Local stability properties of a delayed SIR model with relapse effect

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***Abstract*:** This paper investigates a time-delayed SIR model with a non-linear Beddington-DeAngelis type incidence rate function and relapse. Immunity of some diseases is temporary, that is, the recovered individuals may return to the infected class after a certain period of time. The local stability properties of the disease-free equilibrium and the endemic equilibrium were completely analyzed by utilizing the characteristic equations at particular equilibrium. It can be seen that local stability properties of the model is totally based on the basic reproduction number which depend on the latent time delay. It was obtained that the disease dies out when the basic reproduction number less than unity, and the disease remains endemic when greater than one.

***Index Terms*-** Characteristic equation; Equilibrium; Epidemic model; Local stability.

I. INTRODUCTION

Recently, considerable attention on the dynamics analysis of various models such as SIR, SEI, SEIR, SIRS and epidemiological disease transmission models transmitted by vectors have been paid by the researchers [9,11,16,18,17]. Most of the models in literature have been considered without taking into account the exposed class which has individuals who are infected but not infective (see, for example, [6]). The authors of the research articles [3,4] have explicitly included exposed class into their models. Further, large numbers of models found from literature have considered only the dynamics of disease using ordinary differential equations, that is, without time delay. However, dynamical behaviors of models that of utilizing past information of the model reflects more realistic dynamical behaviors. Hence, it is more reasonable to incorporate time delays into the models though it has complicated mathematical analysis. In epidemiological models, the bilinear incidence rate, that is, βSI and the standard incidence rate, that is, $\beta SI / N$ are frequently used, which are based on the mass action law. Where $\beta > 0$ is the contact rate, N is the total human population, S is the number of susceptible individuals who are subjected to the disease and I is the number of infective individuals who can transmit the disease. To place the epidemic models in biologically more sensible ground Capasso and Serio [1] introduced the saturation incidence rate instead of standard and bilinear incidence rate. Furthermore, some authors (see, for example, [2,10,13,14] and the references therein) used Beddington-DeAngelis type incidence rate function. A more general incidence rate function of the form $f(S,I)$ has been used by the authors in [5] in epidemiology and have obtained stability results.

Relapse effect of the diseases of humans such as tuberculosis, bovine and herpes are common in epidemiology, and its mathematical behavior have been studied by several authors van den Driessche[12] and Xu[15]. Motivated by the article of Xu[15] and Hattaf et. al [5], in this study we used Beddington-DeAngelis type incidence rate function in the following model with relapse effect and investigated the stability behavior, and the results obtained are presented in this article.

$$\begin{cases} \dot{S}(t) = \Lambda - \dfrac{\beta S(t) I(t)}{1 + aS(t) + bI(t)} - \mu_1 S(t), \\ \dot{E}(t) = \dfrac{\beta S(t) I(t)}{1 + aS(t) + bI(t)} - \dfrac{\beta e^{-\mu_2 \tau} S(t-\tau) I(t-\tau)}{1 + aS(t-\tau) + bI(t-\tau)} - \mu_2 E(t), \\ \dot{I}(t) = \dfrac{\beta e^{-\mu_2 \tau} S(t-\tau) I(t-\tau)}{1 + aS(t-\tau) + bI(t-\tau)} + \delta R(t) - (\mu_3 + \gamma + \alpha) I(t), \\ \dot{R}(t) = \gamma I(t) - (\mu_4 + \delta) R(t). \end{cases} \tag{1.1}$$

In model (1.1), the number of individuals at time t in the susceptible, exposed, infectious and recovered classes is denoted by $S(t)$, $E(t)$, $I(t)$ and $R(t)$, respectively. The death rates related to above classes are respectively denoted by positive real number μ_1, μ_2, μ_3 and μ_4. The removal rate and disease induced death rate are respectively denoted by the parameters γ and α which are

non-negative constants. We also assume that the influx of susceptible comes from a constant recruitment is given by Λ. Moreover, the non-negative parameter δ represents the rate at which an individual in the recovered class reverts to the infective class. The term $\beta e^{-\mu_2\tau}S(t-\tau)I(t-\tau)/(1+aS(t-\tau)+bI(t-\tau))$ represents the individuals surviving in the latent period τ and becoming infective at time t, where $\tau \geq 0$ represents the time delay describing the latent period of the disease. Further, $a, b \geq 0$.

As usual, the initial conditions for model (1.1) is chosen as below

$$S(\theta)=\phi_1(\theta), \quad E(\theta)=\phi_2(\theta), \quad I(\theta)=\phi_3(\theta) \quad R(\theta)=\phi_4(\theta), \quad (\tau \leq \theta \leq 0) \tag{1.2}$$

For the continuity of the initial conditions, it is assumed that

$$E(0)=\int_{-\tau}^{0} \beta e^{\mu_2\theta} \frac{\phi_1(\theta)\phi_2(\theta)}{1+a\phi_1(\theta)+b\phi_2(\theta)}\,d\theta,$$

where $\phi=(\phi_1(\theta),\phi_2(\theta),\phi_3(\theta),\phi_4(\theta))^T \in C,$ and C denotes the Banach space $C([-\tau,0],R_+^4)$ of continuous functions mapping the interval $[-\tau,0]$ into R_+^4. To place the initial conditions in biologically more sensible we use $\phi_i(0)>0$ for $i=1,2,3,4.$

We can consider the dynamics of the following model instead of model (1.1), as $E(t)$ is not appeared explicitly in the first, third and fourth equations.

$$\begin{cases} \dot{S}(t)=\Lambda-\dfrac{\beta S(t)I(t)}{1+aS(t)+bI(t)}-\mu_1 S(t), \\ \dot{I}(t)=\dfrac{\beta e^{-\mu_2\tau}S(t-\tau)I(t-\tau)}{1+aS(t-\tau)+bI(t-\tau)}+\delta R(t)-(\mu_3+\gamma+\alpha)I(t), \\ \dot{R}(t)=\gamma I(t)-(\mu_4+\delta)R(t). \end{cases} \tag{1.3}$$

Non-negativity and boundedness results of model (1.1) are shown in the following theorem.

Theorem 1. Under the initial condition (1.2), the solution $(S(t),\ E(t),\ I(t),\ R(t))^T$ of model (1.1) is existent, unique and non-negative and bounded on $[0,+\infty)$.

Proof. The existence and uniqueness of the solution $(S(t),\ E(t),\ I(t),\ R(t))^T$ of model (1.1) can be easily proved by using the well-known theorems in [7]. Let us show that the solution $(S(t),\ E(t),\ I(t),R(t))^T$ of model (1.1) is non-negativity and ultimately bounded.

First let us show the non-negativeness of solutions on $[0,T)$ where $0<T<\tau$. For $t\in[0,\tau]\cap(0,T)$, from the first equation of model (1.1), we have that

$$\begin{aligned} \dot{S}(t) &= \Lambda-\frac{\beta S(t)I(t)}{1+aS(t)+bI(t)}-\mu_1 S(t) \\ &\geq -\left[\frac{\beta I(t)}{1+aS(t)+bI(t)}+\mu_1\right]S(t). \end{aligned}$$

Hence, one has that

$$S(t) \geq \phi_1(0)e^{-\int_0^t\left(\frac{\beta I(s)}{1+aS(s)+bI(s)}+\mu_1\right)ds}.$$

For $t\in[0,\tau]\cap(0,T)$, from the third equation of model (1.1), one has that

$$\begin{aligned} \dot{I}(t) &= \frac{\beta e^{-\mu_2\tau}S(-\tau)I(-\tau)}{1+aS(-\tau)+bI(-\tau)}+1\delta\phi_4(0)-(\mu_3+\gamma+\alpha)I(t), \\ &\geq -(\mu_3+\gamma+\alpha)I(t). \end{aligned}$$

Hence, one has that

$$I(t) \geq \phi_3(0)e^{-(\mu_3+\gamma+\alpha)t}.$$

For $t \in [0,\tau] \cap (0,T)$, again from the fourth equation of model (1.1), one has that

$$\begin{aligned}\dot{R}(t) &= \gamma I(t) - (\mu_4+\delta)R(t),\\ &\geq -(\mu_4+\delta)R(t).\end{aligned}$$

Hence, one has that

$$R(t) \geq \phi_4(0)e^{-(\mu_4+\delta)t}.$$

From the second equations

$$E(t) = \beta\int_{t-\tau}^{t} \frac{e^{-\mu_2(t-\theta)}S(\theta)I(\theta)}{1+aS(\theta)+bI(\theta)}\,d\theta.$$

Therefore , by step by step method, one can show that the solutions $(S(t), E(t), I(t), R(t))^T$ of model (1.1) are existence, unique and nonnegative in $[0,+\infty)$.

For $t \geq 0,$ define $N(t)$,

$$N(t) = S(t)+E(t)+I(t)+R(t).$$

Taking the derivation along the solution of model (1.1), for $t \geq 0,$ we have that

$$\begin{aligned}\dot{N}(t) &= \Lambda - \mu_1 S(t) - \mu_2 E(t) - (\mu_3+\alpha)I(t) - \mu_4 R(t),\\ &\leq \Lambda - MN(t).\end{aligned}$$

It follows that $\limsup_{t\to+\infty} \leq \frac{\Lambda}{M}$. □

II. EQUILIBRIA AND STABILITY

Model (1.3) has always a disease-free equilibrium $E_0(S_0 = \Lambda/\mu_1, 0)^T$. The characteristic equation of the linearization of model (1.3) near the disease-free equilibrium $E_0(S_0, 0)^T$ is

$$\begin{vmatrix} \lambda+\mu_1 & \dfrac{\beta S_0}{1+aS_0} & 0 \\ 0 & \lambda+(\mu_3+\gamma+\alpha)-\dfrac{\beta S_0 e^{-(\mu_2+\lambda)\tau}}{1+aS_0} & -\delta \\ 0 & -\gamma & \lambda+\mu_4+\delta \end{vmatrix} = 0. \tag{2.4}$$

Define the basic reproduction number

$$R_0 = \frac{\Lambda\beta e^{-\mu_2\tau}}{(a\Lambda+\mu_1)B},$$

where $B = \mu_3+\gamma+\alpha-\gamma\delta/(\mu_4+\delta)$. Hence, one can obtain the following theorem for local stability of the disease-free equilibrium E_0. Therefore, the eigenvalues satisfy $\lambda = -\mu_1 - \frac{1}{(1+aS_0)^2} = 0,$ which is always negative and others are given by

$$f(\lambda,\tau) = \lambda^2 + P_1(\tau)\lambda + P_0(\tau) + (Q_1(\tau)\lambda + Q_0(\tau))e^{-\lambda\tau} = 0. \tag{2.5}$$

where

$$\begin{aligned} &P_0(\tau) = (\mu_4+\delta)(\mu_3+\gamma+\alpha) - \delta\gamma, \quad P_1(\tau) = \mu_4+\delta+\mu_3+\gamma+\alpha,\\ &Q_0(\tau) = -\frac{\Lambda\beta(\mu_4+\delta)e^{-\mu_2\tau}}{a\Lambda+\mu_1}, \qquad Q_1(\tau) = -\frac{\Lambda\beta e^{-\mu_2\tau}}{a\Lambda+\mu_1}.\end{aligned}$$

Theorem 2. The disease-free equilibrium E_0 is locally asymptotically stable provided that $R_0 < 1,$ and unstable provided that $R_0 > 1.$

Proof. Let us first consider the case $\tau = 0.$ Hence, we have that

$$P_0(0) + Q_0(0) = (\mu_4 + \delta)(\mu_3 + \gamma + \alpha) - \delta\gamma - \frac{\Lambda\beta(\mu_4 + \delta)}{a\Lambda + \mu_1}$$
$$= (\mu_4 + \delta)B(1 - R_0), \tag{2.6}$$

$$P_1(0) + Q_1(0) = (\mu_4 + \delta) + (\mu_3 + \gamma + \alpha) - \frac{\Lambda\beta}{a\Lambda + \mu_1}$$
$$= (\mu_4 + \delta) + \frac{\delta\gamma}{(\mu_4 + \delta)} + B(1 - R_0), \tag{2.7}$$

From equations (2.7) and (2.6), one can see that $P_0(0) + Q_0(0) > 0$ and $P_1(0) + Q_1(0) > 0$ if $R_0 < 1.$ Then, by Ruth Hurwitz criterion, it has that equation (2.5) has roots with negative real parts if $R_0 < 1.$ Hence, if $R_0 < 1,$ the disease-free equilibrium E_0 is locally asymptotically stable.

Next, we consider case $\tau > 0.$ Then, let us take $\lambda = \omega i$ is a root of equation (2.5) where $\omega \geq 0.$ It has by separating real and imaginary parts that

$$\omega^2 - P_0(\tau) = Q_1(\tau)\omega \sin\omega\tau + Q_0 \cos\omega\tau,$$
$$P_1(\tau)\omega = Q_0(\tau)\sin\omega\tau - Q_1\omega\cos\omega\tau.$$

By squaring and taking addition, we have that

$$\omega^4 + [(P_1(\tau))^2 - 2P_0(\tau) - (Q_1(\tau))^2]\omega^2 + (P_0(\tau))^2 - (Q_0(\tau))^2 = 0. \tag{2.8}$$

$$(P_0(\tau))^2 - (Q_0(\tau))^2 = [(\mu_4 + \delta)(\mu_3 + \gamma + \alpha) - \delta\gamma]^2 - \left[\frac{\Lambda\beta(\mu_4 + \delta)e^{-\mu_2\tau}}{a\Lambda + \mu_1}\right]^2$$
$$= B(1 - R_0)(\mu_4 + \delta)^2\left[B + \frac{\Lambda\beta e^{-\mu_2\tau}}{a\Lambda + \mu_1}\right],$$

$$(P_1(\tau))^2 - 2P_0(\tau) - (Q_1(\tau))^2 = \left[(\mu_4 + \delta) + (\mu_3 + \gamma + \alpha)\right]^2 - 2\left[(\mu_4 + \delta)(\mu_3 + \gamma + \alpha) - \delta\gamma\right]$$
$$-\left[\frac{\Lambda\beta e^{-\mu_2\tau}}{a\Lambda + \mu_1}\right]^2$$
$$= (\mu_4 + \delta)^2 + \delta\gamma\left[2 + \frac{\mu_3 + \gamma + \alpha}{\mu_4 + \delta} + \frac{B}{\mu_4 + \delta}\right] + B^2(1 - R_0^2),$$

Equation (2.8) does not have positive real roots, if $R_0 < 1.$ In case $R_0 < 1,$ the disease-free equilibrium E_0 is locally asymptotically stable from Theorem 3.4.1 in [7].

If $R_0 > 1,$ we have that

$$f(0, \tau) = (\mu_4 + \delta)B(1 - R_0) < 0, \quad \lim_{\lambda \to +\infty} f(\lambda, \tau) = +\infty.$$

Therefore, $f(\lambda, \tau) = 0$ has at least one positive root. □

When $R_0 > 1,$ model (1.3) has a unique endemic equilibrium $E^*(S^*, I^*, R^*)^T$ other than the disease-free equilibrium $E_0,$ where

$$I^* = \frac{B(a\Lambda + \mu_1)(R_0 - 1)e^{-\mu_2\tau}}{\mathrm{B}((\beta + \mu_1 b)e^{-\mu_2\tau} - aB)}, \quad S^* = \frac{(1+bI^*)B}{\beta e^{-\mu_2\tau} - aB}, \quad R^* = \frac{\gamma I^*}{\mu_4 + \delta}.$$

The characteristic equation of the linearization of model (1.3) near the endemic equilibrium E^* is

$$\begin{vmatrix} (\lambda + \mu_1 + \frac{(1+bI^*)\beta I^*}{(1+aS^*+bI^*)^2}) & \frac{(1+aS^*)\beta S^*}{(1+aS^*+bI^*)^2} & 0 \\ -\frac{(1+bI^*)\beta I^* e^{-(\lambda+\mu_2)\tau}}{(1+aS^*+bI^*)^2} & \lambda + \mu_3 + \gamma + \alpha - \frac{(1+aS^*)\beta S^* e^{-(\lambda+\mu_2)\tau}}{(1+aS^*+bI^*)^2} & -\delta \\ 0 & -\gamma & \lambda + \mu_4 + \delta \end{vmatrix} = 0.$$

By introducing the following notations,

$$a_2(\tau) = \mu_1 + \frac{(1+bI^*)\beta I^*}{L} + \mu_3 + \gamma + \alpha + \mu_4 + \delta,$$

$$a_1(\tau) = \left[\mu_1 + \frac{(1+bI^*)\beta I^*}{L}\right]\left[\mu_3 + \gamma + \alpha + \mu_4 + \delta\right] + (\mu_4 + \delta)B,$$

$$a_0(\tau) = \left[\mu_1 + \frac{(1+bI^*)\beta I^*}{L}\right](\mu_4 + \delta)B,$$

$$b_2(\tau) = -\frac{B(1+aS^*)}{1+aS^*+bI^*},$$

$$b_1(\tau) = -\frac{B(1+aS^*)}{1+aS^*+bI^*}(\mu_1 + \mu_4 + \delta),$$

$$b_0(\tau) = -\frac{B(1+aS^*)}{1+aS^*+bI^*}\mu_1(\mu_4 + \delta),$$

$$L = (1+aS^*+bI^*)^2.$$

The characteristic equation becomes,

$$\lambda^3 + a_2(\tau)\lambda^2 + a_1(\tau)\lambda + a_0(\tau) + [b_2(\tau)\lambda^2 + b_1(\tau)\lambda + b_0(\tau)]e^{-\lambda\tau} = 0. \tag{2.9}$$

Theorem 3. The endemic equilibrium E^* is locally asymptotically stable, if it existence.

Proof. Let us consider case $\tau = 0$,

$$a_0(0) + b_0(0) = (\mu_4 + \delta)B\left\{\frac{\mu_1 bI^*}{1+aS^*+bI^*} + \frac{(1+bI^*)\beta I^*}{L}\right\}, \tag{2.10}$$

$$a_1(0) + b_1(0) = \frac{(1+bI^*)\beta I^*}{L}(\mu_3 + \gamma + \alpha + \mu_4 + \delta) + \mu_1(\mu_4 + \delta) + \frac{(\mu_4 + \delta)BbI^*}{1+aS^*+bI^*}$$
$$+ \frac{\mu_1\gamma\delta(1+aS^*)}{(1+aS^*+bI^*)(\mu_4 + \delta)} + \mu_1\frac{bI^*}{(1+aS^*+bI^*)}(\mu_3 + \gamma + \alpha), \tag{2.11}$$

$$a_2(0)+b_2(0)=\mu_1+\mu_4+\delta+\frac{(1+bI^*)\beta I^*}{L}+\frac{\delta\gamma}{\mu_4+\delta}+\frac{BbI^*}{1+aS^*+bI^*}, \tag{2.12}$$

$$\begin{aligned}(a_2(0)+b_2(0))(a_1(0)+b_1(0))-(a_0(0)+b_0(0))\\ =\frac{\mu_1(1+bI^*)\beta I^*}{L}(\mu_3+\gamma+\alpha)+\frac{(\mu_4+\delta)^2BbI^*}{1+aS^*+bI^*}\\ +(\mu_1+\mu_4+\delta)\left\{\frac{(1+bI^*)\beta I^*}{L}(\mu_4+\delta)+\mu_1(\mu_4+\delta)\right.\\ \left.+\frac{\mu_1}{1+aS^*+bI^*}\left[\frac{\gamma\delta(1+aS^*)}{(\mu_4+\delta)}+(\mu_3+\gamma+\alpha)bI^*\right]\right\}.\end{aligned} \tag{2.13}$$

By equations (2.10), (2.11), (2.12) and (2.13) and by Ruth Hurwitz criterion, one can see that equation (2.9) has roots with negative real parts. Hence, the endemic equilibrium E^* is locally asymptotically stable when $\tau=0.$

Next, we consider case $\tau>0$. Then, let us take $\lambda=\nu i$ be a root of equation (2.9) where $\nu\geq 0.$ It has by separating real and imaginary parts that

$$-\nu^3+a_1(\tau)\nu=(b_0(\tau)-b_2(\tau)\nu^2)\sin\nu\tau-b_1(\tau)\cos\nu\tau,$$
$$a_2(\tau)\nu^2-a_0(\tau)=b_1(\tau)\nu\sin\nu\tau+(b_0(\tau)-b_2(\tau)\nu^2)\cos\nu\tau.$$

By squaring both sides and taking the addition, we have that

$$\nu^6+(a_2^2(\tau)-2a_1(\tau)-b_2^2(\tau))\nu^4+(a_1^2(\tau)-2a_0(\tau)a_2(\tau)+2b_0(\tau)b_2(\tau)-b_1^2(\tau))\nu^2+a_0^2(\tau)-b_0^2(\tau)=0. \tag{2.14}$$

$a_0^2(\tau)-b_0^2(\tau)=(a_0(\tau)-b_0(\tau))(a_0(\tau)+b_0(\tau))$. We have already proven that $a_0(\tau)+b_0(\tau)>0.$ Further, from the expressions of $a_0(\tau)$ and $b_0(\tau)$, we can easily see that $a_0(\tau)-b_0(\tau)>0.$ It shows that $a_0^2(\tau)-b_0^2(\tau)>0$.

$$\begin{aligned}a_2^2(\tau)-2a_1(\tau)-b_0^2(\tau)=(\mu_4+\delta)^2+\left[\mu_1+\frac{(1+bI^*)\beta I^*}{L}\right]^2+\frac{\gamma\delta}{\mu_4+\delta}\frac{B(1+aS^*)^2}{L}\\ +\gamma\delta\left[2+\frac{(\mu_3+\gamma+\alpha)}{(\mu_4+\delta)}\frac{(1+aS^*)^2}{L}\right]\\ +\frac{(\mu_3+\gamma+\alpha)bI^*}{L}(bI^*+2(1+aS^*))>0,\end{aligned}$$

$$
\begin{aligned}
&a_1^2(\tau) - 2a_0(\tau)a_2(\tau) + 2b_0(\tau)b_2(\tau) - b_1^2(\tau) \\
&= \frac{bI^*(bI^* + 2(1+aS^*))}{L}(\mu_1^2 + (\mu_4+\delta)^2 B^2) \\
&+ \left[\mu_1 + \frac{(1+bI^*)\beta I^*}{L}\right]^2 \left((\mu_4+\delta)^2 + 2\gamma\delta\right) \\
&+ \left[2\mu_1 + \frac{(1+bI^*)\beta I^*}{\mu_4+\delta}\right]\frac{(1+bI^*)\beta I^*}{\mu_4+\delta}(\mu_3+\gamma+\alpha)^2 \\
&+ \frac{\mu_1^2\gamma\delta}{\mu_4+\delta}\frac{(1+aS^*)^2}{L}\left(\mu_3+\gamma+\alpha+B\right) > 0.
\end{aligned}
$$

It can be seen that from above calculation that equation (2.14) does not have positive real roots for ν^2. Hence, the endemic equilibrium E^* is locally asymptotically stable from Theorem 3.4.1 in [7]. □

III. Discussion

Now In this study, we have developed a SEIR epidemic model with relapse and time delay. We have established that existence of endemic equilibrium and stability properties to completely depend on the basic reproduction number. It is necessary to mention here that the author in the research article [15] has considered a SEIR epidemic model with saturation incidence rate function and time delay, and in which the author have given complete local and global stability properties. However, in this paper, we have used non-separable incidence rate function of variables S and I. Therefore, it is complicated to define a Liapunov function to show the global dynamics for the endemic equilibrium. Instead of global dynamics, we have given complete local dynamic properties for both disease free equilibrium and endemic equilibrium. It is worthy to note that the basic reproduction number in this paper is less than that of in the paper [15], by which we can see that use of Beddington-DeAngelis incidence rateis better to avoid disease becoming endemic. We left studying of global stability properties of both equilibria as future work.

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