On the comparative Study of Compactness and some of its relative notion in Metric and topological spaces.

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Abstract- In this paper, we compared compactness and its related properties in metric and topological spaces and determine what topological spaces can do that metric spaces cannot.

Index Terms- Metric Spaces, Topological Spaces, Compact Spaces, Lindelöf space, Countably compact Spaces, Separable Space, Sequentially Compact Space, Second Countable Space, Complete Space.

I. INTRODUCTION AND DEFINITIONS

The general topology has become foundation of knowledge for all branches of mathematics. Its methods have enriched other fields of mathematics and also give enough clues to their new development.

In this work, we compared compactness property and its related notions within metrics and topological spaces, we further study the implication of the notions. As we know, sequences are not sufficient for the study of general topological spaces, that is why sequential compactness is not a best notion. Compactness is the most important property of topological space, it has very wild application to both analysis and functional analysis [1]. [5] The following definition below leads us to the main work as we can see in [5], [7].

Definition 1.1: Suppose $X$ is a topological space. A cover of $X$ is a family $O \subseteq p(X)$ of subset of $X$ such that $\bigcup O = \bigcup_{n \in O} = X$. This cover is open if every $A \in O$ is closed.

Definition 1.2: Suppose $X$ is a topological space. A subfamily $B \subseteq O$ of a cover $O$ is said to be a subcover iff $\bigcup B = X$.

Definition 1.3: A topological space $X$ is said to be compact if every open cover of $X$ has a finite subcover. i.e for each open cover $\{O\}_{i=1}^n$ of $X$ exists a finite subfamily set $\{\alpha_{1}, \alpha_{2}, ..., \alpha_{n}\}$ such that $\bigcup_{i=1}^{n} \alpha_{i} = X$.

Definition 1.4: [6] A topological space $X$ is Lindelöf if every open cover has a countable subcover. Example: (i) Every finite space is compact (ii) Closed and bounded interval $[a, b]$ in $\mathbb{R}$ is compact while bounded interval $(a, b)$ in $\mathbb{R}$ is not compact. To show this, let $E = b - a$ and consider the collection $U = \{U_{n} : n \in \mathbb{N}\}$ of open subsets of $(a, b)$ given by $U_{n} = \left(n + \frac{1}{n}, b\right)$. Then, $U$ is a cover of $(a, b)$ infact $a < x < b \Rightarrow x - a > \frac{1}{n}$ for some integer $n \geq 1$ and then $x \in U_{n}$. However, if $F$ is any finite subcollection of $U$, then $a + \frac{E}{M+1} \notin U_{n}, U_{n} \in F$, where $M = \max\{n : U_{n} \in F\}$, so $F$ is not a cover of $(a, b)$.

(iii) No any half-open interval $(a, b]$ or $[a, b)$ is compact (iv) The real line $\mathbb{R}$ is not compact for $(-n, n)$ is an open cover of $\mathbb{R}$ that has no finite subcover of $\mathbb{R}$.

(v) The discrete topology on a countable infinite set gives an example of a space which is Lindelöf but not compact.

(vi) The sorgenfrey line is Lindelöf (vii) Countable and cocomplete spaces are Lindelöf

Definition 1.5: [6] A collection $F$ of sets is said to have the finite intersection property if $F$ is nonempty and each nonempty finite subcollection of $F$ has nonempty intersection. [2]

Lemma: [3] A topological space $X$ is compact iff each collection of closed subsets of $X$ having the finite intersection property itself has nonempty intersection.

Two generalization of compactness will be derive by weakening the requirement that subcovers must be finite. A topological space is $\sigma$ - compact if it can be express as the union of countably many compact sets, by comparing this definition of $\sigma$ - compact to definition 1.4 above, we see that every compact space is $\sigma$ - compact and every $\sigma$ - compact space is Lindelöf. A topological space $X$ is countably compact if every countable open cover has a finite subcover and two other compactness notions are closely related, but not equivalent to countable compactness a topological space $X$ is said to be seqeuncially compact if every sequence in $X$ has a convergent subsequence and $X$ is weakly countably compact if every infinite set has a limit point [4]. These are the view among other compactness properties we would start comparing between metric and topological spaces to determine what topological spaces can do that metric space cannot.

II. COMPACTNESS PROPERTIES AND ITS RELATED NOTIONS.

The following charts (Fig. A and B) below demonstrate the comparative study of compactness properties and its related notions in topological spaces and metrics spaces respectively. We would study notions like $\sigma$ - compactness, Lindelöf, countably compactness and determine that they are all equivalent to compactness on metrics spaces and show where they failed to do so in topological spaces, that is, what topological notions can do.
that metric notions cannot. We start our comparative study from theorem 2.1 below.

**Theorem 2.1:** Every \( \sigma \) - compact space is Lindelöf

\( \sigma \) – compactness \( \Rightarrow \) Lindelöf

**Proof:** Suppose \( X \) is a \( \sigma \) - compact space. Then there exist a countable family \( \{ F_n : n = 1, 2, \ldots \} \) of compact subsets of \( X \) such that \( X = \bigcup \{ F_n : n = 1, 2, \ldots \} \). Now, consider any open cover \( U \) of \( X \) so that for each \( n = 1, 2, \ldots \) \( \exists \) a finite subfamily \( U_n \) of cover \( U \) which cover a countable family \( F_n \) is compact, \( U \{ U_n : n = 1, 2, \ldots \} = \bigcup \{ U_n : n = 1, 2, \ldots \} \) is a countable subfamily of cover \( U \) which cover \( X \). Therefore \( X \) is Lindelöf.

**Proposition 2.1:** Sequentially compactness properties implies countably compactness in a topological space. i.e. if a topological space \( X \) is sequentially compact, then is countably compact.

**Proof:** First, we shall show that if \( \{ x_n \} \) is a sequence which has a subsequence \( \{ x_{n_k} \} \) converging to \( t \) in a topological space \( X \), the \( t \) is an accumulation point of the sequence \( \{ x_n \} \). By let \( O \) be an open neighbourhood of \( t \), provided that \( t \) is an accumulation point of subsequence \( \{ x_{n_k} \}, \exists N \in \mathbb{N} \) such that \( k \geq N \) implies that \( x_{n_k} \in O \), that means there are infinitely many \( n \) such that \( x_n \in O \). So, \( t \) is an accumulation point. Therefore, the sequentially compactness implies that every sequence in a topological space \( X \) has an accumulation point.

From the figure B below, we have the Lemma 2.1 below.

**Lemma 2.1:** A metric space is totally bounded if it is sequentially compact.

Totally boundedness \( \Rightarrow \) Sequentially boundedness.

**Proof:** We shall provide proof of this Lemma from contradictory point of view. Suppose a metric space \( X \) is not totally bounded, that means there does not exist finite many points for \( \varepsilon > 0 \) such that \( \bigcup_{i=1}^{n} B(x_i, \varepsilon) = X \), that is \( X \) cannot be covered by finitely many \( \varepsilon \)-balls. Let \( \{ x_n \} \) be a sequence such that \( x_n \) is a member of \( X \setminus \bigcup_{i=1}^{n} B(x_i, \varepsilon) \), \( \forall n \Rightarrow d(x_i, x_j) \geq \varepsilon \forall i > j \). This provides that \( i \neq j \) gives that \( d(x_i, x_j) \geq \varepsilon \), so the sequence \( \{ x_n \} \) is not Cauchy.

Conversely, suppose \( \{ x_n \} \) is a sequence in a metric space \( X \). Since \( X \) is totally bounded, that means it can be covered by finitely many \( \varepsilon \)-balls of radius \( 1 \). Let \( B_1 \) be one of the ball and \( x_n \in B_1 \) for \( n > 0 \). Then \( B_1 \) can be covered by finitely many balls of radius \( 1 \). Also, let \( B_2 \) be another ball which satisfy that \( B_1 \cap B_2 \) holds and \( x_n \in B_1 \cap B_2 \) for \( n > 0 \). If we continue like this, we have a sequence \( B_i \) of an open balls of radius \( 1 \) such that \( B_1 \cap B_2 \cap \ldots \cap B_m \) holds and \( x_n \in B_1 \cap B_2 \cap \ldots \cap B_m \) for \( n > 0 \). Thus, if we take a subsequence \( \{ x_{n_k} \} \) such that for each \( k, B_i \cap B_j \cap \ldots \cap B_m \) contains \( x_{n_k} \). Now, if \( j \geq k \), both \( x_{n_j} \) and \( x_{n_k} \) are also contained in \( B_k \), then \( d(x_{n_k}, x_{n_j}) < \frac{2^j}{i} \). Therefore, the subsequence \( x_{n_k} \) contains in \( (n = 1, B_1 \cap B_2 \cap \ldots \cap B_m \) which converges to \( k \) is a Cauchy sequence and since the subsequence \( x_{n_k} \) converge to \( k \), therefore, the metric space \( X \) is sequentially compact.

**Proposition 2.3:** Suppose \( (X, d) \) is a metric space. Then, the following must satisfy: (i) sequentially compactness \( \Rightarrow \) completeness and sequentially compactness \( \Rightarrow \) totally boundedness.

**Proof:**
Suppose \( X \) is a metric space, we have from the Fig. B below

Compactness property \( \Rightarrow \) Lebesgue property \( \Rightarrow \) completeness.

**Theorem 2.2:** If \( X \) is compact metric space then \( X \) is complete. i.e. If \( X \) is compact metric space, Compactness \( \Rightarrow \) completeness.

**Proof:** Suppose \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in the subspace \( A \subset X \) and \( S \subset A \) be the following subset \( S = \{x_1, x_2, \ldots, x_n, \ldots\} \subset A \). If we take \( S \) to be infinite then, we follow the proof as follows:

**Step A:** \( \exists \) an accumulation point \( z \in A \), for each \( \epsilon > 0, B_z(\epsilon) \cap S \neq \emptyset \) and also \( B_{z,E}(\epsilon) \cap S = \{z\} \). Suppose there does not exist such \( z \in A \), then for each \( x \in A \) \( \exists \) some \( \varepsilon(x) > 0 \) such that \( B_{z_k}(\varepsilon(x)) \cap S = \emptyset \) and also \( \{x\} \). As shown in Lemma 2.1 above, the collection of open balls \( \{B_{x,\varepsilon(x)}\}_{x \in A} \) covers set \( A \) and is open, so, by compactness, it will contain a finite subcover; for this, \( \exists\ x_1, x_2, \ldots, x_k \) such that \( S \subset \bigcup_{i=1}^{k} B_{x_i}(\varepsilon(x_i)) \). Assume that \( S \subset \bigcup_{i=1}^{k} B_{x_i}(\varepsilon(x_i)) \)

------------------ (*)

But since \( S \) is infinite and \( B_{z,E}(\epsilon) \cap S \neq \emptyset \) and also \( \{x\}, \forall x \in A \), then (*) above is not possible.

**Step B:** Suppose \( z \in A \) is an accumulation point as we shown in step A above. Then the sequence has a subsequence \( \{x_n\} \) where \( k \in \mathbb{N} \) and has \( z \) as its accumulation point. If we let \( z \notin S \), this part is done by using an induction method. Now, let \( n(1) \in \mathbb{N} \) such that \( x_{n(1)} \in B_{z_1}(1) \cap S \). Assume that we have \( n(1) < n(2) < \cdots < n(k) \) such that \( x_{n(1)} \in B_{z_1}(1 \cap S \). t = 1, \ldots, k \) and if

\[
E = \min \left\{ \frac{1}{k^{1/2}}, d_{x_1}(z, x_1), d_{x_2}(z, x_2), \ldots, d_{x_k}(z, x_n(k)) \right\}
\]

Provided that \( x_{n_k} \rightarrow z \) \( \forall n_k \), we have that \( E > 0 \) and by part A above, \( \exists \) some \( n(k+1) \in \mathbb{N} \) such that \( x_{n(k+1)} \in B_{E}(1) \cap S \). Then, we see that choice of \( E \) provides that \( n(k) < n(k+1) \) and this ends the induction method of Step B.

**Step C:** We have \( z \) as an accumulation point of the sequence \( \{x_n\}_{n \in \mathbb{N}} \) as shown above due to the fact that it is a Cauchy sequence. For this let \( E > 0 \), then there are two reasons to study here

(i) \( N \in \mathbb{N} \) such that, if \( p, q \geq N \), then \( d_{x_p}(z, x_q) \leq E \) and (ii) \( R \in \mathbb{N} \) such that, if \( r \geq R \), then \( d_{x_r}(z, x_r(r)) \leq E \). If we let \( r \in N \) such that \( n(r) \leq N, R \). Therefore if \( n \geq N \), we have \( d_{x_r}(z, x_n(r)) \leq d_{x_r}(z, x_n(r)) + d_{x_n(r)}(z, x_n(r)) \leq 2E \) (by triangle inequality).

Therefore, the sequence \( \{x_n\}_{n \in \mathbb{N}} \) has \( z \) as its accumulation point.

**Definition 2.1:** [6] If \( A \) is an open cover of a metric space \( X \), a real number \( \lambda > 0 \) is called a Lebesgue number for the cover \( A \), if for every \( B \subseteq X \) with \( diam(B) < \lambda \) \( \exists A \subset K \) such that \( B \subseteq K \).

**Theorem 2.3:** [2] Let \( X \) be metric space and \( \{U_{x}\} \) be an open cover of \( X \). Then there is a positive number \( \lambda(\{U_{x}\}) \) called Lebesgue number of the cover \( X \) which satisfy the following property:

Each ball \( B(X, \lambda) \) is contained in at least one \( U_{x} \).

From implication \( \phi \) above, we shall develop this result below by follow theorem 2.3 above.

**Theorem 4.0:** Compactness property \( \Rightarrow \) Lebesgue property for compact metric space \( X \).

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**Theorem 2.7:** In a metric space $X$, if $X$ is separable, then is a second countable space.

**Proof:** If the sequence $\{x_n\}_{n \in \mathbb{N}}$ is a countable dense in $X$. Then for each $x_n$, let $O_n$ be the set of all open balls centred at $x_n$ in which its radius is rational. Then the $O_n$ is countable for each $n$, so, we have that $O = U\{A_n : n \in \mathbb{N}\}$ is a basis. Now, choose an open set $K \subseteq X$, we shall show that $K$ is a union of our basis $O$ defined above. If $x \in K$, we shall show that there is one of basis set $O$ such that $x \in A \subseteq K$. Then, the point $x$ contained in some $\mathcal{E}$ - balls in $K$. If we consider an $\frac{1}{10^2}$ Ball around $x$, this ball must contain some point $x_n$ from dense subset of countable dense set $\{x_n\}$ so that $d(x_n, y) < \frac{1}{10^2}$. Then, if we take a rational radius which is strictly more than $\frac{1}{10^2}$, then, we have a rational radius ball with centre $x_n$ containing $x \in K$.

**Theorem 2.8:** Every compact topological space is $\sigma$ - compact. Compactness $\Rightarrow$ $\sigma$ - compactness.

**Proof:** Suppose $X$ is a compact topological space. Let $K \subseteq X$ be a compact subset of $X$. Let $\{A_n : n \in \mathbb{N}\}$ be the family of compact subset of $X$ that covers $X$, $K \cap A_n$ is a subset of the subspace $A_n \subseteq X$, for every $n \in \mathbb{N}$. For this, $K \cap A_n$ is compact in $K$ for every $n \in \mathbb{N}$. Then $K$ will be union of countable family $\{K \cap A_n : n \in \mathbb{N}\}$ of its compact subsets. Therefore, $K$ is $\sigma$ – compact. This completes the proof.

**Theorem 2.9:** Every totally bounded metric space is separable.

Total boundedness $\Rightarrow$ separability

**Proof:** Let $X$ be a metric space and is totally bounded. Then, there exist $x_{p,1}, x_{p,2}, \ldots, x_{p,n}$ such that $X = \bigcup_{k=1}^{n} B\left(x_{p,q}, \frac{1}{p}\right)$. Then there exist $A \in X$ such that $A = \{x_{p,q} : p \in \mathbb{N}, q = 1, 2, \ldots, n\}$ is countable choose $x \in X$ and $\varepsilon > 0$, also $\varepsilon^{-1}$ such that $p > \varepsilon^{-1}$, so that there exist a such that $d(x, x_{p,q}) < \frac{1}{p} < \varepsilon$. Then, $A$ is dense and the metric space $X$ is separable.

**Theorem 2.10:** Every totally bounded space is bounded.

Totally bounded $\Rightarrow$ Boundedness

**Proof:** Let $X$ be a metric space and is totally bounded. From definition of totally boundedness, there exist finitely many points $x_1, x_2, \ldots, x_n$ such that $U_n B(x_n, 1) = X$ $\Rightarrow$ the open ball of radius 1 covers $X$. By $M = \max_{n,m} d(x_n, x_m)$, by triangle inequality, we have $\text{diam}(X) \leq M + 2 < \infty$. Therefore, $X$ is bounded.

**Theorem 2.11:** Every compact metric space $X$ is totally bounded.

Compactness property $\Rightarrow$ Totally boundedness property

**Proof:** For the radius $r > 0$ of the ball $B(x, r)$, $\exists$ family of open ball $\{B(x, r)\}_{x \in X}$ such that $X = \bigcup_{x \in X} B(x, r)$, by compactness, $\{B(x, r)\}$ cover $X$ and $\exists$ a finite subcover for the cover $\{B(x, r)\}$. Therefore, $X$ is totally bounded.

**Theorem 2.12:** If a metric space $X$ is sequentially compact, then $X$ is Lebesgue.

Sequentially compactness $\Rightarrow$ Lebesgue property.

**Proof:** Let $\lambda > 0$ be a Lebesgue number. Now if $O$ is an open cover that not accept $\lambda$, there exist open sets of diameter $D$ which is arbitrarily small such that $D \not\subset A \in O$. Particularly, for each $n \in \mathbb{N}$ $\exists$ $x_n$ such that $B\left(x_n, \frac{1}{n}\right) \not\subset A, \forall A \subset 0$. Due to sequentially compactness, the sequence $\{x_n\}$ has $x$ as its accumulation point.

Provided that $O$ cover $\lambda$, we have $x \in A$ for some $A \in O$. Since $A$ is open, $\exists$ rarius $r > 0$ such that the metric $d(x, x_n) < \frac{r}{2}$ and $\frac{1}{n} < \frac{r}{4}$. Assume $q \in B\left(x_n, \frac{1}{n}\right)$, then by triangle inequality, $d(q, x) \leq d(q, x_n) + d(x_n, x) < \frac{1}{n} + \frac{r}{2} < \frac{r}{4} + \frac{r}{2} = \frac{3}{4}r < r$. Therefore, the open ball $B\left(x_n, \frac{1}{n}\right) \subseteq A$, which is contrary to the choice of the sequence $\{x_n\}$. This end the proof.

**Definition 2.2:** [7] A topological space $X$ is pseudocompact, if every continuous real – valued function on $X$ is bounded.

**Theorem 2.13:** Every pseudocompact metric space $X$ is countably compact.

Pseudocompactness $\Rightarrow$ Countably compactness

**Proof:** Suppose the implication above is not true, then there exist a sequence $A = \{a_n\}$ which is not converge. Hence $A$ is closed and also discrete. Therefore, the function $f: A \rightarrow \mathbb{N}$ defined by $f(a_n) = n$ is continuous, so by Tietze extension theorem, this function can be extended to the space $X$. But due to the fact that $f(a_n) = n$ is continuous, then, is not bounded which is contrary to pseudocompactness notion, which proved the theorem. The converse of this is also true in topological spaces.
Topological Spaces

- Compactness
- Countably compactness
- Weakly countably compactness property
- Second countable
- Separability
- Lindelof
- Pseudo Compactness
- Sequentially Compactness property

Fig. A
Metric Spaces

Lindelöf property → Separability property → 2nd countable property

Compactly compactness property → Totally boundedness → Boundedness

Lebesgue property → Completeness property

Countably compactness property → Sequentially compactness property

Weakly countably compactness property → Pseudocompactness property

Fig. B
III. CONCLUSION

In this paper, we noticed that there are some compactness properties and some of its relative notions that satisfied in both topological spaces and metric spaces. But out of these properties, there are some that satisfied in arbitrary topological spaces and metric spaces only and some on metric topological spaces.

The figure A and B above provides us those properties that satisfied in topological spaces and in metric spaces respectively. By comparing these two figures, we conclude that:

(i) Completeness, Lebesgue, Boundedness, Total boundedness are not consider in arbitrary topological spaces, this shows that they are properties of metric spaces

(ii) While some other properties mentioned above like $\sigma$ – compactness, compactness, second countability, separability, Lindelöf, etc. Plays roles in both metric and topological spaces.

(iii) Converse of Theorem 2.6 does not hold in topological spaces. Also It does not holds in metric spaces as shown in Fig A and B above.

(iv) Converse of theorem 2.7 does not holds in metric space but holds in topological spaces as in Fig A and B above.

(v) Converse of theorem 2.9, 2.10, 2.11 and 2.12 does not holds.

(vi) Converse of theorem 2.13 holds in topological spaces.

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