Dominator Coloring Number of Some Graphs

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Abstract- Given a graph G, the dominator coloring problem seeks a proper coloring of G with the additional property that every vertex in the graph dominates an entire color class. In this paper, as an extension of Dominator coloring some standard results has been discussed and the solutions for some of the open problems in [2] are also found out.

Index Terms- Coloring, Crown graph, Domination, Dominator Coloring, Dutch Windmill graph, Middle Graph, Windmill graph.

I. Introduction

In graph theory, coloring and dominating are two important areas which have been extensively studied. The fundamental parameter in the theory of graph coloring is the chromatic number $\chi(G)$ of a graph G which is defined to be the minimum number of colors required to color the vertices of G in such a way that no two adjacent vertices receive the same color. If $\chi(G) = k$, we say that G is k-chromatic.

A dominating set S is a subset of the vertices in a graph such that every vertex in the graph either belongs to S or has a neighbor in S. The domination number is the order of a minimum dominating set. Given a graph G and an integer k, finding a dominating set of order k is NP-complete on arbitrary graphs. [5, 6]

Graph coloring is used as a model for a vast number of practical problems involving allocation of scarce resources (e.g., scheduling problems), and has played a key role in the development of graph theory and, more generally, discrete mathematics and combinatorial optimization. A graph has a dominator coloring if it has a proper coloring in which each vertex of the graph dominates every vertex of some color class.
The dominator chromatic number $\chi_d(G)$ is the minimum number of color classes in a dominator coloring of a graph G. A $\chi_d(G)$ -coloring of G is any dominator coloring with $\chi_d(G)$ colors. Our study of this problem is motivated by [3] and [4].

Terminologies

We start with notation and more formal definitions. Let G = (V(G), E(G)) be a graph with n = |V(G)| and m = |E(G)|. For any vertex $v \in V(G)$, the open neighborhood of v is the set $N(v) = \{u | uv \in E(G)\}$ and the closed neighborhood is the set $N[v] = N(v) \cup v$. Similarly, for any set $S \subseteq V(G)$, $N(S) = \cup_{v \in S} N(v) - S$ and $N[S] = N(S) \cup S$. A set S is a dominating set if N[S] = V (G).The minimum cardinality of a dominating set of G is denoted by $\gamma(G)$.

The distance, d(u ,v), between two vertices u and v in G is the smallest number of edges on a path between u and v in G. The eccentricity, $e(v)$, of a vertex v is the largest distance from v to any vertex of G. The radius $rad(G)$ is the smallest eccentricity in G. The diameter $diam(G)$ is the largest eccentricity in G.

A graph coloring is a mapping $f : V(G) \rightarrow C$, where C is a set of colors. A coloring f is proper if, for all $x, y \in V(G)$, $x \in N(y)$ implies $f(x) \neq f(y)$. A k-coloring of G is a coloring that uses at most k colors. The chromatic number of G is $\chi(G) = \min \{k \mid G$ has a proper k –coloring$\}$. A coloring of G can also be thought of as a partition of V(G) into color classes $V_1, V_2, \dots, V_q$, and a proper coloring of G is then a coloring in which each $V_i, 1 \leq i \leq q$ is an independent set of G, i.e., for each i, the subgraph of G induced by V_i contains no edges.

Dominator coloring was introduced in [7] and motivated in [3].

Definition

The Middle graph of G, denoted by $M(G)$ is defined as follows.The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one of the following holds.
1. x, y are in $E(G)$ and x, y are adjacent in G.
2. x is in $V(G)$, y is in $E(G)$ and x, y are incident in G [8].

Definition

The windmill graph $W_n^{(m)}$ is the graph obtained by taking m copies of the Complete Graph K_n with a vertex in common.

Definition

The Dutch windmill graph $D_n^{(m)}$, also called a friendship graph, is the graph obtained by taking *m* copies of the Cycle Graph C_n with a vertex in common.

Definition

The crown graph S_n^0 for an integer $n \geq 3$ is the graph with Vertex Set $\{x_0, x_1, \dots, x_{n-1}, y_0, y_1, \dots, y_{n-1}\}$ and edge set $\{(x_i, y_j): 0 \leq i, j \leq n-1, i \neq j\}$. S_n^0 is therefore equivalent to the complete bipartite graph $K_{n,n}$ with horizontal edges removed.

Proposition

(1) The star $k_{1,n}$ has $\chi_d(k_{1,n}) = 2$.

(2) The complete graph K_n has and $\chi_d(K_n) = n$.

(3) The path P_n of order $n \geq 3$ has

$$\chi_d(P_n) = \begin{cases} 1 + \left\lceil \frac{n}{3} \right\rceil & \text{if } n = 2,3,4,5,7 \\ 2 + \left\lceil \frac{n}{3} \right\rceil & \text{otherwise} \end{cases}$$

(4) The cycle C_n has

$$\chi_d(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil & \text{if } n = 4 \\ \left\lceil \frac{n}{3} \right\rceil + 1 & \text{if } n = 5 \\ \left\lceil \frac{n}{3} \right\rceil + 2 & \text{otherwise} \end{cases}$$

(5) The multi-star $K_n(a_1, a_2, \ldots, a_n)$ has

$\chi_d(K_n(a_1, a_2, \ldots, a_n)) = n + 1$

(6) The wheel $W_{1,n}$ has

$$\chi_d(W_{1,n}) = \begin{cases} 3 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd} \end{cases}.$$

(7) The complete k-partite graph $K_{a_1,a_2,\ldots,a_k}$ has $\chi_d(K_{a_1,a_2,\ldots,a_k}) = k$

(8) The middle graph of cycle [1], G = $M(C_n)$ with $n > 3$ has

$$\chi_d(G) = \begin{cases} \chi(G) + \gamma(G) - 1 & \text{if } n \text{ is even} \\ \chi(G) + \gamma(G) - 2 & \text{if } n \text{ is odd} \end{cases}$$

(9) The middle graph of path[1], G= M(P_n) with $n > 2$ has $\chi_d(G) = \chi(G) + \gamma(G) - 1$.

(10) Let G be a connected graph of order n. Then $\chi_d(G) = n$ if and only if $G = K_n$, for n ∈ N.

Main Results

Theorem 3.1 Let G be any graph, then $\chi_d(K_n \times G) = n$ if and only if G is K_1 or K_2.

Proof

Let $K_n \times G$ is a connected graph. Suppose G is K_1 or K_2. We claim that, $\chi_d(K_n \times G) = n$

If G = K_1, then $K_n \times G$ is itself a K_n.

Therefore $\chi_d(K_n \times G) = n$. If $G = K_2$, let the vertex set of K_n be $\{v_1, v_2, \ldots, v_n\}$ and in $G = \{u_1, u_2\}$.

Then the vertex set in $K_n \times G$ will be $\{(v_1,u_1),(v_1,u_2),(v_2,u_1),(v_2,u_2),\ldots,(v_n,u_1),(v_n,u_2)\}$ and any two vertices (u,u') and (v,v') are adjacent in $K_n \times G$ if and only if either $u = v$ and u' is adjacent with v' in G, or $u' = v'$ and u is adjacent with v in K_n. Now the dominator color class partition is given by

$\{(v_1,u_1),(v_2,u_2)\},\{(v_2,u_1),(v_3,u_2)\},\{(v_3,u_1),(v_4,u_2)\},\ldots,\{(v_n,u_1),(v_1,u_2)\}$, clearly each vertex in the color class dominates atleast one color class. Therefore $\chi_d(K_n \times G) = n$.

Now let $K_n \times G$ is a connected graph with $\chi_d(K_n \times G) = n$.

We claim that, G is K_1 or K_2.

On the contrary, let $G \neq K_1$ or K_2. Suppose the order of G is $m \neq 1,2$. Then $K_n \times G$ contains m copies of K_n such that $1K_n, 2K_n, 3K_n \ldots mK_n$ and some edges between iK_n, jK_n for $i,j = 1,2,3\ldots,m$ and $i \neq j$. Hence there will be mn vertices. Since the dominator coloring number is $\chi_d(K_n \times G) = n$, then the mn vertices can be partitioned into n color classes in such a way that each vertex from each copies. Also by the definition of product graph, there exists atleast one vertex which will not dominate a color class. This contradicts the fact of dominator coloring number. Hence G will be K_1 or K_2

Theorem 3.2 Let G be any graph, then $\chi_d(K_n[G]) = n\chi_d(G)$.

Proof

Let G be any graph. Then the composition of two graphs $K_n[G]$ is a graph such that, the vertex set of $K_n[G]$ is the Cartesian product $V(K_n) \times V(G)$ and any two vertices (u,v) and (x,y) are adjacent in $K_n[G]$ if and only if either u is adjacent with x in K_n or $u = x$ and v is adjacent with y in G.

Also by the definition of composition of graph $K_n[G]$ contains n copy of G with each vertex in one copy is adjacent to all other vertices in the remaining n-1 copies. Since K_n is complete. Then the minimal dominator color class partition of $K_n[G]$ contains n copies of the minimal dominator color class partition of G.

Suppose if the vertex in any two copies of G was in same dominator color class partition, then it contradicts the coloring property. Since each vertex in each copy is adjacent to all other vertex in all other copies. And also each vertex in color class partition dominates atleast one color class. Hence $\chi_d(K_n[G]) = n\chi_d(G)$.

Cherecterization of graphs with Dominator Chromatic number equals Chromatic number

Lemma 4.1.1

Let G be a connected graph. Then $\max\{\chi(G), \gamma(G)\} \leq \chi_d(G) \leq \chi(G) + \gamma(G)$. The bound is sharp.

Lemma 4.1.2

For any graph G, $\chi(G) \leq \chi_d(G)$

Theorem 4.1 Let G be a $(n-2)$ regular graph with even n, then $\chi_d(G) = \chi(G)$

Proof

By lemma 4.1.2, we have $\chi(G) \leq \chi_d(G)$

Also $\gamma(G) = 2$, for $(n-2)$ regular graph and $\chi(G) > 2$.

Hence by lemma 4.1.1,

$\max\{\chi(G),\gamma(G)\} \le \chi_d(G) \le \chi(G) + \gamma(G)$.

That is, $\chi(G) \le \chi_d(G) \le \chi(G) + 2$.

Suppose $\chi_d(G) = \chi(G) + 2$ and $\chi_d(G) = \chi(G) + 1$,

we have more color class in the dominator coloring partition compared to the chromatic coloring. Also each vertex in G is non adjacent to only one vertex; hence each pair of vertex receives different colors. If there are n even vertices in G, then there will be $\frac{n}{2}$ color classes, each with two vertices.

And clearly each vertex dominates atleast one color class. Hence increase in the dominator coloring number than chromatic number will not have proper dominator coloring class.

So $\chi_d(G) = \chi(G)$.

Theorem 4.2 Let G be a graph with $\Delta(G) = n-1$, then $\chi_d(G) = \chi(G)$.

Proof

By lemma 4.1.2, we have $\chi(G) \le \chi_d(G)$. Also γ(G) = 1.
Hence by lemma 4.1.1,
$\max\{\chi(G),\gamma(G)\} \le \chi_d(G) \le \chi(G) + \gamma(G)$.

That is, $\chi(G) \le \chi_d(G) \le \chi(G) + 1$

Suppose $\chi_d(G) = \chi(G) + 1$,

We have more color class in the dominator coloring partition compared to the chromatic coloring. Also there exists a vertex in G is adjacent to all vertex; hence that vertex alone receives a color.

And clearly that color class is dominated by the all other color classes. Hence increase in the dominator coloring number than chromatic number will not have proper dominator coloring class. So $\chi_d(G) = \chi(G)$

Theorem 4.3 Let G_1 and G_2 be any two graphs, then $\chi_d(G_1 + G_2) = \chi(G_1 + G_2)$.

Proof

Let G_1 and G_2 be any two graphs. Then the sum of two graphs $G_1 + G_2$ has all the edges joining the vertices of G_1 to the vertices of G_2.

Also we know that, $\chi(G_1 + G_2) = \chi(G_1 + G_2)$.

And also each vertex in the color class of the G_1 chromatic coloring dominates the color class in the G_2 chromatic coloring. Hence the color class of the chromatic coloring is itself acts as the color class for the dominator coloring.

Therefore $\chi_d(G_1 + G_2) = \chi(G_1 + G_2)$.

Corollary 4.1.3

Let G be any graph, then $\chi_d(G + K_n) = \chi(G) + n$.

Proof

Since $\chi(G_1 + G_2) = \chi(G_1) + \chi(G_2)$,
we have $\chi(G + K_n) = \chi(G) + \chi(K_n)$.
Also we know that $\chi(K_n)$ = n.
Hence $\chi(G + K_n) = \chi(G) + n$

By theorem 4.3, $\chi_d(G + K_n) = \chi(G + K_n) = \chi(G) + n$.

Theorem 4.4 Let $G = W_n^{(m)}$ be a windmill graph, then $\chi_d(G) = \chi(G)$.

Proof

Let $G = W_n^{(m)}$ be the Windmill graph. By the definition of Windmill graph, there exist m copies of K_n with a vertex x in common.

Hence, the vertex x alone posses a color class. Clearly, $\chi(G) = n$. And each vertex in the color class partition dominates the vertex .

Hence $\chi_d(G) = n$. Therefore, $\chi_d(G) = \chi(G)$

Corollary 4.1.4

If $G = D_n^{(m)}$ is a Dutch windmill graph with $n = 3$, then $\chi_d(G) = \chi(G)$.

Theorem 4.5 Let G be a crown graph, then $\chi_d(G) = \chi(G) + \gamma(G) = 4$.

Proof

Let G be the Crown graph. By the definition of Crown graph, it is clear that $\chi(G)$=2 and $\gamma(G) = 2$.

Let the vertex set in the crown graph be $\{x_0, x_1, \dots, x_{n-1}, y_0, y_1, \dots, y_{n-1}\}$.

Now the dominator color class partition is given by $\{\{x_i\},\{y_i\},\{x_0, x_1 \cdots x_{i-1}, x_{i+1} \cdots x_{n-1}\},\{y_0, y_1, \cdots y_{i-1}, y_{i+1} \cdots y_{n-1}\}\}$
Clearly each vertex in the color class partition dominates atleast one color class.

Hence $\chi_d(G) = 4$.

Also, $\chi(G) + \gamma(G) = 4$. Therefore, $\chi_d(G) = \chi(G) + \gamma(G) = 4$.

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