Certain Case of Reducible Hypergeometric Functions of Hyperbolic Function as Argument

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Abstract- In this paper, we specialized parameters and argument, Hypergeometric function F_E (α_1, α_1, α_1, β_1, β_2, β_2; γ_1, γ_2, γ_3; cosh x, cosh y, cosh z) F_G, F_k and F_N can be reduced to the hypergeometric function of Bailey's $F_4(\alpha_1, \beta_2, \gamma_2, \gamma_3$; -coshy, - cosh z) and also discussed their reducible cases into Horn's function. In the journal we consider hypergeometric function of three variables and obtain its interesting reducible case into Bailey's F_4 & Horn's function.
    In the section 2, hypergeometric function of four variables can be reduced to the hypergeometric function of one, two & three variables with some new and interesting particular cases.

Index Terms- Matrix argument, Confluent, Hypergeometric function, Beta and Gamma integrals, M-transform

I.   ON HYPERGEMOTERIC INTEGRALS

1.1 INTRODUCTION

We will study Laplace's double integral for Saran's function F_E (α_1, α_1, α_1, β_1, β_2, β_2; γ_1, γ_2, γ_3; cosh x, cosh y, coshz) which has been reduced to Bailey's F_4 (α_1, β_2, γ_2, γ_3; - cosh y, - cosh z) and pochhamer type of Integrals for F_E, F_G, F_K and F_N and also discussed their reducible cases into Horn's functions. The purpose of studing only the function F_E, F_G, F_K and F_N is mainly due to the function in their integral representation contain Appell's function F_1 or F_2 or the product of Gauss's hypergeometric series which can be reduced by the following relations.

$$_2F_1 (\alpha, \beta; \gamma; \cosh y) = \sum_{k,m=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n m!} \cosh^m y \tag{1.1.1}$$

$$_2F_1 (\alpha, \beta; \gamma; \cosh y) = \sum_{k,m=0}^{\infty} \frac{(\alpha)_n (\beta)_n (e^y - e^{-y})^m}{(\gamma)_n m! 2m} \tag{1.1.2}$$

$$_2F_1 (\alpha, \beta; \gamma; \cosh y) = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \beta)\Gamma(\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-t\cosh y)^{-\alpha} dt \tag{1.1.3}$$

$$_2F_1 (\alpha, \beta; \gamma; \cosh y) = (1\text{- } \cosh y)^{-\alpha} \,_2F_1 (\alpha, \gamma\text{-}\beta; \gamma; \frac{\cosh y}{\cosh y - 1}) \tag{1.1.4}$$

Similarly

$$_2F_1 (\alpha, \beta; \gamma; \cosh y) = (1\text{- } \cosh y)^{-\beta} \,_2F_1 (\gamma - \alpha, \beta; \gamma; \frac{\cosh y}{\cosh y - 1}) \tag{1.1.5}$$

$$_2F_1 (\alpha, \beta; \gamma; \cosh y) = (1\text{- } \cosh y)^{\gamma-\alpha-\beta} \,_2F_1 (\gamma\text{-}\alpha, \gamma\text{-}\beta; \gamma; \cosh y) \tag{1.1.6}$$

$$_2F_1 (\alpha, \beta; \gamma; \cosh y) = (1\text{- } \cosh y)^{-\alpha} \tag{1.1.7}$$

$$F_1(\alpha, \beta, \beta', \alpha: \cosh x, \cosh y) = (1 - \cosh y)^{-\alpha} (1 - \cosh y)^{-\beta} \tag{1.1.8}$$

$F_1(\alpha,\beta,\beta',\beta,\beta'; \cosh x, \cosh y) = (1 - \cosh x - \cosh y)^{-\alpha}$ (1.1.9)

$F_2(\alpha,\beta,\beta',\alpha,\beta'; \cosh x, \cosh y) = (1-\cosh y)^{-\beta-\alpha}(1-\cosh x - \cosh y)^{-\beta}$ (1.10)

where

$|\cosh x| < 1 \mid \cosh y| < 1 | \text{Cosh } z| < 1$ for (1.1.1) to (1.1.10)

1.2 Reduction of integrals of $F_E(.)$ INTO BAIEY'S F_4 (.)

The hypergeometric function F_E is defined by

$F_1(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_2; \gamma_1, \gamma_2, \gamma_3; \cosh x, \cosh y, \cosh z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_{m+n+p}(\beta_1)_m (b_2)_{n+p} \cosh^m x \cosh^n y \cosh^p z}{(\gamma_1)_m (\gamma_2)_n (\gamma_3)_p m!n!p!}$$ (1.2.1)

absolutely convergent if $m + (\sqrt{n} + \sqrt{p})^2 = 1$, $|\cosh x| < 1$, $|\cosh y| < 1$, $|\cosh z| < 1$ and $|\cosh z| < 1$ and It's integral representation S. Saran (1957) is given by

$$F_E = \frac{1}{\Gamma(\alpha_1)\Gamma(\beta_2)} \int_0^\infty \int_0^\infty e^{-p-t} p^{\alpha 1^{-1}} t^{\beta_2 - 1} {}_1F_1(\beta_1; \gamma_1; p\cosh x)$$

$X_0F_1(\gamma_2;- pt \cosh y)_0F_1(\gamma_3;- pt \cosh z)\, dpdt$ (1.2.2)

Where $R(\alpha_1) > 0$ and $R(\alpha_2) > 0$

$F_4(\alpha_1, \beta_2, \gamma_2, \gamma_3; -\cosh y, -\cosh z)$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\beta_2)} \int_0^\infty \int_0^\infty e^{-p-t} p^{\alpha 1^{-1}} t^{\beta_2 - 1} {}_0F_1(\gamma_2;-p\cosh x)$$

$X_0F_1(\gamma_3;- pt \cosh y)\, dp\, dt$ (1.2.3)

Changing the variable t to T by the substitution $t = C^2 T^2/4p$ and writing

$\alpha_1 = \frac{1}{2}(\lambda+\mu+\upsilon-\rho)$ $\beta_2 = \frac{1}{2}(\lambda+\mu+\upsilon+\rho)$

$\gamma_2 = \mu+1 \, \gamma_3 = \upsilon+1$, $\cosh y = \dfrac{a^2}{c^2}$, $\cosh z = \dfrac{b^2}{c^2}$ (1.2.4)

$$F4\left[\frac{1}{2}(\lambda+\mu+v-\rho),\frac{1}{2}(\lambda+\mu+v+\rho),\mu+1,v+1-\frac{a^2}{c^2},-\frac{b^2}{c^2}\right]$$

$$= \frac{e^{(\lambda+\mu+v)}}{2^{(\gamma+\mu+v+\rho-1)}\Gamma\left\{\frac{1}{2}(\gamma+\mu+v-\rho)\right\}\Gamma\left\{\frac{1}{2}(\gamma+\mu+v+\rho)\right\}}$$

$$\int_0^\infty \int_0^\infty e^{-p-\frac{C^2T^2}{4P}} p^{\rho-1} T^{(\lambda+\mu+v+\rho-1)} {}_0F_1(\mu+1;-\frac{1}{4}a^2T^2)_0 F_1(v+1;-\frac{1}{4}b^2T^2)\, dpdt$$

We know that [E.T. Wittaker and G.N. Watson (1902)]

$$\int_0^\infty \exp\{-(cx + a/x)\}dx = 2(c/a)^{\frac{1}{2}v}\, K_v(2\sqrt{ac}\,), \tag{1.2.5}$$

so the p- integral

$$\int_0^\infty e^{-p - \frac{C^2 T^2}{4P}}\, p^{\rho-1}dp = 2(\frac{2}{CT})^\rho k_p(2\sqrt{ac}) \tag{1.2.6}$$

and changing $_0F_1$ in to Bessel function of the kind the relation

$$J_k(z) = \frac{(\frac{1}{2}z)^k}{\Gamma(k+1)}\, _0F_1(k+1; -\frac{1}{4}z^2) \tag{1.2.7}$$

$$\int_0^\infty t^{\lambda-1}J_\mu(at)J_v(bt)J\rho(ct)dt = \frac{2^{\lambda-1}a^\mu b^v \Gamma\left\{\frac{1}{2}(\lambda+\mu+v+\rho)\right\}\Gamma\left\{\frac{1}{2}(\lambda+\mu+v-\rho)\right\}}{c^{\lambda+\mu+v}\Gamma(\mu+1)\Gamma(v+1)}$$

$$F_4\left[\frac{1}{2}(\lambda+\mu+v+p), \frac{1}{2}(\lambda+\mu+v-p), \mu+1, +v+1; \frac{a^2}{c^2}, \frac{b^2}{c^2}\right] \tag{1.2.8}$$

Where F_4 is fourth type of Appell's function

1.3 REDUCTION OF F_E, F_G, F_K AND F_N INTO HORN'S FUNCTION

Let us consider the integral S. Saran (1955) for F_E viz.

$$F_E = \frac{\Gamma(\gamma_2)\Gamma(\gamma_3)\Gamma(2-\gamma_2-\gamma_3)}{(2\pi i)^2}\int_c (-t)^{-\gamma_2}(t-1)^{-\gamma_3}$$

$$F_2(\alpha_1, \beta_1, \beta_2, \gamma_1, \gamma_2+\gamma_3-1; \cosh x, \frac{\cosh y}{t} + \frac{\cosh z}{1-t})\, dt \tag{1.3.1}$$

where

$$|\cosh x| + \left|\frac{\cosh y}{t} + \frac{\cosh z}{1-t}\right| < 1 \text{ along the contour.}$$

Puttting $\beta_1 = \gamma_1$ and $\beta_2 = \gamma_2 + \gamma_3 - 1$ in equation (1.3.1) we get

$$F_E = \frac{\Gamma(\gamma_2)\Gamma(\gamma_3)\Gamma(2-\gamma_2-\gamma_3)}{(2\Pi i)^2}$$

$$\int (-t)^{-\gamma 2}(-t)^{-\gamma 3}(1-\cosh x - \frac{\cosh y}{t} - \frac{\cosh z}{1-t})^{-\alpha 1}\, dt$$

We can expand

$$(1\text{-cosh x} - \frac{\frac{\cosh y}{t} - \frac{\cosh z}{1-t}}{})^{-\alpha 1} = (1\text{-cosh x} - \cosh y)^{-\alpha 1}$$

$$\sum_{m,n=o}^{\infty} \frac{(\alpha_1)_{m+n}}{m!n!} \left(\frac{\cosh y}{1-\cosh x - \cosh y} \cdot \frac{1-t}{t} \right)^m \left(\frac{\cosh z}{1-\cosh x - \cosh y} \cdot \frac{1}{1-t} \right)^m \quad ..... \qquad (1.3.2)$$

Where

$$\left| \frac{\cosh y}{1-\cosh x - \cosh y} \cdot \frac{1-t}{t} \right|_{<1}, \quad \left| \frac{\cosh z}{1-\cosh x - \cosh y} \cdot \frac{1}{1-t} \right|_{<1}$$

along the contour

$$= (1 - \cosh x - \cosh y - \cosh z)^{-\alpha 1}$$

$$\sum_{m,n=o}^{\infty} \frac{(\alpha_1)_{m+n}}{m!n!} \left(\frac{\cosh y}{1-\cosh x - \cosh y - \cosh z} \cdot \frac{1-t}{t} \right)^m \left(\frac{\cosh z}{1-\cosh x - \cosh y - \cosh y} \cdot \frac{1}{1-t} \right)^m \quad ... \ (1.3.3)$$

where

$$\left| \frac{\cosh y}{1-\cosh x - \cosh y - \cosh z} \cdot \frac{1-t}{t} \right|_{<1} \text{ and,} \quad \left| \frac{\cosh z}{1-\cosh x - \cosh y - \cosh z} \cdot \frac{1}{1-t} \right|_{<1}$$

along the contour using (1.3.2) and (1.3.3) and then evaluating the integral, after changing the order of the integration and summation, keeping

$$\int_c (-t)^{\alpha-1}(t-1)^{\beta-1}dt = \frac{(2\pi i)^2}{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(\alpha+\beta)}$$

We will get,

$F_E(\alpha_1, \alpha_1, \alpha_1, \gamma_1, \gamma_2 + \gamma_3 -1, \gamma_2 + \gamma_3 -1, \gamma_1, \gamma_2, \gamma_3, : \cosh x, \cosh y, \cosh z)$
$= (1 - \cosh x - \text{coxh } y)^{-\alpha 1}.$

$$H_1(1\text{-}\gamma_3, \alpha_1, \gamma_2 + \gamma_3\text{-}1, \gamma_2; \frac{\cosh y}{\cosh x + \cosh y \text{ -}1}, \text{''} \frac{\cosh z}{\cosh x + \cosh y \text{ -}1}) \qquad (1.3.4)$$

$= (1\text{-cosh x} - \cosh y - \cosh z)^{-\alpha 1}.$

$$G_1(\alpha_1, 1\text{-}\gamma_2, 1\text{-}\gamma_3, \frac{\cosh y}{1\text{- cosh x} - \cosh y \text{ - cosh z}}, \frac{\cosh z}{1\text{- cosh x} - \cosh y \text{ - cosh z}}) \qquad (1.3.5)$$

When H_1 and G_1 are defined by Horn (1931)

$$G_1(\alpha_1, \beta, \beta'; \cosh x, \cosh y) = \sum_{m,n=o}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{n-m}(\beta')_{m-n}}{m!n!} \cosh^m x \cosh^n y \qquad (1.3.6)$$

$$H_1(\alpha, \beta, \gamma, \delta; \cosh x, \cosh y) = \sum_{m,n=o}^{\infty} \frac{(\alpha)_{m-n}(\beta)_{n+m}(\gamma)_n}{m!n!(\delta)_m} \cosh^m x \cosh^n y \qquad (1.3.7)$$

We also expand $(1\text{-}\cosh x - \dfrac{\cosh y}{t} - \dfrac{\cosh z}{1-t})^{-\alpha 1}$ by taking $=(1\text{-}\cosh x)^{-\alpha 1}$ as factor but that will only reduce F_E to F_4.

Similarly, we have $F_G(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_2, \gamma_3; \cosh x, \cosh y, \cosh z)$

$$= \sum_{m,n=o}^{\infty} \frac{(\alpha_1)_{m+n+p}(\beta_1)_m(\beta_2)_n(\beta_3)_p}{m!n!(\gamma_1)_m(\gamma_2)_{n=p}} \cosh^m x \cosh^n y \cosh^p z \qquad (1.3.8)$$

absolutely convergent if $m + n = 1$ and $m + p = 1$ while

$|\cosh x|<1$, $|\cosh y|<1$, and $|\cosh z|<1$

Its Integral representation is given by S. Saran (1955)

$$F_G = \frac{\Gamma(\rho)\Gamma(\rho_1)\Gamma(2-\rho-\rho_1)}{(2\pi i)^2} \int_c (-t)^{-\rho}(t-1)^{-\rho_1}$$

$${}_2F_1(\rho,\beta_1,:\gamma_1: \frac{\cosh x}{t})F_1(\rho_1,; \beta_2;\beta_2;\gamma_2; \frac{\cosh y}{1-t}+\frac{\cosh z}{1-t})dt \qquad (1.3.9)$$

$${}_2F_1(\alpha,\beta, 2\beta;\cosh x)=(1-\frac{1}{2}\cosh x)^{-\alpha}{}_2F_1\left(\frac{1}{2}\alpha,\frac{1}{2}\alpha+\frac{1}{2},\beta+\frac{1}{2}; \frac{(\cosh x)^2}{(2-\cosh x)^2}\right) \qquad (1.3.10)$$

absolutely convergent if $m + n = 1$ and $m + p = 1$ while $|\cosh x| <1$, $|\cosh y|< 1$, and $|\cosh z|< 1$

It's integral representation is given by

${}_2F_1 (\alpha,\beta,\beta', \beta+\beta'; \cosh x, \cosh y)$

$$(1\text{-}\cosh y)^{-\alpha}{}_2F_1(\alpha,\beta,\beta+\beta'; \frac{\cosh x - \cosh y}{1-\cosh y}) \qquad (1.3.11)$$

using (1.3.10) and then taking the new variable of integration u given by $t = 1/2\cosh x + (1-1/2 \cosh x) u$ in F_G's integral we will get

$$F_G(\gamma_1= 2\beta_1)=(1-\frac{1}{2}Cosh\, x)^{-\alpha 1}\frac{\Gamma(\rho)\Gamma(\rho_1)\Gamma(2-\rho-\rho_1)}{(2\pi i)^2}\int_c(-u)^{-\rho}(u-1)^{-\rho_1}$$

$${}_2F_1\left(\frac{1}{2}\rho,\frac{1}{2}\rho+\frac{1}{2};\beta+\frac{1}{2}; \frac{(\cosh x)^2}{(2-\cosh x)^2 u^2}\right) \qquad (1.3.12)$$

$$F_1(\rho,\beta_2, \beta_3, \gamma_3;(\frac{\cosh x}{(1-\frac{\cosh x}{2})(1-u)},\frac{\cosh z}{(1-\frac{\cosh x}{2})(1-u)})du$$

Now using (1.3.11) after putting $\gamma_2 = \beta_2 + \beta_3$ and introducing a new variable of integration v given by:

$$v = (1\text{-}\cosh z - \frac{1}{2}\cosh x)(1\text{-}v)/ (1 - \frac{1}{2}\cosh x) \qquad (1.3.13)$$

We will get, on Integration

$F_G(\alpha_1, \alpha_1, \alpha_1, \beta_1, \beta_2, \beta_3, 2\beta_1, \beta_2, +\beta_3, \beta_2+ \beta_3;$ cosh x, cosh y, cosh z)

$= (1 - \cosh z - \dfrac{1}{2} \cosh x)^{-\alpha 1}$

$H_4(\alpha_1, \beta_2, \beta_1+ \dfrac{1}{2}, \beta_2, + \beta_3; \dfrac{\mathrm{Cosh}^2 x}{4(2 - \cosh x - 2\cosh z)^2}, \dfrac{\mathrm{Cosh}\, y - \cosh z}{(1 - \cosh z - \frac{1}{2} \cosh x)})$ (1.3.14)

where

$H_4(\alpha, \beta, \gamma, \delta;$ cosh x, cosh y) $= \displaystyle\sum_{m,n=o}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_n}{m!n!(\gamma)_m(\delta)_n} \cosh^m x \cosh^n y$ (1.3.15)

We will now consider the integral S. Saran (1955) for F_k to show that

$F_K(\alpha_1, \alpha_2, \alpha_3, \gamma_1+ \gamma_3 - 1, \gamma_2, \gamma_1+ \gamma_3\text{-}1; \gamma_1, \gamma_2, \gamma_3;$ cosh x, cosh y, cosh z)

$= (1 - \cosh y - \cosh y\, z)^{-\alpha 2}$ (1.3.16)

$H_2(1\text{-}\gamma_1, \alpha_2, \alpha_1, \gamma_1+ \gamma_3\text{-}1, \gamma_3; \dfrac{\cosh x}{\cosh y + \cosh z - 1}, -\cosh x)$

$= (1 - \cosh x)^{-\alpha 1} (1- \cosh y\, y)^{-\alpha 2}$

$H_2(1\text{-}\gamma_3, \alpha_1, \alpha_2, \gamma_1+ \gamma_3\text{-}1, \gamma_1; \dfrac{\cosh z}{\cosh y - 1}, .. \dfrac{\cosh z}{\cosh y - 1})$ (1.3.17)

$= (1\text{-}\cosh x)^{-\alpha 1} (1\text{-}\cosh y - \cosh z)^{-\alpha 2}$

$G_2(\alpha_1, \alpha_2, 1\text{-}\gamma_1, 1\text{-} \gamma_3; \dfrac{\cosh z}{1 - \cosh y}, \dfrac{\cosh z}{1 - \cosh y - \cosh z})$ (1.3.18)

where H_2 and G_2 are defined by

$H_2(\alpha, \beta, \gamma, \delta; \theta;$ cosh x, cosh y) $= \displaystyle\sum_{m,n=o}^{\infty} \frac{(\alpha)_{m-n}(\beta)_m(\gamma)_n(\delta)_n}{m!n!(\theta)_m} \cosh^m x \cosh^n y$ (1.3.19)

and

$G_2(\alpha, \alpha', \beta, \beta'$ cosh x, cosh y) $= \displaystyle\sum_{m,n=o}^{\infty} \frac{(\alpha)_m(\alpha)_n(\beta)_{n-m}(\beta)_{m-n}}{m!n!} \cosh^m x \cosh^n y$ (1.3.20)

$F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1, 2\alpha_1, \beta_2, \beta_3,;$ cosh x, cosh y, cosh z)

$= (1 - \dfrac{1}{2} \cosh x)^{-\beta_1} (1\text{-} \cosh y)^{-\alpha 2}$

$$H_4(\beta_1, \alpha_2, \alpha_1 + \frac{1}{2}, \gamma_3; \frac{\cosh^2 x}{4(2-\cosh x)^2}, \frac{\cosh z}{(1-\frac{1}{2}\cosh x)(1-\cosh y)}) \qquad (1.3.21)$$

This leads to Erdelyi's (1948) when y = 0

To prove these we know that
$F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; \cosh x, \cosh y, \cosh z)$

$$\sum_{m,n=o}^{\infty} \frac{(\alpha_1)_m (\alpha_2)(\beta_1)_{m+p}(\beta_2)_n}{m!\,n!\,p!\,(\gamma_1)_m (\gamma_2)_n (\gamma_3)_p}, \cosh^m x \cosh^n y \cosh^p z \qquad (1.3.22)$$

absolutely convergent if p = (1-m) (1-n) where

|cosh x|< 1, |cosh y|<1, and |cosh z|<1

$F_K(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1; \gamma_1, \gamma_2, \gamma_3; \cosh x, \cosh y, \cosh z)$

$$= \frac{\Gamma(\rho)\Gamma(\rho_1)\Gamma(2-\rho-\rho_1)}{(2\pi i)^2}\int_c (-t)^{-\rho}(t-1)^{-\rho_1}$$

$$_2F_1(\rho, \alpha_1; \gamma_1 + \frac{1}{2}: \frac{\cosh x}{t}).F_2(\alpha_2, \beta_2, \rho_1, \gamma_2, \gamma_3; \cosh y, \frac{\cosh y}{(1-t)})dt \qquad (1.3.23)$$

|1|>|cosh x| and |cosh z/1− t|< 1-|cosh y| along the contour and $\beta_1 = \rho + \rho_1 -1$ putting $\gamma_2 = \beta_2$ and $\gamma_3 = \rho_1$ we have

$$F_2(\alpha_2, \beta_2, \rho_1, \rho_2, \rho_1; \cosh y, \frac{\cosh z}{(1-t)}) = (1\text{-}\cosh y - \frac{\cosh z}{(1-t)})^{-\alpha_2}$$

$$= (1\text{- }\cosh y - \cosh z)^{-\alpha 2}\sum_{m=o}^{\infty}\frac{(\alpha_2)_m}{m!}(\frac{\cosh z}{1-\cosh y - \cosh z}.\frac{t}{1-t})_m \qquad (1.3.24)$$

$$= (1\text{- }\cosh y)^{-\alpha 2}\sum_{m=o}^{\infty}\frac{(\alpha_2)_m}{m!}(\frac{\cosh z}{1-\cosh y - \cosh z}.\frac{t}{1-t})_m \qquad (1.3.25)$$
and

$$_2F_1(\rho, \alpha_1, \rho; \cosh y, \frac{\cosh x}{t}) = (1 - \frac{\cosh x}{t})^{-\alpha_2}$$

$$= (1\text{- }\cosh x)^{-\alpha 1}\sum_{m=o}^{\infty}\frac{(\alpha_2)_m}{m!}(\frac{\cosh x}{1-\cosh x}.\frac{1}{1-t})^m \qquad (1.3.26)$$

Thus if $\beta_1 = \gamma_1 + \gamma_3 -1$ and $\beta_2 = \gamma_2$ we will have

$$F_K = \frac{\Gamma(\gamma_1)\Gamma(\gamma_3)\Gamma(2-\gamma_1-\gamma_3)}{(2\pi i)^2}\int_c (-t)^{-\gamma_1}(t-1)^{-\gamma_3}$$

$$(1 - \frac{\text{Cosh y}}{t})^{-\alpha_1}(1-\cosh y - \frac{\text{Cosh z}}{1\text{-}t})^{-\alpha_2}\,dt \qquad (1.3.27)$$

Now writing down the expansion (1.3.24) and

$$(1-\frac{\cosh y}{t})^{-\alpha 1} = \sum_{m=o}^{\infty} \frac{(\alpha_1)_m}{m!}(\frac{\cosh x}{t})_m \qquad (1.3.28)$$

and the integrating term by term by term we will be table to prove (1.3.16) – (1.3.24) can also be proved by applying (1.3.24) and (1.3.26) and then integrating term by term.

Application of (1.3.25) and (1.3.26) will prove (1.3.17) the proof of (1.3.21) is however, similar to (1.3.14).

We can also prove the following results for F_N

$F_N(\alpha_1, \gamma_2, \gamma_2, \gamma_1+ k-1-\beta_2, \gamma_1+k-1; \gamma_1, \gamma_2,; \gamma_3,; \cosh x, \cosh y, \cosh z)$

$= (1-\cosh x)^{-\alpha 1} (1-\cosh y)^{-\beta 2}$

$$H_2(1-k, \alpha_1,k, \gamma_1+k-1, \gamma_2; \frac{\cosh x}{\cosh x -1}, -\cosh z) \qquad (1.3.29)$$

$= (1-\cosh x)^{-\alpha 1} (1-\cosh y)^{-\beta 2}$

$$G_2(\alpha_1,k,1-\gamma_1,1-k; \frac{\cosh x}{1-\cosh x}, \frac{\cosh z}{1-\cosh z}) \qquad (1.3.30)$$

$F_N(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1, \gamma_1, \gamma_2, \gamma_3,; \cosh x, \cosh y, \cosh z)$

$$= \sum_{m,n=o}^{\infty} \frac{(\alpha_1)_m (\alpha)_2 (\beta_1)_{m+p} (\beta_2)_n}{m! n! p! (\gamma_1)_m (\gamma_2)_{n+p}} \cosh^m x \cosh^n y \cosh^p z \qquad (1.3.31)$$

absolutely convergent if m+p = 1 and n = 1

where |cosh x|<1, |cosh y|<1, and |cosh z|<1 and

it's integral representations

$F_N(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_1, \gamma_1, \gamma_2, \gamma_3,; \cosh x, \cosh y, \cosh z)$

$$= \frac{\Gamma(\rho)\Gamma(\rho_1)\Gamma(2-\rho-\rho_1)}{(2\pi i)^2} \int_c (-t)^{-\rho} (t-1)^{-\rho_1}$$

$$_2F_1(\rho, \alpha_1, \gamma_1; \frac{\cosh x}{t}) F_2(\alpha_2, \beta_2, \rho_1, \gamma_2; \cosh y, \frac{\cosh z}{(1-t)}) dt \qquad (1.3.32)$$

Where $\beta 1= \rho+\rho 1-1$ and |t|>|coshx| and |1-t|>|cosh z| along the contour.

II.   REDUCIBLE CASES FOR THE QARDRUPLE HYPERGEOMETRIC FUNCTION

2.1 INTRODUCTION

The various Hypergeometric function of four variables are studied earlier by H. Exton, H. Srivastava and many others. Here all 21 functions are given below in the terms of a table.

1	K_1 (a,a,a,a;b,b,b,c;d,e_1,e_2,d; x,y,z,t)	$$= \sum \frac{(a,k+m+n+p)(b,k+m+n)(c,p)x^k y^m z^n t^p}{(d,k+p)(e_1,m)(e_1,n)k!m!n!p!}$$
2.	K_2 (a,a,a,a;b,b,b,c;d_1,d_2,d_3,d_4; x,y,z,t)	$$= \sum \frac{(a,k+m+n+p)(b,k+m+n)(c,p)x^k y^m z^n t^p}{(d_1,k)(d_2,m)(d_3,n)(d_4,p)k!m!n!p!}$$
3	K_3 (a,a,a,a;b_1,b_1,b_2,b_2,;c_1,c_2,c_2,c_1; x,y,z,t)	$$= \sum \frac{(a,k+m+n+p)(b_1,k+m)((b_2,n+p)x^k y^m z^n t^p}{(c_1,k+p)(c_2,m+n)k!m!n!p!}$$
4	K_4 (a,a,a,a;b_1,b_1,b_2, b_2;c,d_1,d_2,c; x,y,z,t)	$$= \sum \frac{(a,k+m+n+p)(b_1,k+m)((b_2,n+p)x^k y^m z^n t^p}{(c,k+p)(d_1,m)(d_2,n)k!m!n!p!}$$
5	K_5 (a,a,a,a;b_1,b_1,b_2, b_2;c_1,c_2,c_3,c_4; x,y,z,t)	$$= \sum \frac{(a,k+m+n+p)(b_1,k+m)((b_2,n+p)x^k y^m z^n t^p}{(c_1,k)(c_2,m)(c_3,n)(c_4,p)k!m!n!p!}$$
6	K_6 (a,a,a,a;b,b,c_1,c_2;e,d,d,d; x,y,z,t)	$$= \sum \frac{(a,k+m+n+p)(b,k+m)(c_1,n)(c_2,p)x^k y^m z^n t^p}{(e,k)(d,m+n+p)k!m!n!p!}$$
7.	K_7 (a,a,a,a;b,b,c_1,c_2,d_1,d_2,d_1,d_2; x,y,z,t)	$$= \sum \frac{(a,k+m+n+p)(b,k+m)(c_1,n)(c_2,p)x^k y^m z^n t^p}{(d_1,k+n)(d_2,m+p)k!m!n!p!}$$
8.	K_8 (a,a,a,a;b,b,c_1,c_2;d,e_1,d,e_2; x,y,z,t)	$$= \sum \frac{(a,k+m+n+p)(b,k+m)(c_1,n)(c_2,p)x^k y^m z^n t^p}{(d,k+n)(e_1,m)(e_2,n)k!m!n!p!}$$
9.	K_9 (a,a,a,a;b,b,c_1,c_2,e_1,e_2,d,d; x,y,z,t)	$$= \sum \frac{(a,k+m+n+p)(b,k+m)(c_1,n)(c_2,p)x^k y^m z^n t^p}{(e_1,k)(e_2,m)(d,n+p)k!m!n!p!}$$
10	K_{10} (a,a,a,a;b,b,c_1,c_2;d_1,d_2,d_3,d_4; x,y,z,t)	$$= \sum \frac{(a,k+m+n+p)(b,k+m)(c_1,n)(c_2,p)x^k y^m z^n t^p}{(d_1,k)(d_2,m)(d_3,n)(d_4,p)k!m!n!p!}$$
11	K_{11} (a,a,a,a;b_1,b_2,b_3,b_4,c,c,c,d; x,y,z,t)	$$= \sum \frac{(a,k+m+n+p)(b_1,k)(b_2,m)(b_3,n)(b_4,p)x^k y^m z^n t^p}{(c,k+m+n)(d,p)k!m!n!p!}$$
12	K_{12} (a,a,a,a;b_1,b_2,b_3,b_4,c_1,c_1,c_2,c_2; x,y,z,t)	$$= \sum \frac{(a,k+m+n+p)(b_1,k)(b_2,m)(b_3,n)(b_4,p)x^k y^m z^n t^p}{(c_1,k+m)(c_2,n+p)k!m!n!p!}$$
13	K_{13} (a,a,a,a;b_1,b_2,b_3,b_4,c,c,d_1,d_2; x,y,z,t)	$$= \sum \frac{(a,k+m+n+p)(b_1,k)(b_2,m)(b_3,n)(b_4,p)x^k y^m z^n t^p}{(c,k+m)(d_1,n)(d_2,p)k!m!n!p!}$$
14	K_{14} (a,a,a,c_3;b,c_1,c_1,b;d,d,d,d; x,y,z,t)	$$= \sum \frac{(a,k+m+n)(c_3,p)(b,k+p)(c_1,m+n)x^k y^m z^n t^p}{(d,k+m+n+p))k!m!n!p!}$$

15	$K_{15}(a,a,a,b_5;b_1,b_2,b_3,b_4;c,c,c,c; x,y,z,t)$	$=\sum \dfrac{(a,k+m+n)(b_5,p)(b_1,k)(b_2,m)(b_3,n)(b_4,p)x^k y^m z^n t^p}{(c,k+m+n+p))k!m!n!p!}$
16	$K_{16}(a_1,a_2,a_3,a_4;b; x,y,z,t)$	$=\sum \dfrac{(a_1,k+m)(a_2,k+n)(a_3,n+p)(a_4,n+p)x^k y^m z^n t^p}{(c,k+m+n+p))k!m!n!p!}$
17	$K_{17}(a_1,a_2,a_3,b_1,b_2;c; x,y,z,t)$	$=\sum \dfrac{(a_1,k+m)(a_2,k+n)(a_3,n+p)(b_1,p)(b_2,p)x^k y^m z^n t^p}{(c,k+m+n+p))k!m!n!p!}$
18	$K_{18}(a_1,a_2,a_3,b_1,b_2;c; x,y,z,t)$	$=\sum \dfrac{(a_1,k+m)(a_2,k+p)(a_3,m+n)(b_1,n)(b_2,p)x^k y^m z^n t^p}{(c,k+m+n+p))k!m!n!p!}$
19	$K_{19}(a_1,a_2,b_1,b_2,b_3,b_4;c; x,y,z,t)$	$=\sum \dfrac{(a_1,k+m)(a_2,k+n)(b_1,m)(b_2,n)(b_3,p)x^k y^m z^n t^p}{(c,k+m+n+p))k!m!n!p!}$
20	$K_{20}(a_1,a_2,a_3,a_4;b_1,b_2, a_2, a_2;c,c,c,c; x,y,z,t)$	$=\sum \dfrac{(a_1,k)(a_2,m)(b_3,n)(b_4,p)(b_1,k)(b_2,m)(a_2,n+p)x^k y^m z^n t^p}{(c,k+m+n+p))k!m!n!p!}$
21	$K_{21}(a,a,b_6,b_5;b_1,b_2, b_3, b_4;c,c,c,c; x,y,z,t)$	$=\sum \dfrac{(a,k+m)(b_6,n)(b_5,p)(b_1,k)(b_2,m)(b_3,n)(b_4,p)x^k y^m z^n t^p}{(c,k+m+n+p))k!m!n!p!}$

All results are having convergent conditions and restrictions on parameters including variables due to Exton (1972)

As far as known to me all the results investigation in this journal are new and interesting which have a wide range of application in the field mathematical series

2.2 In this Section Quadruple hypergeometric function reduced to the hypergeometric function of one variable.

(1) Theorem : By specializing the parameters of K_{14}, we obtain the following

$F(1,a,a,1;b_1, b_2, b_1, b_2;b_1, b_2 + b_1+ b_2, b_1+ b_2, b_1+ b_2; x, y,z,t)$

$= (x + y - xy)^{-1} (z + t - zt)^{-1}$.
$[x \, _2F_1(b_1+p,a_2+p;b_1+b_2+p+q;x)+y_2F_1(b_2+q,a_2+p;b_1+b_2+p+q;y)]$
$\quad\quad\quad .[z_2F_1(b_1,a_2;b_1+b_2;z)+y_2F_1(b_2,a_2,b_1+b_2;t)]$ (2.2.1)

condition and restrictions are given in Exton (1972)

proof:

$F(1,a_2,a_2,1;b_1, b_2, b_1, b_2,;b_1+b_2, b_1+ b_2, b_1+ b_2, b_1+ b_2; x, y,z,t)$

$$=\sum_{m,n,p,q=0}^{\infty} \frac{(1)_m (a_2)_{n+p}(1)_q (b_1)_{m+p}(b_2)_{n+q}}{(b_1+b_2)_{m+n+p+q}} \frac{x^m y^n z^p t^q}{m!n!p!q!}$$ (2.2.2)

$$=\sum_{m,n,p,q=0}^{\infty} \frac{m!(a_2)_{n+p}(1)_q (b_1+p)_m (b_1)_p (b_2+q)_n (b_2)_q}{(b_1+b_2+p+q)_{m+n}(b_1+b_2)_{m+n}} \frac{x^m y^n z^p t^q}{m!n!p!q!}$$

$$= \sum_{m,n=0}^{\infty} \left[\frac{(b_1+p)_m (b_2+q)_n x^m y^n}{(b_1+b_2+p+q)_{m+n}} \right] \frac{(a_2+p)_n}{n!} \quad \sum_{p,q=0}^{\infty} \left[\frac{(b_1)_p (b_2)_q z^p t^q}{(b_1+b_2+p+q)_{m+n}} \right] \frac{(a_2)_p (1)_q}{n! q!} \qquad (2.2.3)$$

by virtue of the result the to carlitz formula in the term

$$= \sum_{m,n=0}^{\infty} \left[\frac{(\alpha)_m (\beta)_n x^m y^n}{(\alpha+\beta)_{m+n}} \right] = (x+y-xy)^{-1} \sum_{m=0}^{\infty} \left[\frac{(\alpha)_n x^{n+1}}{(\alpha+\beta)_n} \right] + \sum_{n=0}^{\infty} \left[\frac{(\beta)_n y^{n+1}}{(\alpha+\beta)_n} \right]$$

We have equation (2.2.3) in the term

$$= (x+y-xy)^{-1} \left[\sum_{n=0}^{\infty} \frac{(b_1+p)_n x^{n+1}}{(b_1+b_2+p+q)_n} + \sum_{n=0}^{\infty} \frac{(b_2+q)_n y^{n+1}}{(b_1+b_2+p+q)_n} \right] \frac{(a_2+p)_n}{n!}$$

$$.(z+t-zt)^{-1} \left[\sum_{p=0}^{\infty} \frac{(b_1)_p z^{p+1}}{(b_1+b_2)_p} + \sum_{p=0}^{\infty} \frac{(b_2)_p t^{q+1}}{(b_1+b_2)_p} \right] \frac{(a_2)_p}{p!} \qquad (2.2.4)$$

$$= (x+y-xy)^{-1}(z+t-zt)^{-1} \left[x\sum_{n=0}^{\infty} \frac{(b_1+p)_n (a_2+p)_n x^n}{(b_1+b_2+p+q)_n n!} + y\sum_{n=0}^{\infty} \frac{(b_2+q)_n (a_2+p)_n y^n}{(b_1+b_2+p+q)_n n!} \right]$$

$$. \left[z\sum_{p=0}^{\infty} \frac{(b_1)_p (a_2)_p z^p}{(b_1+b_2)_p p!} + t\sum_{p=0}^{\infty} \frac{(b_2)_p (a_2)_p t^q}{(b_1+b_2)_p p!} \right]$$

then theorem follows

$F(1,a_2,a_2,1;b_1, b_2, b_1, b_2,;b_1+b_2, b_1+b_2, b_1+b_2, b_1+b_2; x, y,z,t)$
$=(x+y-xy)^{-1} (z+t-zt)^{-1}$

$[x_2F_1(b_1+p,a_2+p;b_1+b_2+p+q;x)+y_2F_1(b_2+q,a_2+p;b_1+b_2+p+q;y)]$.
$[z_2F_1(b_1,a_2;b_1+b_2;z)+y_2F_1(b_2,a_2,b_1+b_2;t)]$

The completes the derivation of (2.2.1).

(2) **Theorem** by specializing the parameters of K_{14}, we obtain the following.

(i) $F(1,a_2,a_2,1;b_1, b_2, b_1, b_2,;b_1+b_2, b_1+b_2, b_1+b_2, b_1+b_2; 1, 1,1,1)$

$$= \left[\frac{\Gamma(b_1+b_2+p+q)\Gamma(b_2+q-a_2-p)}{\Gamma(b_2+q)\Gamma(b_1+b_2+q-a_2)} + \frac{\Gamma(b_1+b_2+p+q)\Gamma(b_2+p-q-a_2)}{\Gamma(b_1+q)\Gamma(b_1+b_2+p-a_2)} \right]$$

$$\left[\frac{\Gamma(b_1+b_2)\Gamma(b_2-a_2)}{\Gamma(b_2)\Gamma(b_1+b_2-a_2)} + \frac{\Gamma(b_1+b_2)\Gamma(b_2-a_2)}{\Gamma(b_1)\Gamma(b_1+b_2-a_2)} \right] \qquad (2.2.6)$$

(ii) $F(1,a_2,a_2,1;a_2,a_2,a_2,a_2;2a_2,2a_2,2a_2,2a_2;1,1,1,1)$

$$= \left[\frac{\Gamma(2a_2 + p + q)\Gamma(q - p)}{\Gamma(a_2 + q)\Gamma(a_2 + q)} + \frac{\Gamma(a_2 + p + q)\Gamma(q - p)}{\Gamma(a_2 + q)\Gamma(a_2 + q)} \right] 2 \left[\frac{\Gamma(2a_2)}{\Gamma(a_2)\Gamma(a_2)} \right] \qquad (2.2.7)$$

(iii) $F(1,1,1,1;1,1,1,1;2,2,2,2;1,1,1,1)$

$$= \frac{\Gamma(p + q + 2)}{\Gamma(q + 1)\Gamma(p + 1)} [\Gamma(q+1)+ \Gamma(p+1)] \qquad (2.2.8)$$

(1)   Proof :-
now putting x = y = z = 1 in equation (2.2.1) than we get

$F(1,a_2,a_2,1;b_1,b_2,b_1,b_2;b_1+b_2,b_1+b_2,b_1+b_2,b_1+b_2;1,1,1,1)$

$=[_2F_1(b_1+p,a_2+p;b_1+b_2+p+q;1)+_2F_1(b_2+q,a_2+p;b_1+b_2+p+q;1)].$

$\qquad [_2F_1(b_1,a_2;b_1+b_2;1)+ _2F_1(b_2,a_2,b_1+b_2;1)] \qquad (2.2.9)$

now Apply Gauss's summation theorem (1812)

$$_2F_1(\alpha,\beta,\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} \qquad (2.2.10)$$

using equation (2.2.10) in (2.2.9) we get

$$= \left[\frac{\Gamma(b_1 + b_2 + p + q)\Gamma(b_2 + q - a_2 - p)}{\Gamma(b_2 + q)\Gamma(b_1 + b_2 + q - a_2)} + \frac{\Gamma(b_1 + b_2 + p + q)\Gamma(b_1 + p - q - a_2)}{\Gamma(b_1 + q)\Gamma(b_1 + b_2 + p - a_2)} \right]$$

$$\left[\frac{\Gamma(b_1 + b_2)\Gamma(b_2 - a_2)}{\Gamma(b_2)\Gamma(b_1 + b_2 - a_2)} + \frac{\Gamma(b_1 + b_2)\Gamma(b_2 - a_2)}{\Gamma(b_1)\Gamma(b_1 + b_2 - a_2)} \right] \qquad (2.2.11)$$

This completes the derivation of (2.2.6)

(2)   Proof:- now putting $b_1 = b_2 = a_2$ in equation (2.2.11)

$F(1,a_2,a_2,a_21;a_2,a_2,a_2,a_2;2a_2,2a_2,2a_2,2a_2;1,1,1,1)$

$$= \left[\frac{\Gamma(2a_2 + p + q)\Gamma(q - p)}{\Gamma(a_2 + q)\Gamma(a_2 + q)} + \frac{\Gamma(a_2 + p + q)\Gamma(p - q)}{\Gamma(a_2 + p)\Gamma(a_2 + p)} \right] \left[\frac{\Gamma(2a_2)}{\Gamma(a_2)\Gamma(a_2)} + \frac{\Gamma(2a_2)}{\Gamma(a_2)\Gamma(a_2)} \right]$$

$$= \left[\frac{\Gamma(2a_2 + p + q)\Gamma(q - p)}{\Gamma(a_2 + q)\Gamma(a_2 + q)} + \frac{\Gamma(a_2 + p + q)\Gamma(p - q)}{\Gamma(a_2 + p)\Gamma(a_2 + p)} \right] \left[\frac{\Gamma(2a_2)}{\Gamma(a_2)\Gamma(a_2)} \right] \qquad (2.2.12)$$

This competes the derivation of (2.2.7)

(3)   Proof: now putting $b_1 = b_2 = a_2 = 1$ in equation (2.2.12)

$F(1,1,1,1;1,1,1,1;2,2,2,2;1,1,1,1)$

$$= \left[\frac{\Gamma(p+q+2)}{2\Gamma(q+1)} + \frac{\Gamma(p+q+2)}{2\Gamma(q+1)} \right] \Gamma(2)$$

$$= \frac{\Gamma(p+q+2)}{\Gamma(q+1)\Gamma(p+1)} [\Gamma(q+1)+\Gamma(p+1)] \tag{2.2.13}$$

This completes the derivation of (2.2.8)

2.3 In this Section Quadruple hypergeometric function reduced to the Appell hypergeometric function

(1)   <u>**Theorem**</u> By specializing the parameters of K_{12}, K_{10}, K_{15} we obtain the following

K_{12} (a, a, a, a; $b_1, b_2, b_3, b_4, c_1, c_1, c_2, c_2$; x,y,z,t)

$$= F1 (a, b_1, b_2; c_1; x, y) F_1 (a+m+n, b_3, b_4 ; c_2 ; z,t) \tag{2.3.1}$$

K_{12} (a, a, a, a; $b_1, b_2, b_3, b_4, c_1, c_1, c_2, c_2$; 1,1,1,1)

$$= \frac{\Gamma(c_1)\Gamma(c_1 - a - b_1 - b_2)}{\Gamma(c_1 - a)\Gamma(c_1 - b_1 - b_2)} \cdot \frac{\Gamma(c_2)\Gamma(c_2 - a - m - n - b_3 - b_4)}{\Gamma(c_2 - a)\Gamma(c_2 - b_3 - b_4)} \tag{2.3.2}$$

K_{10} (a, a, a, a; b, b, c_1, c_2 ; d_1, d_2, d_3, d_4; x, y, z, t)

$$= F_4 (a, b; d_1, d_2; x, y) F_2 (a+m+n, c_1, c_2; d_3, d_4; z, t) \tag{2.3.3}$$

K_{15} (a, a, a, b_5; b_1, b_2, b_3, b_4; c, c, c, c; x, y, z, t)

$$= F_1 (a, b_1, b_2; c; x, y) F_3 (a+m+n, b_5, b_3, b_4; c+m+n; z, t) \tag{2.3.4}$$

where (F_1, F_2, F_3, F_4,) are Appell hypergeometric function of two variables.

(1)   **Proof:**
Now Quadruple hypergemotric function can be reduced to Appell function of two variable.

K_{12} (a, a, a, a; $b_1, b_2, b_3, b_4, c_1, c_1, c_2, c_2$; x, y, z, t)

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}(b_1)_m(b_2)_n(b_3)_p(b_4)_q}{(c_1)_{m+n} m! n!} \frac{x^m y^n z^p t^q}{m! n! p! q!} \tag{2.3.5}$$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}(b_1)_m(b_2)_n x^m y^n}{(c_1)_{m+n} m! n!} \frac{(a+m+n)_{p+q}}{(c_2)_{p+q}} \frac{(b_3)_p(b_4)_q z^p t^q}{p! q!}$$

$$= \left[\sum_{m,n=0}^{\infty} \frac{(a)_{m+n+p+q}(b_1)_m(b_2)_n x^m y^n}{(c_1)_{m+n} m! n!} \right] \left[\sum_{p,q=0}^{\infty} \frac{(a+m+n)_{p+q}(b_3)_p(b_4)_q z^p t^q}{(c_2)_{p+q} p! q!} \right]$$

$$= F_1 (a, b_1, b_2; c ; x, y) F_1 (a+m+n, b_3, b_4; c_2; z, t) \tag{2.3.6}$$

where F_1 is Appell function of two variable
This completes the derivation of (2.3.1)

(2) Proof :-
now putting x = y = z = t= 1 in equation (2.3.6)
K_{12} (a, a, a, a; $b_1, b_2, b_3, b_4, c_1, c_1, c_2, c_2$; 1,1,1,1)

$$= F_1(a,b_1,b_2;c_1;\ 1,1)\ F_1\ (a+m+n,b_3,b_4;\ c_2;\ 1,1) \tag{2.3.7}$$

Now Apply Gauss's summation theorem

$$F_1(\alpha,\beta,\beta_1,\gamma;1,1)= \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta-\beta_1)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta-\beta_1)} \tag{2.3.8}$$

Using equation (2.3.7) and (2.3.8)

$$= \frac{\Gamma(c_1)\Gamma(c_1-a-b_1-b_2)}{\Gamma(c_1-a)\Gamma(c_1-b_1-b_2)} \cdot \frac{\Gamma(c_2)\Gamma(c_2-a-m-n-b_3-b_4)}{\Gamma(c_2-a)\Gamma(c_2-b_3-b_4)} \tag{2.3.9}$$

This completes the derivation of (2.3.2)

(3)   Proof:

K_{10} (a, a, a, a; b, b, c_1, c_2; d_1, d_2, d_3, d_4; x, y, z, t)

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}(b)_{m+n}(c_1)_p(c_2)_q}{(d_1)_m(d_2)_n(d_3)_p(d_4)_q} \frac{x^m y^n z^p t^q}{m!n!\,p!q!} \tag{2.3.10}$$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}x^m y^n}{(d_1)_m(d_2)_n m!n!} \frac{(a)_{p+q}(c_1)_p}{(d_3)_p(d_4)_q} \frac{(c_2)_q z^p t^q}{p!q!}$$

$$= \left[\sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}x^m y^n}{(d_1)_m(d_2)_n m!n!}\right]\left[\sum_{p,q=0}^{\infty} \frac{(a+m+n)_{p+q}(c_1)_p(c_2)_q}{(d_3)_p(d_4)_q} \frac{z^p t^q}{p!q!}\right]$$

$$= F_4\ (a, b; d_1, d_2\ ; x, y)\ F_2\ (a+m+n, c_1,c_2; d_3, d_4; z,t) \tag{2.3.11}$$

where F_4 & F_2 are Appell function of two variable
This completes the derivation of (2.3.3)

(4) Proof:

K_{15} (a, a, a, b_5; b_1, b_2, b_3, b_4; c, c, c, c; x, y, z, t)

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p}(b_5)_q(b_1)_m(b_2)_n(b_3)_p(b_4)_q}{(c)_{m+n+p+q}} \frac{x^m y^n z^p t^q}{m!n!\,p!q!} \tag{2.3.12}$$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n x^m y^n}{(c)_{m+n} m!n!} \frac{(a+m+n)_p(b_5)_q(b_4)_q}{(c+m+n)_{p+q}} \frac{(b_3)_p z^p t^q}{p!q!}$$

$$= \left[\sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n x^m y^n}{(c)_{m+n} m!n!}\right]\left[\sum_{p,q=0}^{\infty} \frac{(a+m+n)_p(b_5)_q(b_4)_q}{(c+m+n)_{p+q}} \frac{(b_3)_q z^p t^q}{p!q!}\right]$$

$$= F_1(a, b_1, b_2; c\ ; x, y)\ F_3(a+m+n, b_5,b_3,b_4; c+m+n; z, t) \tag{2.3.13}$$

Where F_1 & F_3 are Appell function of two variable
This completes the derivation of (2.3.4)

2.4   In this Section Quadruple hypergeometric function reduced to the Appell hypergeometric function Lauricella's set and Saran

(1)   Theorem
By specializing the parameters of K_2, K_{12}, K_{15} , K_6 we obtain the following

K_2 (a, a, a, a; b, b, b, c; d_1 d_2, d_3, d_4; x,y,z,t)
= F_E (a, a,a, b,b,c,; d_1, d_2,d_3 ; x,y,t) $_2F_1$(a+m+n+q,b+m+n, d_4; z)   (2.4.1)

K_2, (a, a, a, a; b, b, b, c; d_1, d_2, d_3, d_4; x,y,l,t)

$$\frac{\Gamma(d_4)\Gamma(d_4-a-b-q-2m-2n)}{\Gamma(d_4-a-m-n-q)\Gamma(d_4-b-m-n)}$$ F_E (a,a,a,b,b,c,; d_1,d_2,d_3; x, y, t)   (2.4.2)

K_{12} (a, a, a, a; b_1,b_2, b_3,b_4, c_1, c_1, c_2, c_2; x,y,z,t)
= F_G(a, a, a, b_1,b_3, b_4, c_1,c_2, c_2; x,z,t) $_2F_1$ (a+m+p+q,b_2;c_1+m;y)   (2.4.3)

K_{12} (a, a, a, a; b_1, b_2, b_3, b_4, c_1, c_1,c_2, c_2 ; x,l,z,t)

$$\frac{\Gamma(c_1+m)\Gamma(c_1-a-b_2-p-q)}{\Gamma(c_1+m-b_2)\Gamma(c_1-a-p-q)}$$ F_G (a, a, a, b_1,b_3, b_4,; c_1, c_2, c_2; x, y, z, t)   (2.4.4)

K_{15} (a, a, a, b_5;b_4, b_1,b_2,b_3; c, c, c,c; x, y, z, t)
= F_S(b_5, a, a, b_4,b_1,b_2;c,c,c; x,y,t) $_2F_1$ (a+m+n,b_3 ;c+q+m+n; z)   (2.4.5)

K_{15} (a, a, a, b_5; b_4, b_1, b_2, b_3;c, c, c, c; x, y, z, t)

$$\frac{\Gamma(c+q+m+n)\Gamma(c+q-a-b_3)}{\Gamma(c+q+m+n-b_3)\Gamma(c+q-a)}$$ F_S(b_5, a, a, b_4, b_1, b_2,; c,c,c; x,y,t)   (2.4.6)

K_6 (a,a,a,a; b,b, c_1,c_2;e, d,d,d;x,y,z,t)

= F_F(a, a, a,b, c_1, b, ;e,d,d; x,z,y) $_2F_1$ (a+m+p+n,c_2 ;d+p+n; t)   (2.4.7)

K_6 (a,a,a,a; b,b, c_1,c_2;e, d,d,d;x,y,z,t)

$$= \frac{\Gamma(d+p+n)\Gamma(d-m-a-c_2)}{\Gamma(d-a-m)\Gamma(d+p+n-c_2)}$$ F_F(a, a, a,b, c_1, b, ;e,d,d; x,z,y)   (2.4.8)

Where (F_4,F_{14}, F_8,F_7) & (F_E, F_F, F_G,F_S) are Lauriceila's set & Saran Triple hypergeometric Series.

(1) Proof:-
Now Quadruple hypergeometric function can be reduced to Lauricella's set and Saran Triple hypergometric Series.
K_2 (a,a,a,a; b,b,b,c; $d_1$$d_2$,$d_3$,$d_4$;x,y,z,t)

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}(b)_{m+n+p}(c)_q}{(d_1)_m(d_2)_n(d_3)_p(d_4)_q} \frac{x^m y^n z^p t^q}{m!n!p!q!}$$ (2.4. 9)

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+q}(b)_{m+n}(c)_q x^m y^n t^q}{(d_1)_m(d_2)_n(d_4)_q m!n!} \frac{(a+m+n+q)_p (b+m+n)_p z^p}{(d_3)_p}\frac{}{p!}$$

$$= \left[\sum_{q,m,n=0}^{\infty} \frac{(a)_{m+n+q}(b)_{m+n}(c)_q x^m y^n t^q}{(d_1)_m(d_2)_n(d_4)_q m!n!q!}\right]\left[\sum_{p=0}^{\infty} \frac{(a+m+n+q)(b+m+n)}{(d_3)_p}\frac{z^p}{p!}\right]$$

= F_E (a, a, a, b, b,c,; d_1 , d_2, d_3 , x,y, t) $_2F_1$ (a+m+n+q, b+m+n, d_4; z)   (2.4.10)
This completes the derivation of (2.4.1)

(2)   Proof:
If z = 1, in euqation (2.4.10)

K_2 (a, a, a, a; b, b, b, c; d_1, d_2, d_3, d_4; x,y, l,t)
= F_E(a, a, a, b, b,c,; d_1, d_2, d_3, x,y, t) $_2F_1$ (a+m+n+q, b+m+n, d_4; 1)   (2.4.11)

Now Apply Gauss's summation theorem in equation (2.4.11)

$$F_1(\alpha,\beta,\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta-)}$$

K_2 (a, a, a, a; b, b, b, c; d_1, d_2, d_3, d_4; x,y, l,t)

$$= \frac{\Gamma(d_4)\Gamma(d_4-a-b-q-2m-2n)}{\Gamma(d_4-a-m-n-q)\Gamma(d_4-b-m-n)} F_E(a, a, a,b,b, c,; d_1,d_2,d_3; x,z,t) \quad (2.4.12)$$

This completes the derivation of (2.4.2)

(3)   Proof:

K_{12} (a, a, a, a; b_1, b_2, b_3, b_4,c_1,c_2, c_2 ; x,y, z,t)

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}(b_1)_m(b_2)_n(b_3)_p(b_4)_q}{(c_1)_{m+n}(c_2)_{p+q}} \frac{x^m y^n z^p t^q}{m!n!p!q!} \quad (2.4.13)$$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}(b_1)_m(b_3)_p(b_4)_p x^m y^n t^q}{(c_1)_m(c_2)_{p+q}} \frac{(a+m+p+q)_n(b_2)_n y^n}{m!p!q!(c_1+m)_n n!}$$

$$= \left[\sum_{m,p,q=0}^{\infty} \frac{(a)_{m+n+q}(b_1)_m(b_3)_p(b_4)_q x^m y^p t^q}{(c_1)_m(c_2)_{p+q}m!p!q!} \right] \left[\sum_{p=0}^{\infty} \frac{(a+m+n+q)_n(b_2)_n y^n}{(c_1+m)_n n!} \right]$$

F_G (a, a, a, b_1, b_3, b_4; c_1, c_2,c_2; x, z, t) $_2F_1$ (a+m+p+q; b_2; c_1+m; y)   (2.4.14)

This completes the derivation of (2.4.3)

(4)   Proof:

When y = 1, in equation (2.4.14) Then

K_{12} (a, a, a, a; b_1, b_2, b_3, b_4, c_1, c_1, c_2, c_2; x,I, z,t)

=F_G (a, a, a, b_1, b_3, b_4; c_1, c_2,c_2; x, z, t). $_2F_1$ (a+m+p+q; b_2; c_1+m; 1)   (2.4.15)

Now Apply Gauss's summation theorem in equation (2.4.15)

$$F_1(\alpha,\beta,\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}$$

$$= \frac{\Gamma(c_1+m)\Gamma(c_1-a-b_2-p-q)}{\Gamma(c_1+m-b_2)\Gamma(c_1-a-p-q)} F_G(a, a, a,b_1,b_3, b_4,c_1, c_2, c_2,; x,z,t) \quad (2.4.16)$$

This completes the derivation of (2.4.4)

(5)   Proof :-

K_{15} (a, a, a, b_5; b_1, b_2, b_3, b_4 ; c, c, c, c; x, y, z, t)

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p}(b_5)_q(b_1)_m(b_2)_n(b_3)_p(b_4)_q}{(c)_{m+n+p+q}} \frac{x^m y^n z^p t^q}{m!n!p!q!}$$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(b_5)_q(a)_{m+n}(b_4)_q(b_1)_m(b_2)_n t^q x^m y^n (a+m+n)_p(b_3)_p z^q}{(c)_{q+m+n}q!m!n!(c+q+m+n)_p p!}$$

$$= \left[\sum_{m,p,q=0}^{\infty} \frac{(b_5)_q(a)_{m+n}(b_4)_q(b_1)_m(b_2)_n t^q x^m y^n}{(c)_{m+n}q!m!p!} \right] \left[\sum_{p=0}^{\infty} \frac{(a+m+p+p+q)_n(b_2)_n y^n}{(c+q+m+n)_n n!} \right]$$

= F_S(b_5, a, a, b_4,b_1,b_2;c,c,c; x,y,t) $_2F_1$ (a+m+n,b_3 ;c+q+m+n; z)   (2.4.17)

This completes the derivation of (2.4.5)

(6) Proof :-

When z = 1 in equation (2.4.17) Then

K_{15} (a, a, a, b_5, b_4, b_1, b_2, b_3; c, c, c, c; x, y, 1, t)

= F_s(b_5, a, a, b_4,b_1,b_2;c,c,c; x,y,t) $_2F_1$ (a+m+n,b_3 ;c+q+m+n;1)   (2.4.18)

Now Apply Gauss's summation theorem in equation

$$F_1(\alpha,\beta,\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}$$

$$= \frac{\Gamma(c + q + m + n)\Gamma(c - q - a - b_3)}{\Gamma(c + q + m + n - b_3)\Gamma(c + q - a)} \, F_S(b_5, a,\ a, b_4, b_1,\ b_2, c,\ c,\ c,;\ x, z, t) \qquad (2.4.19)$$

This completes the derivation of (2.4.6)

(7)   Proof:-

K_6 (a, a, a, a; b, b, c_1,c_2 ; e, d, d, d; x, y, z, t)

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}(b)_{m+n}(c_1)_p(c_2)_q}{(e)_m(d)_{n+p+q}} \frac{x^m y^n z^p t^q}{m!n!p!q!} \qquad (2.4.20)$$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+p+n}(b)_{m+n}(c_1)_p x^m z^p y^n}{(e)_m(d)_{n+p+q}} \frac{(a+m+p+n)_q (c_2)_q t^q}{m!n!p!q!}$$

$$= \left[\sum_{m,p,q=0}^{\infty} \frac{(a)_{m+p+n}(b)_{m+n}(c_1)_p x^m z^p y^n}{(e)_m(d)_{p+n} m!p!n!} \right] \left[\sum_{p=0}^{\infty} \frac{(a+m+p+n)_q (c_2)_q t^q}{(d+p+n)_q q!} \right]$$

= F_F(a, a, a,b, c_1, b, ;e,d,d; x,z,y) $_2F_1$ (a+m+p+n,c_2 ;d+p+n; t)   (2.4.21)

This complete the derivation of (2.4.7)

(8) Proof :-

When t = 1 in equation (2.4.21) Then

K_6 (a, a, a, a; b, b, c_1,c_2; e, d, d, d; x, y, z, 1)

= F_F(a, a, a,b, c_1, b,; e,d,d; x,z,y) $_2F_1$ (a+m+p+n,c_2 ;d+p+n; 1)   (2.4.22)

Now Apply Gauss's summation theorem in equation

$$F_1(\alpha,\beta,\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}$$

$$= \frac{\Gamma(d + p + n)\Gamma(d - m - a - c_2)}{\Gamma(d - a - m)\Gamma(d + p + n - c_2)} \, F_F(a,\ a, a, b, c_1,\ b,; e, d, d;\ x, z, y) \qquad (2.4.23)$$

This completes the derivation of (2.4.8)

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