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Schwarzschild-like solution for the gravitational field of an isolated particle on the basis of 7-dimensional metric

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Abstract: Schwarzschild solution is the simplest solution of Einstein's field equations. The solution was first given by Schwarzschild on the basis of 4-dimensional space-time metric or line element. But here we extended our view to the 7-dimensional space-time continuum where 3-usual space components and another 4 time components on the basis of the four fundamental forces of nature. In this write-up especially particular attention is given to the solution of Einstein's field equations on the basis of seven dimensional metric $g_{\mu\nu}$. Using 7-dimensional metric we got the Schwarzschild-like solution of Einstein's field equations for the gravitational field of an isolated particle. The solution gives us some new interesting results and which gives new physical interpretation of the gravitational field of that isolated particle.

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Index Term: 7-dimensional space-time continuum, 4-time components, changing speed or constant, Schwarzschild-like solution

I.INTRODUCTION

Einstein's original field equations representing the law of gravitation in empty space [1-3]

$$R_{\mu\nu} = 0 \qquad (1)$$

The solution of above equations merely consists of finding the line element for interval in empty space surrounding a gravitating point particle which ultimately corresponds to the field of an isolated particle continually at rest at the origin. The solution was first given by Schwarzschild [4, 5].

In the absence of any mass point the space-time would be flat so that the 4-dimensional line element in spherical polar co-ordinates would be expressed as

$$ds^2 = -dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + c^2 dt^2 \qquad (2)$$

But the velocity of light c is taken to be unity in order to use as astronomical unit. Therefore equation (1) becomes,

$$ds^2 = -dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + dt^2 \qquad (3)$$

The presence of the mass point would modify the line element. However since mass is static and isolated, the line element would be spatially spherically symmetric about the point mass and is static. The most general form of such a four dimensional line element may be expressed as

$$ds^2 = -e^{\lambda} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + e^{\nu} dt^2 \qquad (4$$

Where λ and ν are functions of r only; since for spherically symmetric isolated particle the field will depend on $r$alone and not on θ and ϕ.

Finally the line element due to static, isolated gravitating mass point is found

$$ds^2 = -\left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + \left(1 - \frac{2m}{r}\right) dt^2 \qquad (5)$$

The solution was first obtain by Schwarzschild and hence is known as Schwarzschild line element reduces to the line element of flat space-time of special relativity.

Schwarzschild solution is seen to have the following singularities

(i) TheSchwarzschild solution becomes singular at $r = 0$; but this singularity also occurs in Newton's (classical) theory.

(ii) The Schwarzschild solution again becomes singular at a distance$r$ given by $(1-\frac{2m}{r}) = 0$, i.e. $r = 2m$. This value of$r$is known as Schwarzschild radius. For points $0 \le r \le 2m$, $ds^2 < 0$ i.e. the interval is purely space-like. Hence there is a finite singular region for $0 \le r \le 2m$. Thus $r = 2m$ represents the boundary of the isolated particle and the solution holds in empty space outside the spherical distribution of matter (or isolated particle) whose radius must be greater than $2m$. Hence equation (5) is called the Schwarzschild exterior solution for the gravitational field of an isolated particle.

Many authors[6, 7] trying to solve the problems of gravitation on the basis of Schwarzschild solution of the line element or metric of 4-d space-time continuum.

The purpose of this article is simply to solve Einstein's field equations for the gravitational field of an isolated particle on the basis of 7-dimensional metric similar to that of Schwarzschild in 4-dimesional. Taking new idea of time [8, 9] and looking in to the extra dimension of space-time continuum already we have developed a 7-dimensional metric [10] where the 3 space components and 4-time components. The idea of 4-time components has beentaken on the basis of the 4-fundamental forces of nature which are known as Electro-magnetic, Strong, Weak and gravitational forces.

II.MATHEMATICAL FORMULATION

According to our new concept of space-time continuum [10] the physical universe is not 4-dimensional it is considered as 7-dimensional, where the time part has 4-components instead of one. The 4-time components are considered on the basis of four fundamental forces of nature. Therefore the equation (3) becomes

$$ds^2 = -dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + c^2\left[a_1(dt^1)^2 + a_2(dt^2)^2 + a_3(dt^3)^2 + a_4(dt^4)^2\right] \quad (6)$$

Since the time components [10] are

$$c^2 dt^2 = \left[c_1^2(dt^1)^2 + c_2^2(dt^2)^2 + c_3^2(dt^3)^2 + c_4^2(dt^4)^2\right] \quad (7)$$

Here t^1, t^2, t^3, t^4 and c_1, c_2, c_3, c_4 aretime-components and the changing speed or constants due to the four fundamental forces viz. electro-magnetic, strong, weak, gravitational respectively. The equation (7) can be written as,

$$c^2 dt^2 = c^2\left[a_1(dt^1)^2 + a_2(dt^2)^2 + a_3(dt^3)^2 + a_4(dt^4)^2\right] \quad (8)$$

Where $a_1 = \left(\frac{c_1}{c}\right)^2, a_2 = \left(\frac{c_2}{c}\right)^2, a_3 = \left(\frac{c_3}{c}\right)^2, a_4 = \left(\frac{c_4}{c}\right)^2$ and$c$ is the velocity of light. Again c is taken to be unity in order to use as astronomical unit and therefore equation (6) becomes

$$ds^2 = -dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + \left[a_1(dt^1)^2 + a_2(dt^2)^2 + a_3(dt^3)^2 + a_4(dt^4)^2\right]$$

The most general solution of equation (3) is written as equation (4). Therefore putting the value of dt^2 from equation (8) in equation (4) we get,

$$ds^2 = -e^{\lambda} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + e^{\nu}\left[a_1 (dt^1)^2 + a_2 (dt^2)^2 + a_3 (dt^3)^2 + a_4 (dt^4)^2\right] \quad (9)$$

In equation (9) λ and ν are functions of r only; since for spherically symmetric isolated particle the field will depend on r alone and not on θ and ϕ.

Since the gravitational field (i.e. the disturbance from flat-space time) due to a particular diminishes indefinitely as we go to an infinite distance, therefore line element (9) must reduce to Galilean line element (2) at an infinite distance from the particle.

Hence at $r \to \infty; \lambda = 0 = \nu$.

The line element in general relativity is given by

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} \quad (10)$$

Here the co-ordinates are

$$x^1 = r, x^2 = \theta, x^3 = \phi, x^4 = t^1, x^5 = t^2, x^6 = t^3 \ \& \ x^7 = t^4 \quad (11)$$

Comparing equations (9) and (10) with the help of (11) we get

$$g_{\mu\nu} = \begin{bmatrix} -e^{\lambda} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -r^2\sin^2\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 e^{\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_2 e^{\nu} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_3 e^{\nu} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_4 e^{\nu} \end{bmatrix} \quad (12)$$

i.e.

$$\left.\begin{aligned} g_{11} &= -e^{\lambda} \\ g_{22} &= -r^2 \\ g_{33} &= -r^2 \sin^2\theta \\ g_{44} &= a_1 e^{\nu} \\ g_{55} &= a_2 e^{\nu} \\ g_{66} &= a_3 e^{\nu} \\ g_{77} &= a_4 e^{\nu} \\ \& \ g_{\mu\nu} &= 0 \ for \mu \neq \nu \end{aligned}\right\} \quad (13)$$

For our simplicity let we consider,

$$a_4 \approx a_3 \approx a_2 \approx a_1 = 1$$

Then equation (8) becomes

$$dt^2 = \left[(dt^1)^2 + (dt^2)^2 + (dt^3)^2 + (dt^4)^2 \right] \tag{14}$$

And equation (9) becomes

$$ds^2 = -e^{\lambda} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + e^{\nu} \left[(dt^1)^2 + (dt^2)^2 + (dt^3)^2 + (dt^4)^2 \right] \tag{15}$$

And the determinant of $g_{\mu\nu}$ is

$$g = \left| g_{\mu\nu} \right| = -e^{\lambda + 4\nu} r^4 \sin^2 \theta \tag{16}$$

Also $g^{\mu\mu} = \dfrac{1}{g_{\mu\mu}}$ and $g^{\mu\nu} = 0$ for $\mu \neq 0$

i.e.

$$\left.\begin{aligned} & g^{11} = -e^{-\lambda} \\ & g^{22} = -\frac{1}{r^2} \\ & g^{33} = -\frac{1}{r^2 \sin^2 \theta} \\ & g^{44} = g^{55} = g^{66} = g^{77} = e^{-\nu} \\ & \& \, g^{12} = g^{23} = ...= 0 as \mu \neq \nu \end{aligned}\right\} \tag{17}$$

Now the equation (16) can be written as

$$|g| = e^{\lambda + 4\nu} r^4 \sin^2 \theta$$

$$\Rightarrow \sqrt{|g|} = e^{\frac{\lambda + 4\nu}{2}} r^2 \sin \theta$$

$$\left.\begin{aligned} & \therefore \log \sqrt{|g|} = \frac{\lambda + 4\nu}{2} + 2 \log r + \log \sin \theta \\ & and \ \frac{\partial}{\partial r} \left(\log \sqrt{|g|} \right) = \frac{1}{2} \frac{\partial \lambda}{\partial r} + 2 \frac{\partial \nu}{\partial r} + \frac{2}{r} \\ & again \ \frac{\partial^2}{\partial r^2} \left(\log \sqrt{|g|} \right) = \frac{1}{2} \frac{\partial^2 \lambda}{\partial r^2} + 2 \frac{\partial^2 \nu}{\partial r^2} - \frac{2}{r^2} \\ & \frac{\partial}{\partial \theta} \left(\log \sqrt{|g|} \right) = \cot \theta \\ & \frac{\partial^2}{\partial \theta^2} \left(\log \sqrt{|g|} \right) = -\cos ec^2 \theta \\ & and \ \frac{\partial}{\partial \phi} \left(\log \sqrt{|g|} \right) = 0 \end{aligned}\right\} \tag{18}$$

If μ, ν, σ are different suffixes we can now easily get the following possible cases

$$\left.\begin{aligned}
\Gamma^{\mu}_{\mu\mu} &= \frac{1}{2} g^{\mu\mu} \frac{\partial g_{\mu\mu}}{\partial x^{\mu}} = \frac{1}{2} \frac{\partial\left(\log g_{\mu\mu}\right)}{\partial x^{\mu}} \\
\Gamma^{\nu}_{\mu\mu} &= -\frac{1}{2} g^{\nu\nu} \frac{\partial g_{\mu\mu}}{\partial x^{\nu}} \\
\Gamma^{\nu}_{\mu\nu} &= \frac{1}{2} g^{\nu\nu} \frac{\partial g_{\nu\nu}}{\partial x^{\mu}} = \frac{1}{2} \frac{\partial\left(\log_{\nu\nu}\right)}{\partial x^{\mu}} \\
\Gamma^{\sigma}_{\mu\nu} &= 0
\end{aligned}\right\} \qquad (19)$$

Hence we get the following independent non-vanishing 3-index symbols,

$$\left.\begin{aligned}
&\Gamma^{1}_{11} = \frac{1}{2}\frac{\partial\lambda}{\partial r}; \Gamma^{1}_{22} = -re^{-\lambda}; \Gamma^{1}_{33} = -r\sin^2\theta e^{-\lambda} \\
&\Gamma^{1}_{44} = \Gamma^{1}_{55} = \Gamma^{1}_{66} = \Gamma^{1}_{77} = \frac{1}{2} e^{\nu-\lambda} \frac{\partial\nu}{\partial r} \\
&\Gamma^{2}_{33} = -\sin\theta\cos\theta \\
&\Gamma^{2}_{12} = \frac{1}{r}; \Gamma^{3}_{13} = \frac{1}{r}; \Gamma^{3}_{23} = \cot\theta \\
&\Gamma^{4}_{14} = \Gamma^{4}_{15} = \Gamma^{4}_{16} = \Gamma^{4}_{17} = \frac{1}{2}\frac{\partial\nu}{\partial r} \\
&\Gamma^{5}_{14} = \Gamma^{5}_{15} = \Gamma^{5}_{16} = \Gamma^{5}_{17} = \frac{1}{2}\frac{\partial\nu}{\partial r} \\
&\Gamma^{6}_{14} = \Gamma^{6}_{15} = \Gamma^{6}_{16} = \Gamma^{6}_{17} = \frac{1}{2}\frac{\partial\nu}{\partial r} \\
&\Gamma^{7}_{14} = \Gamma^{7}_{15} = \Gamma^{7}_{16} = \Gamma^{7}_{17} = \frac{1}{2}\frac{\partial\nu}{\partial r}
\end{aligned}\right\} \qquad (20)$$

And all others are zero.

We have

$$R_{\mu\nu} = \frac{\partial}{\partial x^{\nu}} \Gamma^{\beta}_{\mu\beta} - \frac{\partial}{\partial x^{\beta}} \Gamma^{\beta}_{\mu\nu} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}$$

Gives us

$$R_{11} = 2\frac{\partial^2\nu}{\partial r^2} + 4\left(\frac{\partial\nu}{\partial r}\right)^2 - \frac{1}{r}\frac{\partial\lambda}{\partial r} - \frac{\partial\lambda}{\partial r}\frac{\partial\nu}{\partial r} \qquad (21)$$

$$R_{22} = \left[e^{-\lambda}\left(1 + 2r\frac{\partial\nu}{\partial r} - \frac{r}{2}\frac{\partial\lambda}{\partial r}\right) - 1\right] \qquad (22)$$

$$R_{33} = \left[e^{-\lambda}\left(1 + 2r\frac{\partial\nu}{\partial r} - \frac{r}{2}\frac{\partial\lambda}{\partial r}\right) - 1\right]\sin^2\theta \quad (23)$$

$$R_{44} = R_{55} = R_{66} = R_{77} = -\frac{1}{2} e^{\nu-\lambda}\left[\frac{\partial^2\nu}{\partial r^2} + 2\left(\frac{\partial\nu}{\partial r}\right)^2 - \frac{1}{2}\frac{\partial\nu}{\partial r}\frac{\partial\lambda}{\partial r} + \frac{2}{r}\frac{\partial\nu}{\partial r}\right] \qquad (24)$$

Obviously equation (23) is a mere repetition of equation (22).

Thus the only Einstein's field equations for empty space to be satisfied by λ and ν are

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$$2\frac{\partial^2 \nu}{\partial r^2}+4\left(\frac{\partial \nu}{\partial r}\right)^2-\frac{1}{r}\frac{\partial \lambda}{\partial r}-\frac{\partial \lambda}{\partial r}\frac{\partial \nu}{\partial r}=0 \quad (25)$$

$$\left[e^{-\lambda}\left(1+2r\frac{\partial \nu}{\partial r}-\frac{r}{2}\frac{\partial \lambda}{\partial r}\right)-1\right]=0 \qquad (26)$$

$$\frac{1}{2}e^{\nu-\lambda}\left[\frac{\partial^2 \nu}{\partial r^2}+2\left(\frac{\partial \nu}{\partial r}\right)^2-\frac{1}{2}\frac{\partial \nu}{\partial r}\frac{\partial \lambda}{\partial r}+\frac{2}{r}\frac{\partial \nu}{\partial r}\right]=0 \qquad (27)$$

Dividing equation (27) by $e^{\nu-\lambda}$ and then subtracting equation (25) from the resulting equation, we get

$$\frac{4}{r}\frac{\partial \nu}{\partial r}+\frac{1}{r}\frac{\partial \lambda}{\partial r}=0$$

$$\Rightarrow \frac{\partial}{\partial r}(4\nu)+\frac{\partial}{\partial r}(\lambda)=0$$

$$\Rightarrow \frac{\partial}{\partial r}(4\nu+\lambda)=0$$

Integrating we get

$$4\nu+\lambda=A$$

Where A is constant of integration which may be set equal to zero, without any loss of generality, since at $r\to\infty, \lambda=0$ and $\nu=0$. Hence,

$$\lambda=-4\nu$$

Substituting this in equation (26) we get

$$e^{4\nu}\left(1+4r\frac{\partial \nu}{\partial r}\right)=1$$

$$\Rightarrow \frac{\partial}{\partial r}\left(re^{4\nu}\right)=1$$

Integrating we get

$$re^{4\nu}=r+B$$

B being constant of integration,

i.e. $e^{4\nu}=1+\frac{B}{r}=1-\frac{2m}{r}$ (28)

$$e^{\nu}=\left(1-\frac{2m}{r}\right)^{\frac{1}{4}}\cong\left(1-\frac{m}{2r}\right) \quad \text{approximately}$$

Here we have put $B=-2m$. This is done in order to facilitate the physical interpretation of m as the mass of the gravitating particle. Thus the line element due to a static, isolated gravitating mass point, the equation (15) becomes,

$$ds^2=-\left(1-\frac{2m}{r}\right)^{-1}dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2+\left(1-\frac{m}{2r}\right)\left[(dt^1)^2+(dt^2)^2+(dt^3)^2+(dt^4)^2\right] (29)$$

Considering equation (14) and putting in above equation (29) we get,

$$ds^2 = -\left(1-\frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + \left(1-\frac{m}{2r}\right) dt^2 \quad (30)$$

III. CONCLUSIONS

In equation (30) the Schwarzschild-like solution becomes singular at a distance r is given by $\left(1-\frac{m}{2r}\right)=0$ i.e. $r=\frac{m}{2}$. For points $0 \le r \le \frac{m}{2}$, $ds^2 < 0$ i.e. the interval is purely space-like. Hence there is a finite singular region for $0 \le r \le \frac{m}{2}$. Thus $r=\frac{m}{2}$ represents the boundary of singular region inside the spherical distribution of matter.

In equation (30) the term $\left(1-\frac{2m}{r}\right)=0$ i.e.r=2$m$the region $\frac{m}{2} < r \le 2m$ represents where $dt^2$is not zero. This region is time-like and very interesting i.e. time is there and meaning is that in that region still the four fundamental forces are interacting. Thus *r=2m*known as Schwarzschild radius represents the boundary of the isolated particle and the solution holds in empty space outside the spherical distribution of matter whose radius must be greater than 2m.

So the equation (30) holds for both interior and exterior solution.

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