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The Trigonometric Representation of Complex Type Number System

ThayamathyPio Jude Navinthan

[*] Department of Mathematics, Eastern University, Sri Lanka.

Abstract- An extension to the classical complex number system from the space of real sequences was proposed. We show that the new complex type number system forms a field over the real or complex field with infinite dimension. We explore some possible exponential and trigonometric formulations this new complex type numbers and investigate their properties. The complex type numbers are used to represent three dimensional physical spaces with Cartesian coordinates.

Index Terms- Cauchy-Riemann conditions, Euler's formula, Harmonic polynomials, Hyper-complex numbers.

I. Introduction

The real number system, naturally identified and known from the early period of philosophical era and now denoted by $\mathbb{R}$, is a field with two basic binary operations, namely, addition and multiplication, having the well-known properties shortly listed as closed-ness, commutative, associative and distributive properties, existence of identities and inverses for both operations along with the assumption that the additive and multiplicative identities are distinct.

The development of mathematics is so heavily dependent on this fundamental number system that any other number systems soughtare evaluated for their usability based on having these basic field properties—some or all of them. In this connection, the only number system having the all of theabove mentioned field properties is the complex number system.

Inspired by the vast usability of complex number system in many areas, search for higher dimensional extensions of this kind had been an active research interest in the past.

The early extensions were defined on finite dimensional spaces. As it is now known and proved that the only finite dimensional spaces with all the field properties are the real the complex number systems, these attempts for higher dimensional number systems have obviously failed. Hamilton[4] dedicated his entire life for this failed attempt. Finally, in 1847, he came up with 4-dimensional quaternions which lack one of the properties of field – the commutative property.

Since then higher dimensional extensions of complex numbers were focused on relaxing some of the field properties. The commutative and associative properties are relaxed in quaternion[5] and octonion[5] respectively. Clifford Algebra also relaxes the commutative property. Recently, Fleury et al[5] defined a multi-complex number system relaxing the existence of inverse of all non-zero muti-complex numbers.

In this paper, we propose a class of number systems of infinite dimension which retains all the properties of field. Although this number system is infinite dimensional, it can be applied to represent points and functions in three and higher dimensional Cartesian spaces. We call our new number system the Complex Type Number system.

The motivation of the definition of complex type system comes from an attempt for a generalization of the fact that the complex valued analytic functions of the two dimensional Cartesian coordinate variables x and y constitute a pair of real valued harmonic functions. Our complex type representation gives a canonical extension of this property to functions of higher dimensional coordinate variables. Beyond these, by the complex type field with some generalizations, other basic properties of complex type numbers such as trigonometric functions representation, De Moivre's formula and Euler's formula for exponential representations are satisfied.

In any finite dimensional space, our construction of complex type numbers can be readily extended to satisfy the harmonic function property for functions. In this paper we restrict our discussion to function in three dimensional variables only which can be directly extended to higher dimensions. In this context, we see that our complex type representation is a natural extension of the classical complex space $\mathbb{C}$ in the sense that our complex type space include $\mathbb{C}$ and that the fundamental algebraic, trigonometric, exponential representations and holomorphic properties of function are carried out to the extended complex type space with some modifications.

This paper is organized as follows: In Section 2 the definition of complex type number system and its basic properties are given. In Section 3, the extended trigonometric and exponential representation and their identities are established. In Section 4, the analytic properties of complex type valued functions based on derivatives of these functions are described and finally, a conclusion is drawn in Section 5.

II. Definition And Basic Properties

Assume t be a symbolic variable and $\mathbb{R}(t)$ be the space of infinitely differentiable functions with real coefficients. We define a $\mathbb{C}(t)$as the set of pairs of functions $(a(t), b(t)) \in \mathbb{R}(t)^2$.

Definition 2.1. The complex type space $\mathbb{C}(\mathrm{t})$ is defined as the set $\mathbb{C}(\mathrm{t}) = \{(\mathrm{a}(\mathrm{t}), \mathrm{b}(\mathrm{t})) | \mathrm{a}(\mathrm{t}), \mathrm{b}(\mathrm{t}) \in \mathbb{R}(\mathrm{t}) \}$ with operations defined by

1.$(\mathrm{a}_1, \mathrm{b}_1) + (\mathrm{a}_2, \mathrm{b}_2) = (\mathrm{a}_1 + \mathrm{a}_1, \mathrm{b}_2 + \mathrm{b}_2)$ (1)

2. $(\mathrm{a}_1, \mathrm{b}_1)(\mathrm{a}_2, \mathrm{b}_2) = (\mathrm{a}_1\mathrm{a}_2 - (1+\mathrm{t}^2)\mathrm{b}_1\mathrm{b}_2, \mathrm{a}_1\mathrm{b}_2 + \mathrm{a}_2\mathrm{b}_1)$, where $\mathrm{a}_\mathrm{I}, \mathrm{b}_\mathrm{i} \in \mathbb{R}(\mathrm{t})$,
$\mathrm{i} = 1{,}2.$ (2)

Here, we have omitted the argument symbol (t) in the functions in $\mathbb{R}(\mathrm{t})$and denote $\mathrm{a}(\mathrm{t})$ as a for brevity of notations.

Although we define the space consisting of functions of the symbol *t*, we are mainly interested in the coefficients of the powers of *t* in the infinite series representation of the functions. In this context, the functions are regarded as generating functions of their coefficients. We denote the sequence of coefficients of a function $a(t)$ by a itself so that when $a = \{a_n\}_{n=0}^{\infty}$, we have$a(t) = \sum_{n=0}^{\infty} a_n t^n$.

We refer the pairs of functions or the pairs of sequences as the "complex type numbers" and denote $(a(t); b(t))$ for the function form and $(a; b)$for the sequence form. We use both notations interchangeably as they point to the same concept.

Alternatively, the elements of the space $\mathbb{C}(t)$ can also be expressed in the complex type form $a(t) + jb(t)$, where the 'imaginary' symbol $j$satisfies $j^2 = -(1+t^2)$. For a complex type number $a + jb$, we call a and b, the real type and imaginary type parts of f respectively.

The addition and multiplication for complex type numbers are in such a way that the rules of addition and multiplication are consistent with the rules of real numbers.

Thus if $f_1 = a_1 + jb_1$ and $f_2 = a_2 + jb_2$, we have

$$f_1 + f_2 = (a_1 + a_2) + j(b_1 + b_2)$$

and

$$f_1 f_2 = (a_1 + jb_1)(a_2 + jb_2) = a_1a_2 - (1+t^2)b_1b_2 + j(a_1b_2 + a_2b_1).$$

Our problem is to see whether the complex type space can be a candidate for a higher dimensional number system that allows analysis of function defined on the higher dimensional spaces.

Proposition 2.2. The space of complex type numbers $\mathbb{C}(t)$ forms a field under the defined operations. The elements of $\mathbb{C}(t)$ can be expressed in the complex type form $a + jb$, where
$j^2 = -(1+t^2)$.

Proof: By algebraic manipulations closed, associative, commutative and distributive properties of the operations are immediate. The zero element is $(0(t); 0(t))$, where $0(t) = 0$ and the unit element is $\big(1(t), 0(t)\big)$, where $1(t) = 1$. The inverse of a non-zero complex type number $f = (a; b)$ is $f^{-1} = \dfrac{(a;-b)}{a^2 + (1+t^2)b^2}$.

The second elementary basis element $j = (0(t); 1(t))$ satisfies a coupled relation
$j^2 = -(1+t^2)\big(1(t), 0(t)\big) = -(1+t^2)$.

It should be pointed out that the fundamental imaginary unit *j* is a function of the symbolic variable *t*. There is a main difference between the present complex type formulation and the previous other formulation that the fundamentalunit*e*satisfies the condition of the form$e^n = -1$for some power *n*.

Definition 2.3.Given$f = a + jb$, we define its complex type conjugate f^* by $f^* = a - jb$ and its modulus (absolute size)$|f|$ by $|f| = +\sqrt{a^2 + (1+t^2)b^2}$, where a and b, the real type and imaginary type parts of f respectively.

From the above definition, we see that the system of complex numbers has canonical extensions of many properties of their complex counterparts.

Proposition 2.4.For the complex type numbers$w, v \in \mathbb{C}(t)$we have

i. $|w^*| = |w|$, *ii.* $|wv| = |w||v|$, *iii.* $ww^* = |w|^2$ *iv.* $w^{-1} = \dfrac{w^*}{|w|^2}, w \neq 0$

.*v.* $wv = 0$ if and only if $w = 0$ *or* $v = 0$ *vi.* $|w| = 0$ if and only if$w = 0$.

Proof: By algebraic manipulation we can proof i. to v.

To proof vi., it is convenient to define a generating function $a_n(t)$ for a sequence $a = \{a_n\}_{n=0}^{\infty}$ as follows: For each n, $a_n(t) = a_n + a_{n+1}t + \cdots$, so that we have $a(t) = a_0(t)$ and $a_n(t) = a_n + ta_{n+1}(t), n = 0,1,2,\cdots$.

Now, for $w = a(t) + jb(t),$ with $|w| = 0$, we prove inductively that $a_n = b_n = 0$ for all $n = 0,1,2,\cdots,$

$$
\begin{aligned}
|w|^2 &= a_0(t)^2 + (1+t^2)b_0(t)^2 = (a_0 + ta_1(t))^2 + (1+t^2)(b_0 + tb_1(t))^2 \\
&= [a_0^2 + 2t\,a_0a_1(t) + t^2a_1(t)^2] + (1+t^2)[b_0^2 + 2t\,b_0b_1(t) + t^2b_1(t)^2] \\
&= (a_0^2 + b_0^2) + 2t[a_0a_1(t) + b_0b_1(t)] + t^2[a_1(t)^2 + b_1(t)^2] + t^2[b_0^2 + 2t\,b_0b_1(t) + t^2b_1(t)^2] \\
&= (a_0^2 + b_0^2) + 2t[a_0a_1(t) + b_0b_1(t)] + t^2[a_1(t)^2 + b_1(t)^2] + t^2(b_0 + b_1(t))^2 \\
&= a_0^2 + b_0^2 + 2t(a_0a_1(t) + b_0b_1(t)) + t^2(a_1(t)^2 + b_1(t)^2 + b_0(t)^2) \\
&= 0.
\end{aligned}
$$

Equating the constant term to zero, we get $a_0^2 + b_0^2 = 0$. Hence $a_0 = b_0 = 0$.

Now consider

$$
\begin{aligned}
w &= a(t) + jb(t) \\
&= a_0(t) + jb_0(t) \\
&= (a_0 + ta_1(t)) + j(b_0 + tb_1(t)) \\
&= [a_0 + t(a_1 + ta_2(t))] + j[b_0 + t(b_1 + tb_2(t))] \\
&= [a_0 + ta_1 + t^2a_2(t)] + j[b_0 + tb_1 + t^2b_2(t)] \\
&= [a_0 + ta_1 + t^2a_2 + \ldots + t^{n-1}a_{n-1} + t^na_n(t)] + j[b_0 + tb_1 + t^2b_2 + \ldots + t^{n-1}b_{n-1} + t^nb_n(t)]
\end{aligned}
$$

Suppose inductively, that $a_i = b_i = 0$ for $i = 0,1, \ldots, n-1$. We then get $w = t^na_n(t) + jt^nb_n(t)$.

Then, $|w| = 0$ gives

$$
\begin{aligned}
&|w| = (t^na_n(t))^2 + (1+t^2)(t^nb_n(t))^2 = 0 \\
&\Rightarrow t^{2n}[(a_n + ta_{n+1}(t))^2 + (1+t^2)(b_n + tb_{n+1}(t))^2] = 0 \\
&\Rightarrow [a_n^2 + 2ta_na_{n+1}(t) + t^2a_{n+1}(t)^2] + (1+t^2)[b_n^2 + 2tb_nb_{n+1}(t) + t^2b_{n+1}(t)^2] = 0 \\
&\Rightarrow a_n^2 + b_n^2 + 2t(a_na_{n+1}(t) + b_nb_{n+1}(t)) + t^2[(a_{n+1}(t)^2 + b_{n+1}(t)^2) + \\
&\qquad\qquad (b_n^2 + 2tb_nb_{n+1}(t) + t^2b_{n+1}(t)^2)] = 0 \\
&\Rightarrow a_n^2 + b_n^2 + 2t(a_na_{n+1}(t) + b_nb_{n+1}(t)) + t^2[(a_{n+1}(t)^2 + b_{n+1}(t)^2) + (b_n + tb_{n+1}(t))^2] = 0 \\
&\Rightarrow a_n^2 + b_n^2 + 2t(a_na_{n+1}(t) + b_nb_{n+1}(t)) + t^2(a_{n+1}(t)^2 + b_{n+1}(t)^2 + b_n(t)^2) = 0.
\end{aligned}
$$

Thus, we have $a_n^2 + b_n^2 = 0$ and hence $a_n = b_n = 0.$ This completes the proof.

To prove v., we see from ii. that

$$wv = 0 \Rightarrow |wv| = |w||v| = 0 \Rightarrow |w| = 0 \; or \; |v| = 0 \Rightarrow w = 0 \; or \; v = 0.$$

We see that our new complex type space $\mathbb{C}(t)$ is an extension of the complex space $\mathbb{C}$ in the sence that $\mathbb{C}$ is included in $\mathbb{C}(t)$. Specifically, the subspace $\mathbb{C}(0)$ of $\mathbb{C}(t)$ is the complex space consisting the constant coefficients of the real and imaginary type parts of complex type numbers.

III. Exponential And Trigonometric Representations

In our complex type numbers formulation, we can formulate polar form through the functional approach. The following results show that the trigonometric and exponential functions representations and their properties in the classical complex space have canonical extensions in the complex type space.

For this purpose, we consider defining a complex type valued function of complex type variable as follows:

Definition3.1. Let $\mathrm{F}(.)$ be an infinitely differentiable real valued function of one variable and $\mathrm{w} \equiv \mathrm{w}(\mathrm{t}) = \mathrm{a}(\mathrm{t}) + \mathrm{jb}(\mathrm{t})$ be a complex type variable in $\mathbb{C}(\mathrm{t})$. The function $\mathrm{F}(\mathrm{w})$ is defined as

$$\mathrm{F}(\mathrm{w}) = \sum_{n=0}^{\infty} \frac{\mathrm{F}^{(n)}(\alpha)}{n!}(\mathrm{w}-\alpha)^{n},$$

where α is a real number in the domain of F.
Since $\mathbb{C}(t)$ is a field, one can easily see that $F(w)$ is a complex type valued function.
Now we will prove the following results concerning the exponential and logarithmic functions of a complex type variables.
Lemma 3.2.The exponential function$exp(w)$ satisfy the relation$e^{u+v} = e^{u}e^{v}$.
Proof. By Definition 3.1, the exponential function is given by the series form

$$e^{w} = \sum_{n=0}^{\infty} \frac{w^{n}}{n!}.$$

Now, for the complex type variable u, v, we consider

$$e^{u+v} = \sum_{n=0}^{\infty} \frac{(u+v)^{n}}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a+b=n}^{n} n! \frac{u^{a}}{a!}\frac{u^{b}}{b!} = \sum_{a=0}^{\infty} \frac{u^{a}}{a!} \sum_{b=0}^{\infty} \frac{v^{b}}{b!} = e^{u}e^{v}.$$

Theorem 3.3.The complex type numbers$w \in \mathbb{C}(t)$ can be expressed in a polar type exponentialformulations $w = a + jb = \rho e^{j\theta}$, where$\theta \equiv \theta(t), \rho \equiv \rho(t)$are functions of the symbolic variable$t$with real valued coefficients.
Proof. We define $\ln w$ using Definition 3.1 with $\alpha = 1$. Hence, $\ln w$ can be expressed in a complex type form $\ln w = \lambda(t) + j\theta(t)$.

In the meantime, it is easy to see that $\exp(\ln w) = w$ by the same Definition above by the expansion of the real valued function $\exp(\ln x) = x$ about $\alpha = 1$ again.
This gives, $\exp(\ln w) = 1 + w - 1 + 0 + 0 + \cdots = w$.
Hence we have, by Lemma 3.2

$$w = e^{\lambda(t) + j\theta(t)} = e^{\lambda(t)}e^{j\theta(t)} = \rho e^{j\theta}$$

Where $\theta \equiv \theta(t), \rho \equiv \rho(t)$.
Theorem 3.4:A complex type number$w = a + jb$can be expressed in a trigonometric type Euler formulation given by

$$w = \rho(\cos_t \theta + \sin_t \theta)$$

and the trigonometric type functions are defined as

$$\cos_t \theta := \cos(\sqrt{1+t^2}\theta) = \frac{a}{\rho}, \tag{1}$$

$$\sin_t \theta := \frac{\sin(\sqrt{1+t^2}\theta)}{\sqrt{1+t^2}} = \frac{b}{\rho}.$$

and (2)
Further, the trigonometric type functions satisfy the De Moivre’s formula:
For an integer *n*,

$$(\cos_t \theta + j\sin_t \theta)^n = \cos_t n\theta + j\sin_t n\theta.$$

Proof. In view of Theorem 3.3, we have

$$w = \rho e^{j\theta} = \rho e^{i\sqrt{1+t^2}\theta} = \rho\left(\cos\left(\sqrt{1+t^2}\theta\right) + i\sin\left(\sqrt{1+t^2}\theta\right)\right)$$
$$= \rho(\cos_t \theta + j\sin_t \theta)$$

where the trigonometric type functions are defined.
By proposition 2.4, $\left|e^{j\theta}\right| = \left|e^{i\sqrt{1+t^2}\theta}\right| = 1$, and hence$\rho = |w|$. Here, the relations between the exponential and trigonometric type functions are establish regarding them as infinite series.
The De Moivre’s type formula is then an immediate consequence by induction.
It is not recognized whether the real coefficients of $\theta(t)$ have any geometric or trigonometric meaning because we have identified $\theta(t)$ from the concepts of infinite series representations of the functions involved. It is, however, observed that the first coefficient $\theta(0)$ is the classical angular argument of the complex number $f(0)$.

We derive some identities for the trigonometric functions that are similar to the corresponding identities to their trigonometric counterparts.

Theorem 3.5.The trigonometric type functions satisfy the following identities:

1. $\cos_t^2\theta + (1+t^2)\sin_t^2\theta = 1,$
2. $\cos_t(\theta_1 \pm \theta_2) = \cos_t\theta_1\cos_t\theta_2 \mp (1+t^2)\sin_t\theta_1\sin_t\theta_2,$
3. $\sin_t(\theta_1 \pm \theta_2) = \sin_t\theta_1\cos_t\theta_2 \pm \cos_t\theta_1\sin_t\theta_2,$
4. $\cos_t 2\theta = \cos_t^2\theta - (1+t^2)\sin_t^2\theta = 1 - 2(1+t^2)\sin_t^2\theta = 2\cos_t^2\theta - 1,$
5. $\sin_t 2\theta = 2\sin_t\theta\cos_t\theta$.

Proof: It can be proved by direct manipulation of their classical version.

IV. Conclusion

A complex type space for a real sequence is proposed. The complex type space is the canonical extension of the classical complex space in the sense that the fundamental trigonometric, algebraic and exponential properties are carried to the proposed space with some extensions.

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Authors

First Author – ThayamathyPio Jude Navinthan, Department of Mathematics, Eastern University, Sri Lanka.