Abstract. Motivated by some results on derivations on rings, and the generalizations of BCK and BCI algebras, in this paper, we define $f$-derivations on BP-algebras and investigate some important results.

Key Words: BP-algebras, derivations on BP-algebras, $f$-derivations on BP-algebras.

Subject Classification: AMS(2000) 06F35, 03G25, 06D99, 03B47

1. Introduction

BCK and BCI algebras are two new classes of algebras based on propositional calculi or logic introduced by Imai and Isaki[5]. In [6] K.Isaki and K.Tanaka introduced the theory of BCK-algebras. In [3,4] Q.P.Hu and X.Li and introduced a wider class of abstract algebras BCH-algebras. The class of BCI-algebras is a proper subclass of the class BCH-algebras. J.Neggers and H.S.Kim [9] introduced the notion of d-algebras which is another generalization of BCK-algebras.

In S.S.Ahn and J.S.Han [1] introduce the notion of a BP-algebras. In 2004 Y.B.Jun and X.L..Xin [7] introduced the notion of derivations of BCI-algebras, which was motivated from a lot of workdone on derivations of rings. Since then many authors worked on the notion of derivations on several algebras such as d-algebras and TM-algebras [2,8] motivated by this paper introduce the notion of f-derivations on BP-algebras.

2. Preliminaries

In this section we recall some basic definitions that are required in our work.

Definition 2.1. [6] Let $X$ be a set with a binary operation $*$ and a constant $0$. Then $(X, *, 0)$ is called a BCK-algebras if it satisfies the following axioms:

1. $x * x = 0$
2. $0 * x = 0$
3. $((x * y) * (x * z)) * (z * y) = 0$
4. $(x * (x * y)) * y = 0$
5. $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y, z \in X$

Definition 2.2. [7] Let $X$ be a set with a binary operation $*$ and a constant $0$. Then $(X, *, 0)$ is called a BCI-algebra if it satisfies the following axioms:
(1) \((x \star y) \star (x \star z)) \star (z \star y) = 0\)
(2) \((x \star (x \star y)) \star y = 0\)
(3) \(x \star x = 0\)
(4) \(x \star y = 0 \text{ and } y \star x = 0 \Rightarrow x = y \forall x, y, z \in X\)

**Definition 2.3.** Let \(x\) be a BCI-algebra. Two elements \(x\) and \(y\) in \(X\) are said to be comparable if \(x \leq y\) or \(y \leq x\). Here \(x \leq y\) if and only if \(x \star y = 0\). Also we define \(y \star (y \star x)\) by \(x \wedge y\).

**Definition 2.4.** \([9]\) A \(d\)-algebra is a non-empty set \(X\) with a constant 0 and binary operation \(\ast\) satisfying the following axioms:

(1) \(x \ast x = 0\)
(2) \(0 \ast x = 0\)
(3) \(x \ast y = 0 \text{ and } y \ast x = 0 \Rightarrow x = y\). \(\forall x, y, z \in X\)

**Definition 2.5.** \([1]\) Let \(X\) be a set with a binary operation \(\ast\) and a constant 0. Then \((X, \ast, 0)\) is called a BP-algebra if it satisfies the following axioms.

(1) \(x \ast x = 0\)
(2) \(x \ast (x \ast y) = y\)
(3) \((x \ast z) \ast (y \ast z) = x \ast y\) for any \(x, y, z \in X\).

**Definition 2.6.** \([9]\) Let \(X\) be a \(d\)-algebra and \(I\) be a subset of \(X\), then \(I\) is called \(d\)-ideal of \(X\) if it satisfies the following conditions.

(1) \(0 \in I\)
(2) \(x \ast y \in I \text{ and } y \in I \Rightarrow x \in I\)
(3) \(x \in I \text{ and } y \in X \Rightarrow x \ast y \in I\) (ie) \(I \ast X \subseteq I\)

**Example 2.7.** Let \(X = \{0, 1, 2, 3\}\), \((X, \ast, 0)\) be a set with the following cayley table

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Then \((X, \ast, 0)\) is a BP-algebra.

**Definition 2.8.** Let \(X\) be a \(d\)-algebra. A map \(\theta : X \rightarrow X\) is a left - right derivation (briefly, \((l, r)\)-derivation) on \(X\), if it satisfies the identity 
\[\theta(x \ast y) = (\theta(x) \ast y) \wedge (x \ast \theta(y))\] for all \(x, y \in X\).

If \(\theta\) satisfies the identity 
\[\theta(x \ast y) = (x \ast \theta(y)) \wedge (\theta(x) \ast y)\] for all \(x, y \in X\), then \(\theta\) is called a right-left derivation (briefly, \((r, l)\)-derivation) on \(X\).

If \(\theta\) is both an \((l, r)\) and an \((r, l)\)-derivation, then \(\theta\) is called a derivation on \(X\).

3. \(f\)-DERIVATIONS ON BP-ALGEBRA

In this section, we define the notion of \(f\)-derivations and regular of \(f\)-derivations on BP-algebras and prove some results. Throughout this section we assume that \(f\) is an endomorphism of the BP-algebra \((X, \ast, 0)\).
Definition 3.1. Let $X$ be a BP-algebra. By a left - right $f$ - derivation (briefly, $(l,r)$ - $f$ - derivation ) on $X$, we mean a self map $\theta_f$ of $X$ satisfies the identity 
\[ \theta_f(x \ast y) = (\theta_f(x) \ast f(y)) \land (f(x) \ast \theta_f(y)) \] 
for all $x, y \in X$.
If $\theta_f$ satisfies the identity 
\[ \theta_f(x \ast y) = (f(x) \ast \theta_f(y)) \land (\theta_f(x) \ast f(y)) \] 
then it is said that $\theta_f$ is a right - left $f$ - derivation (briefly, $(r,l)$ - $f$ - derivation) of $X$. If $\theta_f$ is both an $(r,l)$ - and an $(l,r)$ - $f$ - derivation, then $\theta_f$ is said to be a $f$ - derivation.

Example 3.2. Let $X = \{0, 1, 2, 3\}$ be a BP-algebra with the following cayley table

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(1) Define an endomorphism $f$ of $X$ by $f(0)=0, f(1)=3, f(2)=2,$ and $f(3)=1.$
and a self map $\theta_f : X \rightarrow X$ by $\theta_f(0)=1, \theta_f(1)=0, \theta_f(2)=3,$ and $\theta_f(3)=2.$
Then it is easily checked that $\theta_f$ is a $(l,r)$ - $f$ - derivation of $X$.
(2) Define an endomorphism $f$ of $X$ by $f(0)=0, f(1)=3, f(2)=2,$ and $f(3)=1.$
and a self map $\theta_f : X \rightarrow X$ by $\theta_f(0)=2, \theta_f(1)=1,$ and $\theta_f(3)=3.$
Then it is easily checked that $\theta_f$ is a $(r,l)$ - $f$ - derivation of $X$.

Definition 3.3.
An $f$ - derivation $\theta_f$ on a BP-algebra $X$ is said to be regular if
$\theta_f(0) = 0$.

Proposition 3.4.
Every $(r,l)$ - $f$ - derivation $(l,r)$ - $f$ - derivation) of a BP-algebra is regular.

Proof:
Let $X$ be a BP-algebra and $\theta_f$ be a $(r,l)$ - $f$ - derivation on $X$. Then for all $x \in X,$
we have
\[ \theta_f(0) = \theta_f(x \ast x) \]
\[ = (f(x) \ast \theta_f(x)) \land (\theta_f(x) \ast f(x)) \]
\[ = (f(x) \ast \theta_f(x)) \ast ((\theta_f(x) \ast f(x)) \ast (f(x) \ast \theta_f(x))) \]
\[ = f(x) \ast \theta_f(x) \]
\[ \because x \ast x = 0 \]
\[ = 0. \]

Let $\theta_f$ be a $(l,r)$ - $f$ - derivation on $X$.
Then for all $x \in X$, we have
\[ \theta_f(0) = \theta_f(x \ast x) \]
\[ = (\theta_f(x) \ast f(x)) \land (f(x) \ast \theta_f(x)) \]
\[ = (f(x) \ast \theta_f(x)) \ast ((f(x) \ast \theta_f(x)) \ast (\theta_f(x) \ast f(x))) \]
\[ = \theta_f(x) \ast f(x) \]
\[ = 0. \]
One can easily prove that the following result gives a necessary and sufficient condition for the derivation $\theta_f$ to be regular.

**Proposition 3.5.**

Let $\theta_f$ be a self map of a BP-algebra on $X$, then the following hold:

1. If $\theta_f$ is an $(l,r)-f$-derivation on $X$, then $\theta_f(x) = \theta_f(x) \wedge f(x)$ for all $x \in X$ if and only if $\theta_f(0) = 0$.

2. If $\theta_f$ is an $(r,l)-f$-derivation on $X$, then $\theta_f(x) = f(x) \wedge \theta_f(x)$ for all $x \in X$ if and only if $\theta_f(0) = 0$.

**Proposition 3.6.**

Let $\theta_f$ be a $(l,r)-f$-derivation on a BP-algebra $X$. Then $\theta_f(x) = \theta_f(0) \ast (0 \ast f(a))$, for all $a \in X$.

**Proof:** Let $\theta_f$ be an $(l,r)-f$-derivation on a BP-algebra $X$.

Now,

$$\theta_f(a) = \theta_f(0 \ast (0 \ast a)) \; (\because 0 \ast (0 \ast x) = x)$$

$$= (\theta_f(0) \ast f(0 \ast a)) \wedge (f(0) \ast \theta_f(0 \ast a))$$

$$= (f(0) \ast \theta_f(0 \ast a)) \ast ((f(0) \ast \theta_f(0 \ast a)) \ast (\theta_f(0) \ast f(0 \ast a)))$$

$$= \theta_f(0) \ast f(0 \ast a)$$

$$= \theta_f(0) \ast (f(0) \ast f(a))$$

$$= \theta_f(0) \ast (0 \ast f(a))$$

$$\therefore \theta_f(a) = \theta_f(0) \ast (0 \ast f(a)).$$

**Proposition 3.7.**

Let $\theta_f$ be a self map on a BP-algebra $X$ and $\theta_f$ be an $(r,l)-f$-derivation on $X$. Then $\theta_f(x) = f(x)$, for all $x \in X$ if and only if $\theta_f(0) = 0$.

**Proof:** Let $\theta_f$ be an $(r,l)-f$-derivation on $X$.

Assume that $\theta_f(0) = 0$.

Now,

$$\theta_f(x) = \theta_f(x \ast 0) \; (\because x \ast 0 = x)$$

$$= (f(x) \ast \theta_f(0)) \wedge (\theta_f(x) \ast f(0))$$

$$= (\theta_f(x) \ast f(0)) \ast ((\theta_f(x) \ast f(0)) \ast (f(x) \ast \theta_f(0)))$$

$$= f(x) \ast \theta_f(0)$$

$$= f(x).$$
Coversely, assume that \( \theta_f(x) = f(x) \).

Now,
\[
\theta_f(0) = \theta_f(x \ast x) \\
= (f(x) \ast \theta_f(x)) \land (\theta_f(x) \ast f(x)) \\
= (\theta_f(x) \ast f(x)) \ast ((\theta_f(x) \ast f(x)) \ast (f(x) \ast \theta_f(x))) \\
= f(x) \ast \theta_f(x) \\
= f(x) \ast f(x) \quad (\therefore \theta_f(x) = f(x)) \\
= 0.
\]

\( \therefore \theta_f(0) = 0. \)

**Definition 3.8.**

An ideal \( A \) on a BP-algebra \( X \) is said to be an \( f \)-ideal if \( f(A) \subseteq A \).

**Example 3.9.** Let \( X = \{0, 1, 2, 3\} \) be a BP-algebra with the following cayley table.

Consider the ideal \( A = \{0, 3\} \) of \( X \).

\[
\begin{array}{cccc}
* & 0 & 1 & 2 & 3 \\
0 & 0 & 2 & 1 & 3 \\
1 & 1 & 0 & 3 & 2 \\
2 & 2 & 3 & 1 & 0 \\
3 & 3 & 1 & 2 & 0 \\
\end{array}
\]

If \( \theta_f : X \to X \) is defined by \( \theta_f(0) = 0, \ \theta_f(1) = 2, \ \theta_f(2) = 1, \ \theta_f(3) = 3 \) and define an endomorphism \( f \) of \( X \) by \( \theta_f(x) = f(x) \).

Since \( f(0) = 0, f(3) = 3, f(A) \subseteq A \) proving that \( A \) is an \( f \)-ideal on \( X \).

**Definition 3.10.**

Let \( \theta_f \) be a self map of a BP-algebra \( X \). An \( f \)-ideal on \( X \) is said to be \( \theta_f \)-invariant if \( \theta_f(A) \subseteq A \).

**Example 3.11.**

Example(3.9), \( \theta_f(0) \in A \) and \( \theta_f(3) = 3 \in A \).

Hence \( \theta_f(A) \subseteq A \), showing that \( A \) is \( \theta_f \)-invariant.

**Theorem 3.12.**

Let \( \theta_f \) be a regular \((r,l) - f \)-derivation on a BP-algebra \( X \). Then \( f \)-ideal \( A \) on \( X \) is \( \theta_f \) invariant.

**proof:**

Let \( \theta_f \) be a regular \((r,l) - f \)-derivation on \( X \).

Now,
\[
\theta_f(x) = \theta_f(x \ast 0) \\
= (f(x) \ast \theta_f(0)) \land (\theta_f(x) \ast f(0)) \\
= (f(x) \ast 0) \land (\theta(x) \ast 0) \\
= f(x) \land \theta_f(x) \\
= \theta_f(x) \ast (\theta_f(x) \ast f(x)) \\
= f(x), \forall x \in X.
\]
Let \( y \in \theta_f(A) \) then \( y = \theta_f(x) \) for some \( x \in A \).

It follows that \( y \ast f(x) = \theta_f(x) \ast f(x) = 0 \in A \).

Since \( x \in A \), then \( f(x) \in f(A) \subseteq A \) as \( A \) is an \( f \)-ideal.

It follows that \( y \in A \) since \( A \) is an ideal on \( X \).

Hence \( \theta_f(A) \subseteq A \).

Thus \( A \) is \( \theta_f \)-invariant.

4. Composition of \( f \)-derivation

**Definition 4.1.**

Let \( X \) be a BP-algebra and \( \theta_f, \theta'_f \) be two self maps on \( X \). We define

\[ \theta_f \circ \theta'_f : X \to X \]

as

\[ (\theta_f \circ \theta'_f)(x) = \theta_f(\theta'_f(x)) \text{ for all } x \in X. \]

**Proposition 4.2.**

Let \( X \) be a BP-algebra and \( \theta_f, \theta'_f \) are the \((l,r)\)-\( f \)-derivations on \( X \).

Let \( f^2 = f \circ f = f \), then \( \theta_f \circ \theta'_f \) is also a \((l,r)\)-\( f \)-derivation on \( X \).

Proof:

Let \( X \) be a BP-algebra, and \( \theta_f \) and \( \theta'_f \) are the \((l,r)\)-\( f \)-derivations on \( X \).

\[
(\theta_f \circ \theta'_f)(x \ast y) = \theta_f(\theta'_f(x \ast y))
\]

\[
= \theta_f[(\theta'_f(x) \ast f(y)) \land (f(x) \ast \theta'_f(y))]
\]

\[
= \theta_f[(f(x) \ast \theta'_f(y)) \ast ((f(x) \ast \theta'_f(y)) \ast \theta'_f(x) \ast f(y))]
\]

\[
= \theta_f(\theta'_f(x) \ast f(y)) \quad (\because y \ast (y \ast x) = x)
\]

\[
= (\theta_f(\theta'_f(x)) \ast f^2(y)) \land (f(\theta'_f(x)) \ast \theta_f(f(y)))
\]

\[
= \theta_f(\theta'_f(x)) \ast f^2(y)
\]

\[
(\theta_f \circ \theta'_f)(x \ast y) = (\theta_f(\theta'_f(x)) \ast f(y))
\]

\[
= (f(x) \ast \theta_f(\theta'_f(y))) \ast ((f(x) \ast \theta_f(\theta'_f(y))) \ast \theta_f(\theta'_f(x) \ast f(y))]
\]

\[
= (f(x) \ast (\theta_f \circ \theta'_f)(y) \ast
\]

\[
[([f(x) \ast (\theta_f \circ \theta'_f)(y)] \ast (\theta_f \circ \theta'_f)(x) \ast f(y)]
\]

\[
= (\theta_f \circ \theta'_f)(x) \ast f(y) \land (f(x) \ast (\theta_f \circ \theta'_f)(y)).
\]

Which implies that \( (\theta_f \circ \theta'_f) \) is a \((l,r)\)-\( f \)-derivation on \( X \).

One can easily prove that the following proposition.

**Proposition 4.3.**

Let \( X \) be a BP-algebra, \( \theta_f \) and \( \theta'_f \) are the \((r,l)\)-\( f \)-derivations on \( X \) such that \( f^2 = f \circ f = f \). Then \( \theta_f \circ \theta'_f \) is also a \((r,l)\)-\( f \)-derivation on \( X \).
Thus we have for all $x,y$. But $\theta f$ is also a $f$-derivation on $X$.

One can easily prove that the following proposition that the composition of derivations is commutative.

**Proposition 4.5.**

Let $X$ be a BP-algebra and $\theta_f, \theta'_f$ be two $f$-derivations on $X$ such that $f \circ \theta_f = \theta_f \circ f$, $\theta'_f \circ f = f \circ \theta'_f$. Then $\theta_f \circ \theta'_f = \theta'_f \circ \theta_f$.

**Proof:**

Let $X$ be a BP-algebra and $\theta_f, \theta'_f$ be the $f$-derivations on $X$.

Since $\theta'_f$ is a $(l,r)$ - $f$-derivation on $X$, then for all $x, y, \in X$.

\[
(\theta_f \circ \theta'_f)(x \ast y) = \theta_f(\theta'_f(x \ast y)) = \theta_f((\theta'_f(x) \ast f(y)) \land (f(x) \ast \theta'_f(y))) = \theta_f(\theta'_f(x) \ast f(y))
\]

But $\theta_f$ is a $(r,l)$ - $f$-derivation on $X$.

\[
(\theta_f \circ \theta'_f)(x \ast y) = \theta_f((\theta'_f(x) \ast f(y)) = (f(\theta'_f(x)) \ast \theta_f(f(y))) \land (\theta_f(\theta'_f(x)) \ast f^2(y)) = (f(\theta'_f(x)) \ast \theta_f(f(y))) = (f \circ \theta'_f)(x) \ast (\theta_f \circ f)(y)
\]

Thus we have for all $x, y \in X$, $(\theta_f \circ \theta'_f)(x \ast y) = (f \circ \theta'_f)(x) \ast (\theta_f \circ f)(y)$.

Also since $\theta_f$ is a $(r,l)$ - $f$-derivation on $X$ then for all $x, y \in X$.

\[
(\theta'_f \circ \theta_f)(x \ast y) = \theta'_f(\theta_f(x \ast y)) = \theta'_f((f(x) \ast \theta_f(y)) \land (\theta_f(x) \ast f(y))) = \theta'_f(\theta_f(x) \ast f(y))
\]

But $\theta'_f$ is a $(l,r)$ - $f$-derivation on $X$.

\[
(\theta_f \circ \theta'_f)(x \ast y) = (\theta'_f(f(x)) \ast f(\theta_f(y))) \land (f^2(x) \ast \theta'_f(\theta_f(y))) = (\theta'_f(f(x)) \ast f(\theta_f(y))) = (\theta'_f \circ f)(x) \ast (\theta_f \circ f)(y) = (f \circ \theta'_f)(x) \ast (\theta_f \circ f)(y)
\]

Thus we have for all $x, y \in X$, $(\theta'_f \circ \theta_f)(x \ast y) = (f \circ \theta'_f)(x) \ast (\theta_f \circ f)(y)$.

From (1) and (2) we get for all $x, y \in X$, $(\theta_f \circ \theta'_f)(x \ast y) = (\theta'_f \circ \theta_f)(x \ast y)$.

By putting $y = 0$ we get for all $x \in X$,

\[
(\theta_f \circ \theta'_f)(x) = (\theta'_f \circ \theta_f)(x)
\]

which implies that $(\theta_f \circ \theta'_f) = (\theta'_f \circ \theta_f)$.
Definition 4.6. \( X \) be a BP-algebra and \( \theta_f, \theta'_f \) be two self maps on \( X \). We define \( \theta_f \cdot \theta'_f : X \rightarrow X \) as 
\[
(\theta_f \cdot \theta'_f)x = \theta_f(x) \cdot \theta_f(x) \quad \text{for all} \quad x \in X.
\]

Proposition 4.7. \( X \) be a BP-algebra and \( \theta_f, \theta'_f \) are \( f \)-derivations on \( X \). Then 
\[
(f \circ \theta'_f) \cdot (\theta_f \circ f) = (\theta_f \circ f) \cdot (f \circ \theta'_f)
\]

Proof: \( X \) be a BP-algebra and \( \theta_f, \theta'_f \) be two derivations on \( X \). Since \( \theta'_f \) is a \((l,r)\) - \( f \)-derivation on \( X \). Then for all \( x,y \in X \).
\[
(\theta_f \circ \theta'_f)(x \cdot y) = \theta_f(\theta'_f(x \cdot y))
\]
\[
= \theta_f(\theta'_f(x) \cdot f(y)) \land (f(x) \cdot \theta'_f(y)))
\]
\[
= \theta_f(\theta'_f(x) \cdot f(y))
\]

But \( \theta_f \) is a \((r,l)\) - \( f \)-derivation on \( X \).
\[
\theta_f(\theta'_f(x) \cdot f(y)) = (f(\theta'_f(x)) \cdot \theta_f(f(y))) \land (\theta_f(\theta'_f(x)) \cdot f^2(y))
\]
\[
= (f(\theta'_f(x)) \cdot \theta_f(f(y)))
\]
\[
= (f \circ \theta'_f)(x) \cdot (f \circ f)(y)
\]
\[
(\theta_f \circ \theta'_f)(x \cdot y) = (f \circ \theta'_f)(x) \cdot (f \circ f)(y) \quad \text{for all} \quad x,y \in X \cdot \cdot \cdot (1)
\]
Also we have that \( \theta'_f \) is a \((r,l)\) - \( f \)-derivation on \( X \), then for all \( x,y \in X \).
\[
(\theta_f \circ \theta'_f)(x \cdot y) = \theta_f(\theta'_f(x \cdot y))
\]
\[
= \theta_f[(f(x) \cdot \theta'_f(y)) \land (\theta'_f(x) \cdot f(y))]
\]
\[
= \theta_f(f(x) \cdot \theta'_f(y))
\]

But \( \theta_f \) is a \((l,r)\) - \( f \)-derivation on \( X \).
\[
\theta_f(f(x) \cdot \theta'_f(y)) = (\theta_f(f(x)) \cdot f(\theta'_f(y))) \land (f^2(x) \cdot \theta_f(\theta'_f(y)))
\]
\[
= (\theta_f(f(x)) \cdot f(\theta'_f(y)))
\]
\[
(\theta_f \circ \theta'_f)(x \cdot y) = (\theta_f \circ f)(x) \cdot (f \circ \theta'_f)(y), \forall x,y \in X \cdot \cdot \cdot (2).
\]
From (1) and (2) we get for all \( x \in X \) (By putting \( y = x \))
\[
(f \circ \theta'_f)(x) \cdot (f \circ f)(x) = (\theta_f \circ f)(x) \cdot (f \circ \theta'_f)(x)
\]
\[
(f \circ \theta'_f) \cdot (f \circ f)(x) = (\theta_f \circ f) \cdot (f \circ \theta'_f)(x)
\]

which implies that \( (f \circ \theta'_f) \cdot (f \circ f) = (\theta_f \circ f) \cdot (f \circ \theta'_f) \)

Notation: \( \text{Der}_f (X) \) denotes the set of all \( f \)-derivations on \( X \).
Definition 4.8.
Let $\theta_f, \theta'_f \in \text{Der}_f(X)$. Define the binary operation $\wedge$ as
$$(\theta_f \wedge \theta'_f)(x) = \theta_f(x) \wedge \theta'_f(x).$$

Proposition 4.9.
Let $X$ be a BP-algebra and $\theta_f, \theta'_f$ are $(l,r)$ - $f$ - derivations on $X$. Then $\theta_f \wedge \theta'_f$ is also a $(l,r)$ - $f$ - derivation on $X$.

Proof:
Let $X$ be a BP-algebra and $\theta_f, \theta'_f$ are $(l,r)$ - $f$ - derivations on $X$. We have

$$(\theta_f \wedge \theta'_f)(x \ast y) = \theta_f(x \ast y) \wedge \theta'_f(x \ast y)$$

$$= \{(\theta_f(x) \ast f(y)) \wedge (f(x) \ast \theta_f(y))\} \wedge \{(\theta'_f(x) \ast f(y)) \wedge (f(x) \ast \theta'_f(y))\}$$

$$= (\theta_f(x) \ast f(y)) \wedge (\theta'_f(x) \ast f(y))$$

$$= (\theta_f(x) \ast \theta'_f(x)) \ast f(y)$$

$$= (\theta_f(x) \wedge \theta'_f(x)) \ast f(y)$$

$$= (f(x) \ast (\theta_f(x) \wedge \theta'_f(x))) \ast f(y)$$

$$= ((\theta_f \wedge \theta'_f)(x) \ast f(y)) \wedge (f(x) \ast (\theta_f \wedge \theta'_f)(y))$$

This shows that $(\theta_f \wedge \theta'_f)$ is a $(l,r)$ - $f$ - derivation on $X$. This completes the proof.

In the similar fashion, we can establish the following.

Proposition 4.10.
Let $X$ be a BP-algebra and $\theta_f, \theta'_f$ are $(r,l)$ - $f$ - derivations on $X$. Then $\theta_f \wedge \theta'_f$ is also a $(r,l)$ - $f$ - derivation on $X$.

Theorem 4.11.
If $\theta_f, \theta'_f \in \text{Der}_f(X)$, $\theta_f \wedge \theta'_f \in \text{Der}_f(X)$. Also $(\theta_f \wedge (\theta_f \wedge \theta'_f))(x \ast y) = ((\theta_f \wedge \theta'_f)(x) \wedge (\theta_f \wedge \theta'_f)(y))(x \ast y)$.

Proof:
If $\theta_f, \theta'_f \in \text{Der}_f(X)$, then $\theta_f$ is both a $(l,r)$ and a $(r,l)$ derivation. Similarly $\theta'_f$ is both a $(l,r)$ and a $(r,l)$ derivation. By proposition (4.9) and (4.10), we observe that $\theta_f \wedge \theta'_f$ is both a $(l,r)$ and a $(r,l)$ derivation. Hence $\theta_f \wedge \theta'_f \in \text{Der}_f(X)$. 

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To show the associativity, choose $\theta_f, \theta'_f, \theta''_f \in Der_f (X)$.

\[
((\theta_f \land \theta'_f) \land \theta''_f)(x \ast y) = (\theta_f \land \theta'_f)(x \ast y) \land (\theta''_f)(x \ast y) \\
= ((\theta''_f)(x \ast y)) \ast ((\theta''_f)(x \ast y)) \ast \\
((\theta_f \land \theta'_f)(x \ast y)) \\
= (\theta_f \land \theta'_f)(x \ast y) \\
= (\theta_f)(x \ast y) \land (\theta'_f)(x \ast y) \\
= [(\theta_f(x) \ast f(y)) \land (f(x) \ast \theta_f(y))] \land \\
[(\theta'_f(x) \ast f(y)) \land (f(x) \ast \theta'_f(y))] \\
= (\theta_f(x) \ast f(y)) \land (\theta'_f(x) \ast f(y)) \\
= (\theta_f(x) \ast f(y))
\]

Also,

\[
(\theta_f \land (\theta'_f \land \theta''_f))(x \ast y) = (\theta_f(x \ast y) \land (\theta'_f \land \theta''_f)(x \ast y) \\
= \theta_f(x \ast y) \land [(\theta'_f(x \ast y) \land \theta''_f(x \ast y)] \\
= \theta_f(x \ast y) \land (\theta'_f)(x \ast y) (x \ast (x \ast y) = y) \\
= [(\theta_f(x) \ast f(y)) \land (f(x) \ast \theta_f(y))] \land \\
[(\theta'_f(x) \ast f(y)) \land (f(x) \ast \theta'_f(y))] \\
= (\theta_f(x) \ast f(y)) \land (\theta'_f(x) \ast f(y)) \\
= (\theta_f(x) \ast f(y))
\]

This shows that,

\[
((\theta_f \land \theta'_f) \land \theta''_f)(x \ast y) = (\theta_f \land (\theta'_f \land \theta''_f))(x \ast y)
\]

which implies that \(((\theta_f \land \theta'_f) \land \theta''_f) = (\theta_f \land (\theta'_f \land \theta''_f)

From the above theorem, we conclude that $Der_f (X)$ is closed under the binary composition $\land$ defined in (4.8) which is also associative. Hence we have the following theorem.

**Theorem 4.12.**

$Der_f (X)$ is a semigroup under the binary composition $\land$.

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