On the Domain of Four Dimensional Pascal Matrix in the Space $l_q^2$

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Abstract- We introduce the double sequence space $p_q^2 = \mathbb{P}(l_q^2)$ as the domain of four dimensional Pascal matrix $P$ in the space $l_q^2$, for $1 \leq q < \infty$. Furthermore, we show that $p_q^2$ is a BK-space, Banach space, establish its Schauder basis, topological properties, isomorphism and some inclusions.

Index Terms- 4-dimensional Pascal Matrix, Isomorphism and inclusions

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I. BASIC NOTATIONS AND BACKGROUND

Let $\alpha: \mathbb{N} \times \mathbb{N} \rightarrow \tau$ be a function, where $\tau$ may stand for any nonempty set and $\mathbb{N}$ a set of counting numbers. Then $(j, k) \rightarrow \alpha(j, k) = x_{jk}$ can be termed to be a double sequence.

In the paper by Basar & Sever [2] the Banach space $l_q^2$ of double sequences corresponding to the well-known $l_q$ of single sequences was introduced, its properties were studied, and its $\beta(\nu)$-dual determined and the established that the $\alpha$- and $\delta$-duals of the space coincide with the $\beta(\nu)$-dual; where $1 \leq q < \infty$ and $\nu \in \{p, b, p, \tau\}$. If $P$ denotes the Pascal mean (a four dimensional matrix), then $p_0^2, p_1^2, p_2^2$ and $p_3^2$ are collections of all double sequences whose $P$-transforms are in the spaces $l_0^2, c_2, c_0^2$ and $c_0^2$ respectively; where $l_0^2, c_2, c_0^2$ and $c_0^2$ are the double spaces of bounded, convergent, both bounded and convergent and null sequences respectively in the Pringsheim’s sense, see Moricz [3]. We introduce a new double sequence space $p_2^2$ of Pascal as the set of all double sequences whose $P$-transforms are in the space $l_2^2$. Next, fixing some notations is necessary.

Let $\omega^2$ be a vector space of all real or complex valued double sequences for which coordinatewise addition and scalar multiplications are defined, see Moricz [3]. Further, a vector subspace of $\omega^2$ is termed as a double sequence space, see Moricz [3]. The space $l_0^2$ denotes the space of all bounded sequences with norm $\|x\|_0 = \sup_{j, k \in \mathbb{N}} |x_{jk}| < \infty, N = \{1, 2, 3, \ldots\}$. If $x = x_{jk} \in \mathbb{C}$, then $x$ is convergent to a number $l$ in Pringsheim’s sense if for every $\epsilon > 0$, there exists a number $n_0 = n_0(\epsilon) \in \mathbb{N}$ and $l \in \mathbb{C}$ such that $|x_{jk} - l| < \epsilon \forall j, k \geq n_0$ and we write $\lim_{j, k \rightarrow \infty} x_{jk} = l, \mathbb{C}$ being the complex field, see Pringsheim [4]. $c_2$ is used to denote the space of all convergent double sequences in Pringsheim’s sense, see Moricz [3]. In Moricz [3], it is pointed out that there are some double sequences in $c_2$ that are not in $l_0^2$; for example, the double sequence $x = (x_{jk})$ defined by $x_{jk} := \begin{cases} n, & j = 0, k \in \mathbb{N}, \\ 0, & j \geq 1, k \in \mathbb{N}. \end{cases}$

In Moricz [3], it can be seen that $x$ is convergent in Pringsheim’s sense but not bounded, since $\lim_{j,k \rightarrow \infty} x_{jk} = 0$ but $\|x\|_0 = \infty$.

Hence, we consider the space $c_0^2$ of all double sequences which are both convergent in Pringsheim’s sense and bounded; that is, $c_0^2 = c_2 \cap l_0^2$, Moricz [3]. So, $c_0^2$ is the space of all double sequences converging to zero in Pringsheim’s sense, and $c_0^2$ is the space of all double sequences that are bounded and converging to zero in Pringsheim’s sense, i.e., $c_0^2 = c_0^2 \cap l_0^2$, Moricz [3]. Also, Basar and Sever [2] defined the space $l_q^2$ by

$$l_q^2 := \left\{x = (x_{jk}) \in \omega^2 : \sum_{j, k = 0}^{\infty} |x_{jk}|^q < \infty \right\}, (1 \leq q < \infty).$$

For $1 \leq q < \infty$, Basar and Sever [2] proved that the double sequence space $l_q^2$, corresponds to $l_q$ of single sequences, and is a Banach space with the norm

$$\|x\|_{l_q^2} := \left(\sum_{j, k = 0}^{\infty} |x_{jk}|^q\right)^{\frac{1}{q}},$$

$$1 \leq q < \infty$$

(1)

In Basar and Sever [2], in case $q \in (0, 1)$ in (1), then the space $l_q^2$ is a $q$-norm with the $q$-norm

$$\|\|x\|\|_{l_q^2} := \left(\sum_{j, k = 0}^{\infty} |x_{jk}|^q\right)^{\frac{1}{q}}.$$ 

Let $X$ and $Y$ be two double sequence spaces and $A = (a_{mnjk})$ be any four-dimensional infinite matrix of complex numbers as in Robison [16]. In Robison [16], it was also indicated that $A$ is said defines a matrix mapping from $X$ into $Y$ and write $A: X \rightarrow Y$ for every $x = (x_{jk}) \in X$, so that the $A$ -transform of $x = (x_{jk})$ is $Ax = (Ax)_{mn} = \sum_{j, k = 0}^{\infty} a_{mnjk} x_{jk}$ for each $m, n \in \mathbb{N}$, exists. For matrix domains, Yesilkayagil & Basar [10], the $v$ -matrix domain $\chi_A^{(v)}$ of $A$ in $X$ is defined by

\[ v \text{ -matrix domain } \chi_A^{(v)} \text{ of } A \text{ in } X \text{ is defined by } \]
\[ X_A^{(v)} = \left\{ x = (x_{jk}) : \omega^2; P - \sum_{j,k} a_{mnjk}x_{jk} \text{ exists and is in } Y \right\}. \]

Clearly, (2) suggests that \( A \) maps \( X \) into \( Y \) if \( X \subset Y_A^{(v)} \); and \( (X:Y) \) can denote the set of all four-dimensional matrices transforming \( X \) into \( Y \), see Yesilkayagil & Basar [10]. \( A = (a_{mnjk}) \in (X:Y) \) if, and only if the double series on the right of (1) converges in Pringsheim’s sense for each \( m, n \in \mathbb{N} \), that is, \( A_{mn} \in c_0^{(v)} \) for all \( j, k \in \mathbb{N} \) and any \( x \in X \) have \( Ax \in Y \), see Yesilkayagil & Basar [10]. It is well known, for example in Cooke [15], that \( A = (a_{mnjk}) \) is a triangular matrix if \( a_{mnjk} = 0 \) for \( j > m, k > m \) or both, and \( a_{mnjk} \neq 0 \) for all \( m, n \in \mathbb{N} \) and every triangular matrix has a unique inverse which also happens to be a triangular matrix too.

II. THE PASCAL DOUBLE SEQUENCE SPACE \( p_q^2 \)

Pascal matrix of finite order existed for a very long time as pointed out by Aggarwala & Lamoureux [5], where the authors declared that there was no reason, whatsoever, to stop at a finite matrix of this type for, one can extend the Pascal matrix of finite order to an infinite lower triangular matrix. We felt that this extension aroused Polat [1] to introduce some Pascal sequence spaces, each which is a matrix domain via infinite Pascal matrix as follows:

- \( (l_{\infty})_p = p_{\infty} = \left\{ x = (x_k) \in \omega^2 : \sup_n \left| \sum_{k}^{n} (\frac{n}{n-k}) x_k \right| < \infty \right\} \)
- \( (c)_p = p_c = \left\{ x = (x_k) \in \omega^2 : \lim_{n \to \infty} \sum_{k}^{n} (\frac{n}{n-k}) x_k \exists \right\} \)
- \( (c_0)_p = p_0 = \left\{ x = (x_k) \in \omega^2 : \lim_{n \to \infty} \sum_{k}^{n} (\frac{n}{n-k}) x_k = 0 \right\} \)

Recently, Kiltho et al. [17] introduced Pascal double sequence spaces, \( p_0^2, p_c^2, p_{bc}^2 \) and \( p_2^2 \) as matrix domains of four-dimensional Pascal matrix, as an extension of the work of Polat [1]. This paper will therefore wish to introduce the Pascal double sequence \( p_q^2 \), the set of all double sequences whose \( P \)-transforms are in the space \( l_q^2 \). We define the four-dimensional Pascal matrix \( P = (p_{mn}) \) as follows:

\[ p_{mn}^{jk} = \begin{cases} \frac{m}{m-j} \left( \frac{n}{n-k} \right), & 0 \leq j \leq m, 0 \leq k \leq n \\ 0, & j > m \text{ and } k > n. \end{cases} \]

with inverse \( P^{-1} = Q = (q_{mn}) \) defined by

\[ q_{mn}^{jk} = \begin{cases} (-1)^{(m-j)(n-k)} \left( \frac{m}{m-j} \right) \left( \frac{n}{n-k} \right), & 0 \leq j \leq m, 0 \leq k \leq n \\ 0, & j > m \text{ and } k > n. \end{cases} \]

Now, we introduce the space \( p_q^2 \) as the collection of all double sequences such that its \( P \)-transform is in the space \( l_q^2 \), as follows:

\[ p_q^2 = \left\{ x = (x_{jk}) \in \omega^2 : \sum_{m,n}^{m,n} \sum_{j,k=0}^{m} \left( \frac{m}{m-j} \right) \left( \frac{n}{n-k} \right) x_{jk} q^{j+k} \right\}. \]

Following Yesilkayagil & Basar [10] the space \( p_q^2 \) is linear with coordinatewise addition and scalar multiplication, where we are going to show that it is a complete \( q \)-normed space with the \( q \)-norm:

\[ \|x\|_{p_q^2} = \left( \sum_{m,n}^{m,n} \sum_{j,k=0}^{m} (m-j) (n-k) x_{jk} q^{j+k} \right)^{\frac{1}{2}}. \]

Let \( x_{mn} = (x_{jk}) \in \omega^2 : Ax \in X \) be a matrix domain of a four-dimensional matrix \( A \), then the Pascal sequence space in (6) is also a matrix domain, as \( p_q^2 = (l_q^2)_p = P(l_q^2)_p \); while \( P \)-transform of a double sequence space \( = (x_{jk}) \) in (6) can be defined as

\[ y_{mn} = (P x)_{mn} = \sum_{m,n}^{m,n} (m-j) (n-k) x_{jk} \]

for all \( m, n \in \mathbb{N} \). The terms of the double series \( x = (x_{mn}) \) and \( y = (y_{mn}) \) are assumed to be connected with the relation (8).

Definition 1: A Banach space is called a \( BK \)-space provided each of the maps \( p_{jk} : X \in C \) defined by \( p_{jk} = x_{jk} \) is continuous for all \( j, k \in \mathbb{N} \), see Choudhary & Nanda [12].

Lemma 1: If \( A \) is a triangle and \( X \) is a \( BK \)-space, then \( X_A \) is a \( BK \)-space with the norm given by \( \|x\|_{X_A} = \|x\|_X \) for all \( x \in X_A \), see Boos [11].

By considering the notion of \( BK \)-space, one can say that the sequence space \( l_q^2 \) is a \( BK \)-space according to its \( l_q^2 \)-normed defined by \( \|x\|_{l_q^2} = \left( \sum_{m,n}^{m,n} |x_{jk}|^q \right)^{\frac{1}{q}} \), where \( 1 \leq q < \infty \).

Next, we present our results viz:

Theorem 1: The Pascal sequence \( p_q^2 \) is a \( BK \)-space according to the norm defined by

\[ \|x\|_{p_q^2} = \|P x\|_{l_q^2} = \left( \sum_{m,n}^{m,n} |x_{jk}|^q \right)^{\frac{1}{q}}, \quad 1 \leq q < \infty \]
\[\sum_{j,k=0}^{m,n} Q_{m,n} = x.\] This shows that the coordinates are continuous on \(p^2_q\). Hence \(p^2_q\) is a BK space.

**Theorem 2:** The set \(p^2_q\) becomes a linear space with the coordinatewise addition and scalar multiplication and the following statements hold:

i) If \(q \in (0,1)\), then \(p^2_q\) is a complete \(q\)-normed space with

\[\|x\|_{p^2_q} = \left(\sum_{m,n} \left(\sum_{j,k=0}^{m,n} m \cdot (n - k) x_{jk}\right)^q\right)^{\frac{1}{q}}\]

which is \(q\)-norm isomorphic to the space \(l^2_q\).

ii) If \(q \in [1,\infty)\), then \(p^2_q\) is a Banach space with

\[\|x\|_{p^2_q} = \left(\sum_{m,n} \left(\sum_{j,k=0}^{m,n} m \cdot (n - k) x_{jk}\right)^q\right)^{\frac{1}{q}}\]

which is isomorphic to the space \(l^2_q\).

**Proof:** We are going to give the proof of the second part (ii), since the first part (i) can be proved in a similar way.

The first part of the theorem is a routine verification, where it can be easily seen that (a) \(p^2_q\) is not empty; (b) the sum of any two elements in \(p^2_q\) is also in \(p^2_q\); and (c) the scalar multiplication \(ax \in p^2_q\) \(\forall x \in \mathbb{C}\) and \(x \in p^2_q\). Thus, \(p^2_q\) is a linear space with coordinatewise addition and scalar multiplication. Now, we can show that \(p^2_q\) is a Banach space with the norm defined by (7). Let \((x^\alpha)_{\alpha \in \mathbb{N}}\) be any Cauchy sequence in the space \(p^2_q\), where \(x^\alpha = \{x_{jk}(\alpha)\}_{j,k=0}^\infty\) for every fixed \(\alpha \in \mathbb{N}\). Then for a given \(\varepsilon > 0\), there exists a positive integer \(N = N(\varepsilon)\) such that

\[\|x^\alpha - x^\beta\|_{p^2_q} = \left(\sum_{m,n} \left(\sum_{j,k=0}^{m,n} m \cdot (n - k) (x^\alpha_{jk} - x^\beta_{jk})\right)^q\right)^{\frac{1}{q}}< \varepsilon \ \forall \alpha, \beta > N\]

which yields for each \(m,n \in \mathbb{N}\) and applying Minkowki’s inequality, that

\[\|x^\alpha - x^\beta\|_{p^2_q} = \left(\sum_{m,n} \left(\sum_{j,k=0}^{m,n} m \cdot (n - k) x^\alpha_{jk}\right)^q\right)^{\frac{1}{q}} - \left(\sum_{m,n} \left(\sum_{j,k=0}^{m,n} m \cdot (n - k) x^\beta_{jk}\right)^q\right)^{\frac{1}{q}}< \varepsilon\]

This means that \((x^\alpha_{jk})_{j,k=0}^{m,n} = (x^\alpha_{m,n})_{j,k=0}^{m,n} \in \mathbb{N}\) is a Cauchy sequence with complex terms for every fixed \(m,n \in \mathbb{N}\). Since \(\mathbb{C}\) is complete, it converges, i.e.

\[\left(\sum_{m,n} \left(\sum_{j,k=0}^{m,n} m \cdot (n - k) x_{jk}\right)^q\right)^{\frac{1}{q}} \rightarrow \left(\sum_{m,n} \left(\sum_{j,k=0}^{m,n} m \cdot (n - k) x_{jk}\right)^q\right)^{\frac{1}{q}}\]

as \(\alpha \rightarrow \infty\),

such that

\[\lim_{\alpha \rightarrow \infty} \|x^\alpha - x^\beta\|_{p^2_q} = 0.\]

Since, \((\sum_{m,n} \left(\sum_{j,k=0}^{m,n} m \cdot (n - k) x^\alpha_{jk}\right)^q\) \(\in p^2_q\) for each fixed \(\alpha \in \mathbb{N}\), there exists a positive real number \(K_\alpha\) such that

\[\left(\sum_{m,n} \left(\sum_{j,k=0}^{m,n} m \cdot (n - k) x^\alpha_{jk}\right)^q\right)^{\frac{1}{q}} \leq K_\alpha.\]

Therefore, taking summation over \(m,n\) in the following relation

\[\left(\sum_{m,n} \left(\sum_{j,k=0}^{m,n} m \cdot (n - k) x^\alpha_{jk}\right)^q\right)^{\frac{1}{q}} \leq K_\alpha.\]

This shows that \(x = (x_{jk}) \in p^2_q\). Since \((x^\alpha)_{\alpha \in \mathbb{N}}\) is an arbitrary Cauchy sequence, then the space \(p^2_q\) is complete. Thus, \(p^2_q\) is a Banach space with the norm \(\|x\|_{p^2_q} = \left(\sum_{m,n} \left(\sum_{j,k=0}^{m,n} m \cdot (n - k) x_{jk}\right)^q\right)^{\frac{1}{q}}\).

To prove the fact that \(p^2_q\) is linearly isomorphic to \(l^2_q\), we have to show the existence of a linear bijection between the spaces \(p^2_q\) and \(l^2_q\). Consider the transformation \(\tau\) defined from \(p^2_q\) to \(l^2_q\) by \(x \mapsto y = \tau x = (\tau x)_{m,n}\). Clearly, \(\tau\) is linear, \(\tau(u + v) = \tau(u) + \tau(v)\) \(\tau(u) + \tau(v)\) for all \(u, v \in p^2_q\). Further, we can see that \(\tau = 0\), whenever \(\tau = 0\) which shows that \(\tau\) is injective. Now, let \(y = (y_{jk}) \in l^2_q\) and define a sequence \(x = (x_{jk})\) by

\[x_{jk} = \sum_{u,v=0}^{j,k} (-1)^{(j-u)+(k-v)} \left(\frac{j}{k} u_{j-u}(k-v) v_{j-v}\right) \forall u, v \in \mathbb{N}.\]
Hence, by taking into account the hypothesis \( y \in l^2_q \), one can derive by taking summation over \( m,n \in \mathbb{N} \) on the following equality

\[
\begin{align*}
&= \left| \sum_{j,k=0}^{m,n} (m-j)(n-k) \sum_{u,v=0}^{j,k} (-1)^{j-u+(k-v)}(j-u)(k-v)y_{uv} \right| \\
&= |y_{mn}|. 
\end{align*}
\]

That is, \( \|Px\|_q = \|y\|_q \), which implies that \( x \in p^q \). Therefore, \( \tau \) is surjective. Hence, \( p^q \approx l^2_q \).

**Definition 2:** A barreled space is a topological vector space for which every barreled set in the space is a neighbourhood for the zero vector. A barreled set in a topological vector space is a set that is convex, balanced, absorbing, and closed, see [Narici & Beckenstein [13]]

**Lemma 2:** If the sequence space \( X \) is a Banach space or a Frechet space, then it is a barreled space, see Schaefer [14].

**Theorem 3:** The double sequence space \( l^2_q \) is a barreled space for \( 1 \leq q < \infty \).

**Proof:** We have seen that \( l^2_q \) is a Banach space in Theorem 2. Thus, the proof of Theorem 3 is obvious by Theorem 2 and Lemma 2.

**Definition 3:** The space \( X \) of double sequence spaces is monotone if \( xu = (x_\ell u_\ell) \in X \) for every \( x = (x_\ell) \in X \) and \( u = (u_\ell) \in \mathbb{X}^2 \). Then we have \( |x_\ell u_\ell|^q = |x_\ell|^q |u_\ell|^q \leq |x_\ell|^q \) for each \( j,k \in \mathbb{N} \). This simply means that the inequality \( \sum_{j,k=0}^{m,n} |x_\ell u_\ell|^q \leq \sum_{j,k=0}^{m,n} |x_\ell|^q \) holds. That is, \( xu = (x_\ell u_\ell) \in l^2_q \). This shows that \( l^2_q \) is monotone for all \( 1 \leq q < \infty \).

**Theorem 4:** The space \( l^2_q \) is monotone for all \( 1 \leq q < \infty \).

**Proof:** Let \( 1 \leq q < \infty \), \( x = (x_\ell) \in l^2_q \) and \( u = (u_\ell) \in \mathbb{X}^2 \). Then we have \( |x_\ell u_\ell|^q = |x_\ell|^q |u_\ell|^q \leq |x_\ell|^q \) for each \( j,k \in \mathbb{N} \). This simply means that the inequality \( \sum_{j,k=0}^{m,n} |x_\ell u_\ell|^q \leq \sum_{j,k=0}^{m,n} |x_\ell|^q \) holds. That is, \( xu = (x_\ell u_\ell) \in l^2_q \). This shows that \( l^2_q \) is monotone for all \( 1 \leq q < \infty \).

**Definition 4:** A double sequence \( (x_\ell u_\ell)_{j+k=0}^{\infty} \) is called a Schaefer double basis if, for every \( x \in X \), there exists a unique double sequence of scalars \( (\lambda_\ell)_{j+k=0}^{\infty} \) such that \( x = \sum_{j+k=0}^{\infty} \lambda_\ell x_\ell u_\ell \), see Loganathan & Moorthy [8].

**Theorem 5:** Let \( m,n,j,k \in \mathbb{N} \) and define \( q^{(jk)} = \{q^{(jk)}\}_{mn}^{jk} \) by the formula

\[
q^{(jk)} = \begin{cases} 
0 & \text{if } j > m \text{ and } k > n \text{ or both}, \\
(-1)^{(m-j)+(n-k)} \binom{m-j}{n-k} \binom{n}{m-k} & \text{if } 0 \leq k \leq n \text{ and } 0 \leq j \leq m.
\end{cases}
\]

Then the set \( \{q^{(jk)}\} \) is a double basis for the double sequence space \( p^q \) such that any \( x \in p^q \) has a unique representation of the form

\[
q^{(jk)} \sum_{j+k=0}^{\infty} \lambda_\ell x_\ell u_\ell, \quad x \in \sum_{j+k=0}^{\infty} \lambda_\ell x_\ell u_\ell.
\]

**Proof:**

\[
\begin{align*}
&= \sum_{j+k=0}^{\infty} \zeta^{(jk)} \sum_{j+k=0}^{\infty} \lambda_\ell x_\ell u_\ell.
\end{align*}
\]

where \( \zeta^{(jk)} = (Px)^{jk} \forall j,k \in \mathbb{N} \).

**Theorem 6:** The inclusion \( l^2_q \subset p^q \) holds for \( 1 \leq q < \infty \).

**Proof:** Let us take arbitrary \( x = (x_\ell) \in l^2_q \). Then there exists a positive real number \( K \) such that \( \sum_{j+k=0}^{\infty} |x_\ell|^q \leq K \). Applying Holder’s inequality to (8), we have

\[
|\sum_{j+k=0}^{\infty} \lambda_\ell x_\ell u_\ell| \leq \left( \sum_{j+k=0}^{\infty} \left( \sum_{j+k=0}^{\infty} \lambda_\ell x_\ell u_\ell \right)^q \right)^{1/q} \leq \left( \sum_{j+k=0}^{\infty} |x_\ell|^q \right)^{1/q} \leq \left( \sum_{j+k=0}^{\infty} |x_\ell|^q \right)^{1/q} \leq \left( \sum_{j+k=0}^{\infty} |x_\ell|^q \right)^{1/q}.
\]

This implies that

\[
\sum_{j+k=0}^{\infty} |\sum_{j+k=0}^{\infty} \lambda_\ell x_\ell u_\ell| \leq \sum_{j+k=0}^{\infty} |x_\ell|^q \leq K.
\]

This completes the proof.
\begin{equation*}
\sum_{m,n} |x_{jk}|^q \left( \sum_{m,n=0,0}^{\infty} \left( \frac{m}{n-k} \right) \right) \leq K
\end{equation*}

That shows that $x = (x_{jk}) \in p_q^2$ for all $1 \leq q < \infty$. Hence, $l_q^2 \subset p_q^2$ holds for $1 \leq q < \infty$.

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