criteria control problem is confined to the determination of a set of compromise solutions from which a unique optimal solution is chosen based on some heuristic considerations. One such technique employed ensures that the relative decrease in one of several criteria should not exceed the relative increase in the remaining criteria.

Another approach is to determine an ideal point (utopian point) where all the performance measures have their optimal values and then a norm is introduced in the criteria space. The compromise solution obtained in this method is a Pareto – Optimal solution and ensures the closest approach of the criteria to their optimal values [6].

The problem of computing compromise solutions by simultaneously optimizing the design parameters is presented in [9] as an application of compromise solutions obtained. In this paper the compromise solutions are obtained by simultaneously optimizing a design parameter ζ which appears in the state feedback path as an element of the feedback gain matrix.

III. Compromise Solution for Linear Steady State Regulator

The steady state regulator problem with a vector valued performance index is considered. The performance index elements considered are quadratic performance measures with terminal time $t_f = \infty$. Let the dynamic system considered be of the form

$$\dot{x} = Ax + Bu \qquad (3.1)$$

Where *x* is the state vector, *u* is the control vector, *A* is the *nXn* constant state matrix, and B is *nXm* constant control matrix. The initial conditions $x(0) = x_0$.

Also consider the cost function of the form

$$J(x) = \int_0^\infty \left[x^T Q x + u^T R u\right] dt \qquad (3.2)$$

$$where \;\; Q \in R^{nXn},\; R \in R^{nXm}$$

Q and R are positive definite state and control weighting matrices respectively. The optimal feedback control problem in case of single scalar performance index is to find the admissible control *u** so that the cost function *J(x)* is minimized. This is obtained by solving the algebraic Riccati equation in (3.3) where *P* is the Riccati matrix to be solved.

$$PA + Q - PBR^{-1}B^T P + A^T P = 0 \qquad (3.3)$$

The problem presented in this paper is to minimize the vector valued performance index given in equation (3.4) expressed in the form of p^{th} power of the sum of deviations of the individual performance measures from their optimal values. [8]

$$J(u,p) = \left[\{J_1(u) - J_{10}(u)\} + \{J_2(u) - J_{20}(u)\} + \Lambda\; \{J_k(u) - J_{k0}(u)\}\right]^p \qquad (3.4)$$

$$where \;\; 1 \le p < \infty$$

The quadratic performance elements $J_i\,(u)$ have the form

$$J_i(u) = \int_0^\infty (X^T Q_i X + u^T R_i u)\; dt, \qquad (3.5)$$

$$i = 1,2,....k$$

$J_{i0}(u)$ are the optimal values of the individual performance measures obtained by solving the *k* individual optimization problems.

The vector valued performance index in equation (3.4) is expressed as a linear combination of the performance index elements.

$$J(u,p) = \left[\sum_{i=1}^{k} \{J_i(u) - J_{i0}(u)\}^p\right] \qquad (3.6)$$

Equation (3.6) can be rewritten as

$$J(u,p) = \left[\sum_{i=1}^{k} \{J_i(u) - J_{i0}(u)\}^{p-1} \{J_i(u) - J_{i0}(u)\}\right] \qquad (3.7)$$

A set of positive quantities C_i are defined such that

$$C_i = \{J_i(u) - J_{i0}(u)\}^{p-1}, \quad i = 1,2,....k \qquad (3.8)$$

The performance index J can be now viewed as a function of *u* and C_i given in equation (3.9).

$$J(u,C) = \sum_{i=1}^{k} C_i \{J_i(u) - J_{i0}(u)\} \qquad (3.9)$$
$$C = [C_1, C_2, C_3 \;......C_k]$$

For a set of values of C_i, the above expression is equivalent to minimization of $J(u,C) = \sum_{i=1}^{k} C_i J_i(u)$

It is convenient to normalize the weighting constants such that $\sum_{i=1}^{k} C_i = 1;\; C_i > 0$

The set of positive quantities C_i are modified as

$$C_i = \frac{\{J_i(u) - J_{i0}(u)\}^{p-1}}{\sum_{j=1}^{k}\{J_j(u) - J_{j0}(u)\}^{p-1}} \qquad (3.10)$$

The steady state regulator problem addressed here is an ordinary control problem with vector valued performance index. The solution of the regulator problem is obtained by solving the algebraic Riccati equation given in (3.3) where *P, Q, R* are replaced by equations (3.11-3.13) respectively and *P* is the Riccati matrix to be solved.

$$P = C_1P_1 + C_2P_2 + \Lambda + C_kP_k \qquad (3.11)$$
$$Q = C_1Q_1 + C_2Q_2 + \Lambda + C_kQ_k \qquad (3.12)$$
$$R = C_1R_1 + C_2R_2 + \Lambda + C_kR_k \qquad (3.13)$$

Substituting equations (3.11-3.13) in equation (3.3) and differentiating the expression partially with respect to C_1 yields

$$A^TP_1 + P_1A - PBR^{-1}B^TP_1 + PB\left(\frac{\partial R^{-1}}{\partial C_1}\right)B^TP + P_1BR^{-1}B^TP + Q_1 = 0 \qquad (3.14)$$

$$\frac{\partial R^{-1}}{\partial C_1} = \frac{\partial(R^{-1}RR^{-1})}{\partial C_1} = -R^{-1}R_1R^{-1} \qquad (3.15)$$

$$Since\;\frac{\partial R^{-1}}{\partial C_1} = R^{-1}R\frac{\partial R^{-1}}{\partial C_1} + R^{-1}\frac{\partial R}{\partial C_1}R^{-1} + \frac{\partial R^{-1}}{\partial C_1}RR^{-1}$$

$$and\;\frac{\partial R}{\partial C_1} = R$$

Substituting equation (3.15) in equation (3.14) gives

$$A^TP_1 + P_1A - PBR^{-1}B^TP_1 + PB(-R^{-1}R_1R^{-1})B^TP + P_1BR^{-1}B^TP + Q_1 = 0 \qquad (3.16)$$

$$[A - BR^{-1}B^TP]^TP_1 + P_1[A - BR^{-1}B^TP] - PBR^{-1}R_1R^{-1}B^TP + Q_1 = 0 \qquad (3.17)$$

The same procedure is repeated for the 'k' performance measures to yield the k equations given in equation (3.18)

$$\begin{aligned}&[A - BR^{-1}B^TP]^TP_2 + P_2[A - BR^{-1}B^TP] - PBR^{-1}R_2R^{-1}B^TP + Q_2 = 0\\ &\quad\bullet\\ &\quad\bullet\\ &[A - BR^{-1}B^TP]^TP_k + P_k[A - BR^{-1}B^TP] - PBR^{-1}R_kR^{-1}B^TP + Q_k = 0\end{aligned} \qquad (3.18)$$

The k matrix equations are solved to get the matrices P_1, P_2, P_k. Using these matrices the performance index elements are computed using equation (3.18).

$$J_k(u) = x_0^TP_kx_0 \qquad (3.19)$$

IV. Numerical Example

Consider the second order linear steady state regulator represented by the following equations.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - 2x_2 + u$$

Let the initial conditions be $x_1(0) = -1, x_2(0) = 1$

Let the performance index considered be of the form

$$J(x,u) = \int_0^\infty [x^TQx + u^TRu]dt$$

The problem is to minimize the vector valued performance index given as J_1 and J_2 for $1 \le p < \infty$.

$$J(u,p) = [\{J_1(u) - J_{10}(u)\} + \{J_2(u) - J_{20}(u)\}]^p$$

Where the individual performance indices are

$$J_1(x,u) = \int_0^\infty [5x_1^2 + 0.01u^2]dt$$
$$J_2(x,u) = \int_0^\infty [x_2^2 + 0.01u^2]dt$$

The individual optimisation problems are solved and the Riccati matrix P and performance index are obtained as follows

$$P_{10} = \begin{bmatrix} 1.5107 & 0.2138 \\ 0.2138 & 0.0484 \end{bmatrix}$$

$$P_{20} = \begin{bmatrix} 0.0820 & 0.0000 \\ 0.0000 & 0.0820 \end{bmatrix}$$

$$J_{10} = 1.1314 \,;\, J_{20} = 0.1640$$

The iterative procedure for the computation of weighing constant C_i is started with an initial guess of C_1, C_2= 0.5 and p=1.The matrices Q and R takes the values

$$Q = C_1Q_1 + C_2Q_2 = \begin{bmatrix} 5C_1 & 0 \\ 0 & C_2 \end{bmatrix}$$

$$R = C_1R_1 + C_2R_2 = \left[0.01(C_1 + C_2)\right]$$

Using these matrices the combined algebraic Riccati equation given in equation (3.3) is solved by Eigen vector method and the combined Ricatti matrix P is obtained as follows.

$$P = \begin{bmatrix} 2.8586 & 0.2969 \\ 0.2969 & 0.1430 \end{bmatrix}$$

With the value of matrix P and substituting in equations (3.16, 3.17) the individual algebraic Riccati equations are solved to obtain P_1 and P_2 as follows.

$$P_1 = \begin{bmatrix} 1.8548 & 0.2273 \\ 0.2273 & 0.0528 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 1.0039 & 0.0695 \\ 0.0695 & 0.0902 \end{bmatrix}$$

$$J_1 = 1.4529 \,;\, J_2 = 0.9550$$

The calculations are carried out for values of p from 1 to 10 using Matlab and the results are tabulated below in Table 1.

Table 1. Compromise solutions

p	C1	C2	J1	J2
	1	0	1.1314	1.919
	0	1	5.0514	0.1640
1	0.5	0.5	1.4529	0.9550
2	0.4124	0.5876	1.5996	0.8326
3	0.3904	0.6096	1.6431	0.8034
4	0.3806	0.6194	1.6640	0.7901
5	0.3751	0.6249	1.6762	0.7829
6	0.3715	0.6285	1.6843	0.7781
7	0.3690	0.6310	1.6900	0.7748
8	0.3672	0.6328	1.6942	0.7723
9	0.3658	0.6342	1.6975	0.7704
10	0.3646	0.6354	1.7001	0.7689

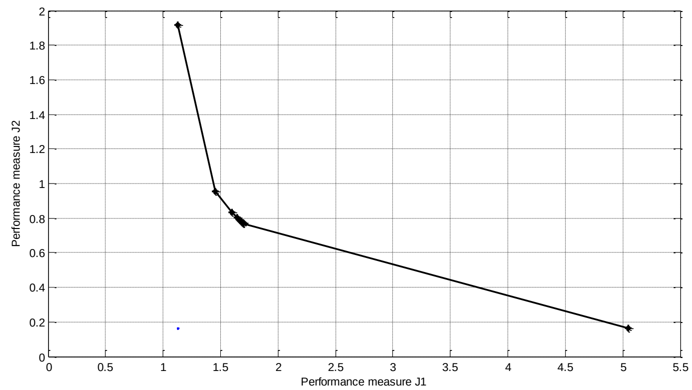

Figure 1. Compromise Solutions

V. Conclusions

The paper presents a method for finding out compromise solutions for vector valued performance criteria. The method is based on the minimization of a specific norm out of a set of norms in the *k* dimensional performance index space. This is equivalent to minimization of a linear combination of *k* performance measures with constant weighing coefficients which is evaluated using an iterative procedure. The choice of the parameter p is made so as to meet a particular design specification for a specified application. The solution method presented is flexible since the choice of parameter *p* can be done depending on the application.

The linear steady state regulator problem with two performance indices is considered in the numerical example. The compromise solutions are obtained and tabulated for values of p varying from 1 to 10.Each point in the two dimensional performance index space is a solution for the problem based on one form of compromise or the other. The curve in the above figure is the locus of non inferior solutions which forms the boundary of the region of admissible controls. The method can be applied to practical systems to obtain solution for multiple criteria optimal control problems.

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