ISSN 2250-3153

On the Number of Zeros of a Polynomial in a Given Domain

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Abstract: In this paper we obtain bounds for the number of zeros of a polynomial in a given domain when the coefficients of the polynomial or their real or imaginary parts are restricted to certain conditions. Our results generalize some known results in the theory of the distribution of zeros of polynomials.

Mathematics Subject Classification: 30C10, 30C15

Keywords and phrases: Coefficient, Polynomial, Zero.

1. Introduction and Statement of Results

In the literature we find a large number of published research papers (e.g.[2,4,5,7,8,9,10,11,12,13,16,17,19,20]) concerning the number of zeros of a polynomial in a circle.

S. K. Singh [18] proved the following result on the number of zeros of a polynomial:

Theorem A: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\min_{0\le j\le n}|a_j|\ge 1$ and $\max_{0\le j\le n}|a_j|\le |a_n|$, then the number of zeros of P(z) in $|z|\le \frac{R}{k}$ does not exceed

$$\frac{2\log\{(n+1)|a_n|R^n\}}{\log k}$$

where

$$R=\max\{\left|\frac{a_{n-1}}{a_n}\right|,\left|\frac{a_{n-2}}{a_n}\right|,\left|\frac{a_{n-3}}{a_n}\right|,.....\}.$$

For the class of polynomials with real coefficients, Q. G. Mohammad [14] proved the following result:

Theorem B: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$$a_n \ge a_{n-1}\ge ......\ge a_1\ge a_0>0.$$

Then the number of zeros of P(z) in $|z| \le \frac{1}{2}$ does not exceed

$$1+\frac{1}{\log 2}\log\frac{a_n}{a_0}.$$

Bidkham an d Dewan [3] generalized Theorem B in the following way:

Theorem C: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$$a_n \le a_{n-1} \le ...... \le a_{k+1} \le a_k \ge a_{k-1} \ge ...... \ge a_1 \ge a_0 > 0,$$

for some $k, 0 \le k \le n$. Then the number of zeros of P(z) in $|z| \le \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2}\log\left\{\frac{|a_n|+|a_0|-a_n-a_0+2a_k}{|a_0|}\right\}.$$

Theorem D: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n with complex coefficients such that for some real α, β, $|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, 0 \le j \le n,$ and for some $0 < t \le 1,\ 0 \le k \le n,$

$$|a_0| \le t|a_1| \le ...... \le t^k|a_k| \ge t^{k+1}|a_{k+1}| \ge ...... \ge t^n|a_n|.$$

Then the number of zeros of P(z) in $|z| \le \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2}\log\left\{\frac{2t^{k+1}|a_k|\cos\alpha + 2\sin\alpha\sum_{j=0}^{n} t^j|a_j| - t^{n+1}|a_n|(\cos\alpha+\sin\alpha-1)}{t|a_0|}\right\}.$$

Ebadian et al [6] generalized the above results by proving the following results:

Theorem E: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$$a_n \le a_{n-1} \le ...... \le a_{k+1} \le a_k \ge a_{k-1} \ge ...... \ge a_0$$

for some k, $0 \le k \le n$.Then the number of zeros of P(z) in $|z| \le \frac{R}{2}$,

R>0, does not exceed

$$\frac{1}{\log 2}\log\left\{\frac{|a_n|R^{n+1}+|a_0|+R^k(a_k-a_0)+R^n(a_k-a_n)}{|a_0|}\right\} \text{ for } R \ge 1$$

and

$$\frac{1}{\log 2}\log\left\{\frac{|a_n|R^{n+1}+|a_0|+R(a_k-a_0)+R^n(a_k-a_n)}{|a_0|}\right\} \text{ for } R \le 1 .$$

Theorem F: Let $P(z)=\sum_{j=0}^{\infty}a_jz^j$ be a polynomial o f degree n with complex coefficients such that for some real α,β, $|\arg a_j-\beta| \le \alpha \le \frac{\pi}{2}, 0 \le j \le n,$ and for some $R>0,\ 0 \le k \le n,$

$$|a_0| \le R|a_1| \le ...... \le R^k|a_k| \ge R^{k+1}|a_{k+1}| \ge ...... \ge R^n|a_n| .$$

Then the number of zeros of P(z) in $|z| \le \frac{R}{2}$ does not exceed

$$\frac{1}{\log 2}\log\left\{\frac{2R^{k+1}|a_k|\cos\alpha+2R\sin\alpha\sum_{j=0}^{n}R^j|a_j|-R^{n+1}|a_n|(\cos\alpha+\sin\alpha-1)}{R|a_0|}\right\}.$$

In this paper we give generalizations of Theorems E and F. More precisely we prove the following results:

Theorem 1: Let $P(z)=\sum_{j=0}^{\infty}a_jz^j$ be a polynomial o f degree n with $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ such that for some $k,\tau,\lambda, 0<k \le 1, 0<\tau \le 1,\ 0 \le \lambda \le n$,

$$k\alpha_n \le \alpha_{n-1} \le ...... \le \alpha_{\lambda+1} \le \alpha_\lambda \ge \alpha_{\lambda-1} \ge ...... \ge \tau\alpha_0 .$$

Then the number of zeros of P(z) in $|z| \le \frac{R}{c}(R>0, c>1)$ does not exceed

$$\frac{1}{\log c}\log\left\{\frac{\begin{array}{l}|a_n|R^{n+1}+|a_0|+R^{\lambda}[\alpha_\lambda-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_\lambda|+2\sum_{j=1}^{\lambda-1}|\beta_j|]\\ +R^n[|\alpha_n|-k(|\alpha_n|+\alpha_n)+\alpha_\lambda+|\beta_\lambda|+|\beta_n|+2\sum_{j=\lambda+1}^{n-1}|\beta_j|]\end{array}}{|a_0|}\right\}$$

for $R\geq 1$

and

$$\frac{1}{\log c}\log\left\{\frac{\begin{array}{l}|a_n|R^{n+1}+|a_0|+R[\alpha_\lambda-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_\lambda|+2\sum_{j=1}^{\lambda-1}|\beta_j|]\\ +R^n[|\alpha_n|-k(|\alpha_n|+\alpha_n)+\alpha_\lambda+|\beta_\lambda|+|\beta_n|+2\sum_{j=\lambda+1}^{n-1}|\beta_j|]\end{array}}{|a_0|}\right\}$$

for $R\leq 1$.

Remark 1: Taking $k=1,\tau=1$, c=2,Theorem 1 reduces to Theorem E if a_j are real i.e. $\beta_j=0,\forall j$.

For different values of the parameters k,τ,λ etc. we get many interesting results e.g. if we take $\tau=1$, we get the following result:

Corollary 1: Let $P(z)=\sum_{j=0}^{\infty}a_j z^j$ be a polynomial o f degree n with $\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j$ such that for some $k, 0<k\leq 1$,

$$k\alpha_n\leq\alpha_{n-1}\leq......\leq\alpha_{\lambda+1}\leq\alpha_\lambda\geq\alpha_{\lambda-1}\geq......\geq\alpha_0,$$

$0\leq\lambda\leq n$. Then the number of zeros of P(z) in $|z|\leq\frac{R}{c}(R>0,c>1)$ does not exceed

$$\frac{1}{\log c}\log\left\{\frac{|a_n|R^{n+1}+|a_0|+R^{\lambda}[\alpha_\lambda-\alpha_0+|\beta_0|+|\beta_\lambda|+2\sum_{j=1}^{\lambda-1}|\beta_j|] + R^n[|\alpha_n|-k(|\alpha_n|+\alpha_n)+\alpha_\lambda+|\beta_\lambda|+|\beta_n|+2\sum_{j=\lambda+1}^{n-1}|\beta_j|]}{|a_0|}\right\}$$

for $R\geq 1$

and

$$\frac{1}{\log c}\log\left\{\frac{|a_n|R^{n+1}+|a_0|+R[\alpha_\lambda-\alpha_0+|\beta_0|+|\beta_\lambda|+2\sum_{j=1}^{\lambda-1}|\beta_j|] + R^n[|\alpha_n|-k(|\alpha_n|+\alpha_n)+\alpha_\lambda+|\beta_\lambda|+|\beta_n|+2\sum_{j=\lambda+1}^{n-1}|\beta_j|]}{|a_0|}\right\}$$

for $R\leq 1$.

Taking $k=1,\lambda=n$ in Theorem 1, we get the following result:

Corollary 2: Let $P(z)=\sum_{j=0}^{\infty}a_jz^j$ be a polynomial o f degree n with $\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j$ such that for some $\tau, 0<\tau\leq 1,$

$$\alpha_n\geq\alpha_{n-1}\geq......\geq\tau\alpha_0.$$

Then the number of zeros of P(z) in $|z|\leq\frac{R}{c}(R>0,c>1)$ does not exceed

$$\frac{1}{\log c}\log\left\{\frac{|a_0|+R^{n+1}[|\alpha_n|+\alpha_n-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+2\sum_{j=1}^{n}|\beta_j|]}{|a_0|}\right\}$$

for $R\geq 1$

and

$$\frac{1}{\log c}\log\left\{\frac{|a_0|+R[|\alpha_n|+\alpha_n-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+2\sum_{j=1}^{n}|\beta_j|]}{|a_0|}\right\}$$

for $R\leq 1$.

Taking $\tau=1,\lambda=0$ in Theorem 1, we get the following result:

Corollary 3: Let $P(z)=\sum_{j=0}^{\infty}a_j z^j$ be a polynomial o f degree n with $\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j$ such that for some $k, 0<k\leq 1$,

$$k\alpha_n\leq\alpha_{n-1}\leq......a_1\leq\alpha_0,$$

then the number of zeros of P(z) in $|z|\leq\frac{R}{c}(R>0,c>1)$ does not exceed

$$\frac{1}{\log c}\log\left\{\frac{|a_0|+R^{n+1}[2|\alpha_n|-k(|\alpha_n|+\alpha_n)+\alpha_0+|\beta_0|+2\sum_{j=1}^{n}|\beta_j|]}{|a_0|}\right\}$$

for $R\geq 1$

and

$$\frac{1}{\log c}\log\left\{\frac{|a_0|+R[2|\alpha_n|-k(|\alpha_n|+\alpha_n)+\alpha_0+|\beta_0|+2\sum_{j=1}^{n}|\beta_j|]}{|a_0|}\right\}$$

for $R\leq 1$.

Applying Theorem 1 to the polynomial –iP(z) , we get the following result:

Theorem 2: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n with $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ such that for some $k, 0<k\le 1, 0<\tau\le 1,$

$$k\alpha\beta_n \le \beta_{n-1} \le \ldots\ldots \le \beta_{\lambda+1} \le \beta_\lambda \ge \beta_{\lambda-1} \ge \ldots\ldots \ge \tau\beta_0,$$

$0\le\lambda\le n$, then the number of zeros of P(z) in $|z|\le \frac{R}{c}(R>0, c>1)$ does not exceed

$$\frac{1}{\log c}\log\left\{\frac{\begin{array}{l}|a_n|R^{n+1}+|a_0|+R^\lambda[\beta_\lambda-\tau(|\beta_0|+\beta_0)+|\beta_0|+|\alpha_0|+|\alpha_\lambda|+2\sum_{j=1}^{\lambda-1}|\alpha_j|]\\ +R^n[|\beta_n|-k(|\beta_n|+\beta_n)+\beta_\lambda+|\alpha_\lambda|+|\alpha_n|+2\sum_{j=\lambda+1}^{n-1}|\alpha_j|]\end{array}}{|a_0|}\right\}$$

for $R\ge 1$

and

$$\frac{1}{\log c}\log\left\{\frac{\begin{array}{l}|a_n|R^{n+1}+|a_0|+R[\beta_\lambda-\tau(|\beta_0|+\beta_0)+|\beta_0|+|\alpha_0|+|\alpha_\lambda|+2\sum_{j=1}^{\lambda-1}|\alpha_j|]\\ +R^n[|\beta_n|-k(|\beta_n|+\beta_n)+\beta_\lambda+|\alpha_\lambda|+|\alpha_n|+2\sum_{j=\lambda+1}^{n-1}|\alpha_j|]\end{array}}{|a_0|}\right\}$$

for $R\le 1$.

Theorem 3: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n with complex coefficients such that for some real α,β, $|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2}, 0\le j\le n,$ and for some $R>0,\ 0\le\lambda\le n, 0<k\le 1, 0<\tau\le 1,$

$$\tau|a_0|\le R|a_1|\le \ldots\ldots \le R^\lambda|a_\lambda|\ge R^{\lambda+1}|a_{\lambda+1}|\ge\ldots\ldots\ge kR^n|a_n|.$$

Then the number of zeros of P(z) in $|z| \le \frac{R}{c} (R > 0, c > 1)$ does not exceed

$$\frac{1}{\log c} \log[\frac{1}{R|a_0|} \{2R^{\lambda+1}|a_\lambda| \cos\alpha + 2R\sin\alpha \sum_{j=1}^{n-1} R^j |a_j| - k|a_n| R^{n+1}(\cos\alpha - \sin\alpha + 1) + 2R^{n+1}|a_n|$$

$$-\tau R|a_0|(\cos\alpha - \sin\alpha + 1) + 2R|a_0|\}].$$

Remark 2: For different values of the parameters k, τ, λ etc., we get many interesting results from Theorem 3 as in case of Theorem 1. For $k = 1, \tau = 1$, c=2,Theorem 3 reduces to Theorem F.

2.Lemmas

For the proofs of the above results we need the following results:

Lemma 1: If f(z) is analytic in $|z| \le R$,but not identically zero, f(0)$\ne$0 and

$f(a_k) = 0, k = 1,2,......,n$, then

$$\frac{1}{2\pi} \int_0^{2\pi} \log|f(\mathrm{Re}^{i\theta}|d\theta - \log|f(0)| = \sum_{j=1}^{n} \log \frac{R}{|a_j|}.$$

Lemma 1 is the famous Jensen's theorem (see page 208 of [1]).

Lemma 2: If f(z) is analytic and $|f(z)| \le M(r)$ in $|z| \le r$, then the number of zeros of f(z) in $|z| \le \frac{r}{c}$ (r>0,c>1)does not exceed

$$\frac{1}{\log c} \log \frac{M(r)}{|f(0)|}.$$

Lemma 2 is a simple deduction from Lemma 1.

Lemma 3: Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n with complex coefficients such that

for some real α, β, $|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, 0 \le j \le n,$ and

$|a_j| \ge |a_{j-1}|, 0 \le j \le n,$ then for any t>0,

$$\left|ta_j - a_{j-1}\right| \le (t\left|a_j\right| - \left|a_{j-1}\right|)\cos\alpha + (t\left|a_j\right| + \left|a_{j-1}\right|)\sin\alpha .$$

Lemma 3 is due to Govil and Rahman [7].

3.Proofs of Theorems

Proof of Theorem 1: Consider the polynomial

F(z) =(1-z)P(z)

$$= (1-z)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0)$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_1 - a_0)z + a_0$$

$$= -a_n z^{n+1} + a_0 + [(ka_n - a_{n-1}) - (k-1)a_n]z^n + \sum_{j=\lambda+1}^{n-1}(a_j - a_{j-1})z^j$$

$$+ \sum_{j=2}^{\lambda}(a_j - a_{j-1})z^j + [(a_1 - \tau a_0) + (\tau - 1)a_0]z$$

For $|z| \le R$, we have, by using the hypothesis

$$\left|F(z)\right| \le \left|a_n\right|R^{n+1} + \left|a_0\right| + [(\alpha_{n-1} - k\alpha_n) + (1-k)\left|\alpha_n\right|]R^n + \sum_{j=\lambda+1}^{n-1}(\alpha_{j-1} - \alpha_j)R^j$$

$$+ \sum_{j=1}^{\lambda}(\alpha_j - \alpha_{j-1})R^j + [(\alpha_1 - \tau\alpha_0) + (1-\tau)\left|\alpha_0\right|]R + \sum_{j=1}^{n}(\left|\beta_j\right| + \left|\beta_{j-1}\right|)R^j .$$

Which gives

$$\left|F(z)\right| \le \left|a_n\right|R^{n+1} + \left|a_0\right| + R^n[\left|\alpha_n\right| - k(\left|\alpha_n\right| + \alpha_n) + \alpha_\lambda + \left|\beta_\lambda\right| + \left|\beta_n\right| + 2\sum_{j=\lambda+1}^{n-1}\left|\beta_j\right|]$$

$$+ R^\lambda[\alpha_\lambda - \tau(\left|\alpha_0\right| + \alpha_0) + \left|\alpha_0\right| + \left|\beta_0\right| + \left|\beta_\lambda\right| + 2\sum_{j=1}^{\lambda-1}\left|\beta_j\right|]$$

for $R \ge 1$

and

$$\left|F(z)\right| \le \left|a_n\right|R^{n+1} + \left|a_0\right| + R^n[\left|\alpha_n\right| - k(\left|\alpha_n\right| + \alpha_n) + \alpha_\lambda + \left|\beta_\lambda\right| + \left|\beta_n\right| + 2\sum_{j=\lambda+1}^{n-1}\left|\beta_j\right|]$$

$$+R[\alpha_\lambda-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_\lambda|+2\sum_{j=1}^{\lambda-1}|\beta_j|]$$

for $R\leq 1$.

Thus

$$\frac{|F(z)|}{|F(0)|}\leq\frac{1}{|a_0|}[|a_n|R^{n+1}+|a_0|+R^n\{|\alpha_n|-k(|\alpha_n|+\alpha_n)+\alpha_\lambda+|\beta_\lambda|+|\beta_n|+2\sum_{j=\lambda+1}^{n-1}|\beta_j|\}$$

$$+R^\lambda[\alpha_\lambda-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_\lambda|+2\sum_{j=1}^{\lambda-1}|\beta_j|]$$

for $R\geq 1$

and

$$\frac{|F(z)|}{|F(0)|}\leq\frac{1}{|a_0|}[|a_n|R^{n+1}+|a_0|+R^n\{|\alpha_n|-k(|\alpha_n|+\alpha_n)+\alpha_\lambda+|\beta_\lambda|+|\beta_n|+2\sum_{j=\lambda+1}^{n-1}|\beta_j|\}$$

$$+R[\alpha_\lambda-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_\lambda|+2\sum_{j=1}^{\lambda-1}|\beta_j|]$$

for $R\leq 1$.

Hence , by Lemma 2,it follows that the number of zeros of F(z) in $|z|\leq\frac{R}{c}(R>0,c>1)$ does not exceed

$$\frac{1}{\log c}\log\left\{\frac{\begin{array}{l}|a_n|R^{n+1}+|a_0|+R^\lambda[\alpha_\lambda-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_\lambda|+2\sum_{j=1}^{\lambda-1}|\beta_j|]\\+R^n[|\alpha_n|-k(|\alpha_n|+\alpha_n)+\alpha_\lambda+|\beta_\lambda|+|\beta_n|+2\sum_{j=\lambda+1}^{n-1}|\beta_j|]\end{array}}{|a_0|}\right\}$$

for $R\geq 1$

and

$$\frac{1}{\log c}\log\left\{\frac{\begin{array}{l}|a_n|R^{n+1}+|a_0|+R[\alpha_\lambda-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+|\beta_\lambda|+2\sum_{j=1}^{\lambda-1}|\beta_j|]\\ +R^n[|\alpha_n|-k(|\alpha_n|+\alpha_n)+\alpha_\lambda+|\beta_\lambda|+|\beta_n|+2\sum_{j=\lambda+1}^{n-1}|\beta_j|]\end{array}}{|a_0|}\right\}$$

for $R\leq 1$.

As the number of zeros of P(z) in $|z|\leq\frac{R}{c}(R>0, c>1)$ does not exceed the number of zeros of F(z) in $|z|\leq\frac{R}{c}(R>0, c>1)$, the theorem follows.

Proof of Theorem 3: Consider the polynomial

F(z) =(R-z)P(z)

$$=(R-z)(a_n z^n+a_{n-1}z^{n-1}+......+a_1 z+a_0)$$

$$=-a_n z^{n+1}+Ra_0+(Ra_n-a_{n-1})z^n+(Ra_{n-1}-a_{n-2})z^{n-1}+......+(Ra_1-a_0)z$$

$$=-a_n z^{n+1}+Ra_0+[(kRa_n-a_{n-1})-(kRa_n-Ra_n)]z^n+(Ra_{n-1}-a_{n-2})z^{n-1}$$

$$+.......+(Ra_{\lambda+1}-a_\lambda)z^{\lambda+1}+(Ra_\lambda-a_{\lambda-1})z^\lambda+......+(Ra_2-a_1)z^2$$
$$+[(Ra_1-\tau a_0)+(\tau a_0-a_0)]z$$

For $|z|\leq R$, we have

$$|F(z)|\leq|a_n|R^{n+1}+R|a_0|+|k-1|R^{n+1}|a_n|+|kRa_n-a_{n-1}|R^n+|Ra_{n-1}-a_{n-2}|R^{n-1}$$

$$+......+|Ra_{\lambda+1}-a_\lambda|R^{\lambda+1}+|Ra_\lambda-a_{\lambda-1}|R^\lambda+......+|Ra_2-a_1|R^2$$
$$+|Ra_1-\tau a_0|R+(\tau-1)|a_0|R,$$

which gives ,by using Lemma 3and the hypothesis

$$|F(z)|\leq|a_n|R^{n+1}+R|a_0|+(1-k)R^{n+1}|a_n|+[(|a_{n-1}|-kR|a_n|)\cos\alpha+(|a_{n-1}|+kR|a_n|)\sin\alpha]R^n$$

ISSN 2250-3153

$$+[(|a_{n-2}|-R|a_{n-1}|)\cos\alpha+(|a_{n-2}|+R|a_{n-1}|)\sin\alpha]R^{n-1}+\ldots\ldots$$
$$+[(|a_{\lambda}|-R|a_{\lambda+1}|)\cos\alpha+(|a_{\lambda}|+R|a_{\lambda+1}|)\sin\alpha]R^{\lambda+1}$$
$$+[(R|a_{\lambda}|-|a_{\lambda-1}|)\cos\alpha+(R|a_{\lambda}|+|a_{\lambda-1}|)\sin\alpha]R^{\lambda}+\ldots\ldots$$
$$+[(R|a_2|-|a_1|)\cos\alpha+(R|a_2|+|a_1|)\sin\alpha]R^2+[(R|a_1|-\tau|a_0|)\cos\alpha$$
$$+(R|a_1|+\tau|a_0|)\sin\alpha]R+(1-\tau)|a_0|R$$

$$=2R^{\lambda+1}|a_{\lambda}|\cos\alpha-k|a_n|R^{n+1}(\cos\alpha-\sin\alpha+1)+2R^{n+1}|a_n|$$

$$-\tau R|a_0|(\cos\alpha-\sin\alpha+1)+2R|a_0|+2R\sin\alpha\sum_{j=1}^{n-1}R^j|a_j|$$

$$=2R^{\lambda+1}|a_{\lambda}|\cos\alpha+2R\sin\alpha\sum_{j=1}^{n-1}R^j|a_j|-k|a_n|R^{n+1}(\cos\alpha-\sin\alpha+1)+2R^{n+1}|a_n|$$

$$-\tau R|a_0|(\cos\alpha-\sin\alpha+1)+2R|a_0|.$$

The rest of the proof can be completed on the same lines as in the proof of Theorem 1.

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