Dominator Coloring Number of Some Graphs

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Abstract- Given a graph G, the dominator coloring problem seeks a proper coloring of G with the additional property that every vertex in the graph dominates an entire color class. In this paper, as an extension of Dominator coloring some standard results have been discussed and the solutions for some of the open problems in [2] are also found out.

Index Terms- Coloring, Crown graph, Domination, Dominator Coloring, Dutch Windmill graph, Middle Graph, Windmill graph.

I. INTRODUCTION

In graph theory, coloring and dominating are two important areas which have been extensively studied. The fundamental parameter in the theory of graph coloring is the chromatic number \( \chi(G) \) of a graph G which is defined to be the minimum number of colors required to color the vertices of G in such a way that no two adjacent vertices receive the same color. If \( \chi(G) = k \), we say that G is k-chromatic.

A dominating set S is a subset of the vertices in a graph such that every vertex in the graph either belongs to S or has a neighbor in S. The domination number is the order of a minimum dominating set. Given a graph G and an integer k, finding a dominating set of order k is NP-complete on arbitrary graphs. [5, 6]

Graph coloring is used as a model for a vast number of practical problems involving allocation of scarce resources (e.g., scheduling problems), and has played a key role in the development of graph theory and, more generally, discrete mathematics and combinatorial optimization. A graph has a dominator coloring if it has a proper coloring in which each vertex of the graph dominates every vertex of some color class.

The dominator chromatic number \( \chi_d(G) \) is the minimum number of color classes in a dominator coloring of a graph G. A \( \chi_d(G) \)-coloring of G is any dominator coloring with \( \chi_d(G) \) colors. Our study of this problem is motivated by [3] and [4].

Terminologies

We start with notation and more formal definitions. Let \( G = (V(G), E(G)) \) be a graph with \( n = \#V(G) \) and \( m = \#E(G) \). For any vertex \( v \in V(G) \), the open neighborhood of \( v \) is the set \( N(v) = \{u | uv \in E(G) \} \) and the closed neighborhood is the set \( N[v] = N(v) \cup v \). Similarly, for any set \( S \subseteq V(G) \), \( N(S) = \bigcup_{v \in S} N(v) \) and \( N[S] = N(S) \cup S \). A set \( S \) is a dominating set if \( N[S] = V(G) \). The minimum cardinality of a dominating set of G is denoted by \( \gamma(G) \).

The distance, \( d(u,v) \), between two vertices \( u \) and \( v \) in \( G \) is the smallest number of edges on a path between \( u \) and \( v \) in \( G \). The eccentricity, \( e(v) \), of a vertex \( v \) is the largest distance from \( v \) to any vertex of \( G \). The radius \( rad(G) \) is the smallest eccentricity in \( G \). The diameter \( diam(G) \) is the largest eccentricity in \( G \).

A graph coloring is a mapping \( f : V(G) \rightarrow \mathbb{C} \), where \( \mathbb{C} \) is a set of colors. A coloring \( f \) is proper if, for all \( x, y \in V(G) \), \( x \neq N(y) \) implies \( f(x) \neq f(y) \). A k-coloring of G is a coloring that uses at most k colors. The chromatic number of G is \( \chi(G) = \min \{k | G \text{ has a proper } k\text{-coloring} \} \). A coloring of G can also be thought of as a partition of \( V(G) \) into color classes \( V_1, V_2, ..., V_q \), and a proper coloring of G is then a coloring in which each \( V_i \) is an independent set of G, i.e., for each \( i \), the subgraph of G induced by \( V_i \) contains no edges.

Dominator coloring was introduced in [7] and motivated in [3].

Definition

The Middle graph of G, denoted by \( M(G) \), is defined as follows. The vertex set of \( M(G) \) is \( V(G) \cup E(G) \). Two vertices \( x \cdot y \) in the vertex set of \( M(G) \) are adjacent in \( M(G) \) in case one of the following holds.
1. \( x, y \) are in \( E(G) \) and \( x \cdot y \) are adjacent in \( G \).
2. \( x \) is in \( V(G) \), \( y \) is in \( E(G) \) and \( x \cdot y \) is incident in \( G \).

Definition

The windmill graph \( W_n(m) \) is the graph obtained by taking \( m \) copies of the Complete Graph \( K_m \) with a vertex in common.

Definition

The Dutch windmill graph \( P_n(m) \), also called a friendship graph, is the graph obtained by taking \( m \) copies of the Cycle Graph \( C_n \) with a vertex in common.

Definition

The crown graph \( S_n^0 \) for an integer \( n \geq 3 \) is the graph with vertex set \( \{x_0, x_1, ..., x_n; y_0, y_1, ..., y_n-1\} \) and edge set \( \{(x_i, y_j) ; 0 \leq i, j \leq n - 1, i \neq j\} \). \( S_n^0 \) is therefore equivalent to the complete bipartite graph \( K_{n,n} \) with horizontal edges removed.

Proposition

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(1) The star $K_{1,n}$ has $\chi_d(K_{1,n}) = 2$.
(2) The complete graph $K_n$ has and $\chi_d(K_n) = n$.
(3) The path $P_n$ of order $n \geq 3$ has
\[ \chi_d(P_n) = \begin{cases} 1 + \left\lfloor \frac{n}{3} \right\rfloor & \text{if } n = 2, 3, 4, 5, 7 \\ 2 + \left\lfloor \frac{n}{3} \right\rfloor & \text{otherwise} \end{cases} \]
(4) The cycle $C_n$ has
\[ \chi_d(C_n) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor & \text{if } n = 4 \\ \frac{n}{2} + 1 & \text{if } n = 5 \\ \frac{n}{2} + 2 & \text{otherwise} \end{cases} \]
(5) The multi-star $K_n(a_1, a_2, \ldots, a_n)$ has
\[ \chi_d(K_n(a_1, a_2, \ldots, a_n)) = n + 1 \]
(6) The wheel $W_n$ has
\[ \chi_d(W_n) = \begin{cases} 3 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd} \end{cases} \]
(7) The complete k-partite graph $K_{a_1, a_2, \ldots, a_k}$ has
\[ \chi_d(K_{a_1, a_2, \ldots, a_k}) = k \]
(8) The middle graph of cycle $[1], G = M(C_n)$ with $n > 3$ has
\[ \chi_d(M(C_n)) = \begin{cases} \chi(G) + \gamma(G) - 1 & \text{if } n \text{ is even} \\ \chi(G) + \gamma(G) - 2 & \text{if } n \text{ is odd} \end{cases} \]
(9) The middle graph of path $[1], G = M(P_n)$ with $n > 2$ has
\[ \chi_d(M(P_n)) = \chi(G) + \gamma(G) - 1 \]
(10) Let $G$ be a connected graph of order $n$. Then
\[ \chi_d(G) = n \] if and only if $G = K_n$, for $n \in \mathbb{N}$.

Main Results

**Theorem 3.1** Let $G$ be any graph, then $\chi_d(K_n \times G) = n$ if and only if $G$ is $K_1$ or $K_2$.

**Proof**

Let $K_n \times G$ is a connected graph. Suppose $G$ is $K_1$ or $K_2$.

We claim that, $\chi_d(K_n \times G) = n$.

If $G = K_1$, then $K_n \times G$ is itself a $K_n$.

Therefore $\chi_d(K_n \times G) = n$. If $G = K_2$, let the vertex set of $K_n$ be $\{v_1, v_2, \ldots, v_n\}$ and in $G$ be $\{u_1, u_2\}$.

Then the vertex set in $K_n \times G$ will be $\{(v_1, u_1), (v_1, u_2), (v_2, u_1), (v_2, u_2), \ldots, (v_n, u_1), (v_n, u_2)\}$ and any two vertices $(u, u')$ and $(v, v')$ are adjacent in $K_n \times G$ if and only if either $u = v$ and $u'$ is adjacent with $v'$ in $G$, or $u' = v$ and $u$ is adjacent with $v$ in $K_n$. Now the dominator color class partition is given by the sets $\{(v_1, u_1), (v_2, u_2)\}, \{(v_2, u_1), (v_2, u_2)\}, \ldots, \{(v_n, u_1), (v_n, u_2)\}$, clearly each vertex in the color class dominates at least one color class. Therefore $\chi_d(K_n \times G) = n$.

Now let $K_n \times G$ is a connected graph with $\chi_d(K_n \times G) = n$.

We claim that, $G$ is $K_1$ or $K_2$.

On the contrary, let $G \neq K_1$ or $K_2$. Suppose the order of $G$ is $m \neq 1, 2$. Then $K_n \times G$ contains $m$ copies of $K_1$ such that $1K_n \times 2K_n \times 3K_n \times \ldots, mK_n$ and some edges between $1K_n \times 1K_n$ for $i, j = 1, 2, 3, \ldots, m$ and $i \neq j$. Hence there will be $mn$ vertices. Since the dominator coloring number is $\chi_d(K_n \times G) = n$, then the $mn$ vertices can be partitioned into $n$ color classes in such a way that each vertex from each copy. Also by the definition of product graph, there exists at least one vertex which will not dominate a color class. This contradicts the fact of dominator coloring number. Hence $G$ will be $K_1$ or $K_2$.

**Theorem 3.2** Let $G$ be any graph, then $\chi_d(K_n[G]) = n\chi_d(G)$.

**Proof**

Let $G$ be any graph. Then the composition of two graphs $K_n[G]$ is a graph such that the vertex set of $K_n[G]$ is the Cartesian product $V(K_n) \times V(G)$ and any two vertices $(u, v)$ and $(x, y)$ are adjacent in $K_n[G]$ if and only if either $u$ is adjacent with $x$ in $K_n$ or $u = x$ and $v$ is adjacent with $y$ in $G$.

Also by the definition of composition of graph $K_n[G]$ contains $n$ copy of $G$ with each vertex in one copy is adjacent to all other vertices in the remaining $n-1$ copies. Since $K_n$ is complete. Then the minimal dominator color class partition of $K_n[G]$ contains $n$ copies of the minimal dominator color class partition of $G$.

Suppose if the vertex in any two copies of $G$ was in the same dominator color class partition, then it contradicts the coloring property. Since each vertex in each copy is adjacent to all other vertex in all other copies. And also each vertex in color class partition dominates at least one color class. Hence $\chi_d(K_n[G]) = n\chi_d(G)$.

Characterization of graphs with Dominator Chromatic number equals Chromatic number

**Lemma 4.1.1**

Let $G$ be a connected graph. Then $\max\{\chi(G), \gamma(G)\} \leq \chi_d(G) \leq \chi(G) + \gamma(G)$. The bound is sharp.

**Lemma 4.1.2**

For any graph $G$, $\chi(G) \leq \chi_d(G)$

**Theorem 4.1** Let $G$ be a $(n-2)$ regular graph with even $n$, then $\chi_d(G) = \chi(G)$

**Proof**

By lemma 4.1.2, we have $\chi(G) \leq \chi_d(G)$

Also $\chi(G) = 2$, for $(n-2)$ regular graph and $\chi(G) > 2$.

Hence by lemma 4.1.1,
That is, \( \chi(G) \leq \chi_d(G) \leq \chi(G) + 2 \),

Suppose \( \lambda_d(G) = \chi(G) + 2 \) and \( \chi_d(G) = \chi(G) + 1 \),

we have more color class in the dominator coloring partition compared to the chromatic coloring. Also each vertex in \( G \) is non adjacent to only one vertex; hence each pair of vertex receives different colors. If there are \( n \) even vertices in \( G \), then there will be \( \frac{n}{2} \) color classes, each with two vertices.

And clearly each vertex dominates atleast one color class. Hence increase in the dominator coloring number than chromatic number will not have proper dominator coloring class.

So \( \chi_d(G) = \chi(G) \).

**Theorem 4.2** Let \( G \) be a graph with \( \Delta(G) = n - 1 \), then \( \chi_d(G) = \chi(G) \).

**Proof**
By lemma 4.1.2, we have \( \chi(G) \leq \chi_d(G) \). Also \( \gamma(G) = 1 \).
Hence by lemma 4.1.1,
\[
\max (\chi(G), \gamma(G)) \leq \chi_d(G) \leq \chi(G) + \chi(G),
\]
That is, \( \chi(G) \leq \chi_d(G) \leq \chi(G) + 1 \).
Suppose \( \chi_d(G) = \chi(G) + 1 \).

We have more color class in the dominator coloring partition compared to the chromatic coloring. Also there exists a vertex in \( G \) is adjacent to all vertex; hence that vertex alone receives a color.

And clearly that color class is dominated by the all other color classes. Hence increase in the dominator coloring number than chromatic number will not have proper dominator coloring class. So \( \chi_d(G) = \chi(G) \).

**Theorem 4.3** Let \( G_1 \) and \( G_2 \) be any two graphs, then \( \chi_d(G_1 + G_2) = \chi(G_1 + G_2) \).

**Proof**
Let \( G_1 \) and \( G_2 \) be any two graphs. Then the sum of two graphs \( G_1 + G_2 \) has all the edges joining the vertices of \( G_1 \) to the vertices of \( G_2 \).
Also we know that, \( \chi(G_1 + G_2) = \chi(G_1 + G_2) \).

And also each vertex in the color class of the \( G_1 \) chromatic coloring dominates the color class in the \( G_2 \) chromatic coloring. Hence the color class of the chromatic coloring is itself acts as the color class for the dominator coloring. Therefore \( \chi_d(G_1 + G_2) = \chi(G_1 + G_2) \).

**Corollary 4.1.3**
Let \( G \) be any graph, then \( \chi_d(G + K_n) = \chi(G) + n \).

**Proof**
Since \( \chi(G_1 + G_2) = \chi(G_1) + \chi(G_2) \),
we have \( \chi(G + K_n) = \chi(G) + \chi(K_n) \).
Also we know that \( \chi(K_n) = n \).
Hence \( \chi_d(G + K_n) = \chi(G) + n \).

By the theorem 4.3,
\[
\chi_d(G + K_n) = \chi(G + K_n) = \chi(G) + n.
\]

**Theorem 4.4** Let \( G = W_n(m) \) be a windmill graph, then \( \chi_d(G) = \chi(G) \).

**Proof**
Let \( G = W_n(m) \) be the Windmill graph. By the definition of Windmill graph, there exist \( m \) copies of \( K_n \) with a vertex \( x \) in common.

Hence, the vertex \( x \) alone posses a color class. Clearly, \( \chi(G) = n \). And each vertex in the color class partition dominates the vertex.
Hence \( \chi_d(G) = n \). Therefore, \( \chi_d(G) = \chi(G) \).

**Corollary 4.1.4**
If \( G = D_n(m) \) is a Dutch windmill graph with \( n = 3 \), then \( \chi_d(G) = \chi(G) \).

**Theorem 4.5** Let \( G \) be a crown graph, then \( \chi_d(G) = \chi(G) + \gamma(G) = 4 \).

**Proof**
Let \( G \) be the Crown graph. By the definition of Crown graph, it is clear that \( \chi(G) = 2 \) and \( \gamma(G) = 2 \).
Let the vertex set in the crown graph be \( \{X_0, X_1, \ldots, X_{n-1}, Y_0, Y_1, \ldots, Y_{n-1}\} \).
Now the dominator color class partition is given by \( \{[X_0], [Y_1], [X_0, X_1, \ldots, X_{n-1}, Y_0, Y_1, \ldots, Y_{n-1}]\} \).
Clearly each vertex in the color class partition dominates atleast one color class.
Hence \( \chi_d(G) = 4 \).
Also, \( \chi(G) + \gamma(G) = 4 \). Therefore, \( \chi_d(G) = \chi(G) + \gamma(G) = 4 \).

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