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Quasi Injective Fuzzy G-Modules On P_r

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Abstract- Representation theory (G-module theory) has had its origin in the 20th century. In the 19th century, groups were generally regarded as subsets of some permutation set, or of the set GL(V) of automorphisms of a vector space V, closed under composition and inverse. Here we consider P_r, the periodic arithmetical functions (mod r), define a fuzzy G-module on it and verify the quasi injective property of its summands.

Index Terms- Injectivity, quasi injectivity, fuzzy set, fuzzy G-module, fuzzy injectivity and quasi fuzzy injectivity.

I. Introduction

In *representation theory*, we consider the embedding of a finite group into a linear group. Here we consider those finite groups, which can be embedded in a ***finite*** *linear group*. The *fuzzy set theory* was introduced by L.A. Zadeh[2] in 1965. Rosenfield[4] started fuzzification of algebraic structures. As a continuation of these works the concept fuzzy finite G-module was introduced and analysed by us in [5].

In this paper, we discuss injectivity and quasi injectivity of fuzzy G-modules. We introduce the G-module P_r of periodic arithmetic functions mod r and discuss quasi injectivity in relation to it.

1. Preliminaries

1.1. Definition [1]. Let G be a finite group, M be a vector space over K (a subfield of **C**) and GL(M) be the group of all linear isomorphisms from M onto itself. A ***linear representation*** of G with representation space M is a homomorphism $T : G \to GL(M)$.

1.2. Example. Let F be a field, K be an extension field of F and a ε K. Let M = F(a), the field obtained by adjoining 'a' to F.

(i.e) $M = F(a)=\{ b_o+b_1a+b_2a^2+.........: b_i \varepsilon F\}$

Let G = (a), the cyclic group generated by 'a'. For j ε **Z**, define $T_j : M \to M$ by

$T_j (\sum \beta_i a^i) = \sum \beta_i a^{i+j}$

Then T_j is an isomorphism of M onto itself. Also the map $T:G\to GL(M)$ defined by

$T(a^j) =T_j, \forall j \varepsilon \mathbf{Z}$

is a homomorphism and hence a linear representation of G.

1.3. Definition [1]. Let G be a finite group. A vector space M over a field K is called a ***G-module*** if for every g ε G and m ε M, there exist a product (called the ***action of G on M***) m.g ε M satisfying the following axioms:

(i) $m.1_G =m, \forall m \varepsilon M$ (1_G being the identity element in G)
(ii) m.(g.h)= (m.g).h, ∀m ε M; g, h ε G; and
(iii) $(k_1m_1+k_2m_2).g = k_1(m_1.g) + k_2(m_2.g), \forall k_1, k_2 \varepsilon K; m_1, m_2 \varepsilon M; g \varepsilon G$

1.4. Example. Let G = {1,-1} and M=**Q**(√2). Then M is a vector space over **Q**, and under the usual addition and multiplication of the elements of M, we can show that, M is a G-module.

1.5. Definition [6]. An ***arithmetical function*** is a complex-valued function defined on the set of positive integers.
For a positive integer r, an arithmetical function f is said to be ***periodic (mod r)*** if f (n + r) = f(n) for all n ∈ N

1.6. Proposition. Let P_r denote the set of all periodic arithmetical functions (mod r). Then P_r is a complex vector space. Also P_r is isomorphic to $\mathbf{C}^r$, the r-dimensional complex space

Proof: Given P_r = {functions f : N→ **C** / f(n + r) = f(n) for all n ∈ N}

Define the operations addition and scalar multiplication in P_r by

[f + g] (n) = f (n) + g (n), n ∈ N
[cf] (n) = c f (n), c ∈ **C**, n ∈ N

Then P_r is a complex vector space. It is an r-dimensional space and is isomorphic to $\mathbf{C}^r$. The set $\{\alpha_k = r^{\frac{-1}{2}} \varepsilon_k : k = 1, 2, \ldots, r\}$ is a basis of P_r, where $\varepsilon_k \in P_r$ is defined by $\varepsilon_k (n) = \exp^{\left(2\pi ikn/_r\right)}$ ■

1.7. Remark. Let G = {1,-1} or G = {1,-1.i.-i}.Then the vector space P_r is a G-module.

1.8. Definition[3] Let $M_1, M_2, \ldots, M_n$ be vectorspaces over a field K. Then the set $\{m_1+m_2+\ldots+m_n : m_i \varepsilon M_i\}$ becomes a vectorspace over K under the operations
$(m_1+m_2+\ldots+m_n) + (m_1'+m_2'+\ldots+m_n') = (m_1+m_1') + (m_2+m_2') +\ldots +(m_n+m_n')$ and
$\alpha(m_1+m_2+\ldots+m_n)=\alpha m_1+ \alpha m_2+\ldots+ \alpha m_n; \alpha \varepsilon K, m_i, m_i' \varepsilon M_i$.

It is called the ***direct sum*** of the vector spaces $M_1,M_2,\ldots,M_n$. It is denoted by $\overset{n}{\underset{i=1}{\oplus}} M_i$.

1.9. Example. The set **Q**(√2,√3) is the field obtained by adjoining the real numbers √2,√3 to **Q**. Then we have **Q**(√2,√3) is a vector space over **Q** and the set { 1,√2,√3,√6} is a basis for **Q**(√2,√3) over **Q**. Let M_1= **Q**, M_2 = **Q**(√2), M_3 = **Q**(√3) and M_4 = **Q**(√6). Then **Q**(√2,√3) = $\bigoplus_{i=1}^{4} M_i$.

1.10. Remark. The G-module M = P_r can be expressed as the direct sum of r G-sub modules as follows:

We have observed that P_r is a complex vector space. It is an r-dimensional space and is isomorphic to $\mathbf{C}^r$. The set {$\alpha_k = r^{\frac{-1}{2}} \varepsilon_k$: k = 1, 2, ….., r} is a basis of P_r , where $\varepsilon_k \in P_r$ is defined by ε_k (n) = exp$\left(2\pi ikn/r\right)$ .Then $\bigoplus_{i=1}^{r} M_i$, where $M_i = \mathbf{C}\alpha_k$.

1.11. Definition[1]. A G-module M is ***injective*** if for any G-module M* and any G-submodule N of M*, every homomorphism from N into M can be extended to a homomorphism from M* into M.

1.12. Example. Let G = {1,-1,i,-i } and M= **C**, which is a vector space over **C**. Then M is a G-module with respect to trivial action. Also, except the zero G-submodule, no proper subset of **C** becomes a G-module.

Let M* be any other G-module. Then following are some prominent cases of M* :

(i) M* = {0}
(ii) M* = $\mathbf{C}^n$ (n≥1) or M* a G-submodule of $\mathbf{C}^n$
(iii) M* = Space of all functions from any set S into **C**
(iv) M* = $\mathbf{C}^{m\times n}$ = Space of all m×n matrices over the field **C** or a G-submodule of M*

Let N be a G-submodule of M* and φ : N → M be a homomorphism .

Case(i) : Here N = M* = {0}, and so 0= ψ : M* → M extends the homomorphism φ .

Case(ii) : Since C^n is n-dimensional, then Dim.M* = k ≤ n. Let Dim.N=m and let {α_1, α_2,........ α_m} be a basis of N such that {α_1, α_2,.. α_m, α_{m+1}.. α_k} is a basis of M*. Then

N = $\mathbf{C}\alpha_1 \oplus \mathbf{C}\alpha_2 \oplus \ldots \oplus \mathbf{C}\alpha_m$ and

M* = $\mathbf{C}\alpha_1 \oplus \mathbf{C}\alpha_2 \oplus \ldots \oplus \mathbf{C}\alpha_m \oplus \mathbf{C}\alpha_{m+1} \oplus \ldots \oplus \mathbf{C}\alpha_k$. The map ψ : M* → M defined by

ψ ($c_1\alpha_1+\ldots+c_m\alpha_m+\ldots c_k\alpha_k$) = φ ($c_1\alpha_1+\ldots+c_m\alpha_m$) is a homomorphism which extends φ .

Case(iii) :Here M* = $M_1 \oplus M_2$, where M_1 is the G-submodule of M* consisting of all odd functions and M_2 is the G-submodule of M* of all even functions. Then as in (ii), there exits a homomorphism ψ :M → M*, which lifts φ .

Case(iv): Since $\mathbf{C}^{m\times n}$ is an mn-dimensional vector space over **C**, Dim.M* ≤mn, and so as in (ii), there exits a homomorphism ψ :M → M*, which lifts φ Similarly for any G-module M* and any G-submodule N* of M*, every homomorphism φ : N*→M can be extended to a homomorphism ψ :M*→M. Therefore M is an injective G – module.

1.13. Definition[1]. Let M and M* be G-modules. Then ***M is M*-injective*** if for every G-submodule N of M*, any homomorphism φ :N→M can be extended to a homomorphism ψ :M* →M.

1.14. Proposition[5]. Let M = $M_1 \oplus M_2$, where M_1 and M_2 are G-submodules of M. Then M is injective if and only if M_1 and M_2 are both injective.

Proof: Let M be injective. Let M* be a G-module and N be any G-submodule of M* and let η : N → M_1 be a homomorphism .Since M is injective, there exists a homomorphism η' : M* → M. Let π : M → M_1 be the projection map. Then η'' = π• η' : M*→M_1 is an extension of η. Therefore M_1 is injective. Similarly we can show that M_2 is injective.

Conversely suppose M_1 and M_2 are injective. Let M* be a G-module and N be any G-submodule of M* and let η : N → M be a homomorphism. Let π_1 and π_2 be the projections of M_1 and M_2 respectively. Since M_1 and M_2 are injective, the mappings π_1•η : N → M_1 and π_2• η: N→ M_2 can be extended to homomorphisms η_1 : M*→M_1 and η_2 : M* → M2 respectively. Define η_3 : M* → M by

η_3 (m) = η_1 (m) + η_2 (m), ∀m ε M*.

Then η_3 is a homomorphism. Also for every m ε M,

η_3 (m)= η_1 (m) + η_2 (m) = π_1• η (m)+ π_2• η (m) = η(m)

Therefore η_3 extends η and hence M is injective ∎

1.15. Proposition[5]. Let M and M* be G-modules such tha M is M*-injective. If N* is a G- submodule of M*, then M is N*-injective and M is M*/N*-injective

Proof: Since N* ⊆ M* and M is M*-injective, it is obvious that M is N*-injective.

Let X*/N* be a G-submodule of M*/N* and φ: X*/N* → M be a homomorphism. Let π: M*→ M*/N* be the canonical map and π' = $\pi|_{X^*}$. Then φ •π': X* → M is a homomorphism. Since M is M*-injective ∋ an extension θ: M* → M of φ •π'. Then θ(N*) = φ •π'(N*) = φ (π'(N*)) = φ (0) = 0. Therefore Ker.π is a G-submodule of Ker.θ and so ∋ a map ψ:M*/N* → M such that ψ •π =θ. Also for any x ε X,

ψ (x+N*)=ψ (π(x)) = θ(x) = (φ •π')(x) = φ (x+N*).

Therefore ψ extends φ. Hence M is M*/N*-injective ∎

1.16. Example. Let M* = $\mathbf{R}^n$. This is n-dimensional vector space over **R**. Let {α_1, α_2,.. α_k,.. α_n} be a basis for M*. Then M* =$\mathbf{R}\alpha_1 \oplus \mathbf{R}\alpha_2 \oplus$ --------$\oplus \mathbf{R}\alpha_n$.

Let M =**R** and G be any finite multiplicative subgroup of **R**. Then both M* and M are G-modules. Let N be any G-submodule M* and φ : N→ M be a homomorphism.

(i). If N = {0}, then φ = 0, then ψ = 0: M→M* extends φ .
(ii). If N = $\mathbf{R}\alpha_j$ (1≤j≤n).

Then ψ: M→M* defined by

ψ($c_1\alpha_1+\ldots+c_j\alpha_j+\ldots+c_n\alpha_n$) = φ ($c_j\alpha_j$) is a homomorphism which extends φ .

(iii). N = $\bigoplus_{i=1}^{n} R_{\alpha j}$ (k≤n), then ψ: M→M* defined by ψ($c_1\alpha_1+\ldots+c_k\alpha_k+\ldots+c_n\alpha_n$) = φ ($c_1\alpha_1+\ldots+c_k\alpha_k$) extends φ. Therefore M is M*- injective .

1.17. Definition[1]. A G-module M is ***Quasi-injective*** if M is M-injective.

1.18. Example. Let S ={1,ω,ω^2},where ω is a complex cube root of unity and G =S_3, the symmetric group of degree three. Let M = span(S) over **R** ={ α + βω + γ ω^2 : α,β,γ ε **R**}. Then M is a vector space over **R**. For each x ε G, define T_x :M → M by

$$T_x(\alpha + \beta\omega + \gamma\omega^2) = \alpha\, x(1)+\beta\, x(\omega)+ \gamma\, x(\omega^2)$$

Then T_x is an isomorphism of M onto itself. Also the map T: G→ GL(M) defined by

$$T(x) = T_x, \forall x \varepsilon G,$$

is a representation of G, and hence M is a G-module. Also the only G-submodules of M are M and {0}. We will show that M is M-injective. Let N be any G-submodule of M. Then N={0} or N=M. Let φ: N → M be any homomorphism.

Case(i). N={0}: Then the map ψ : M→ M defined by ψ (x)=0, ∀ x ε M extends φ .

Case(ii). N=M: In this case, φ is a homomorphism from M into itself; and hence ψ = φ is the required extension.

Thus, in both cases, φ:N → M can be extended to a homomorphism ψ :M → M. Therefore M is M-injective; and hence quasi-injective .

II. FUZZY G-MODULE INJECTIVITY

2.1. Definition (Fuzzy set) [2]. The characteristic function of a crisp set (classical set or non-fuzzy sets) assigns a value of either 1 or 0 to each individual element in the universal set, thereby discriminating between members and non-members of the crisp sets under consideration. This function can be generalised in such a way that the values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements in the set in question. Larger values denote the higher degrees of the set membership. Such a function is called a ***membership function***, and the set defined by it a ***fuzzy set***.

The most commonly used range of values of membership functions is the unit interval **[0,1].**
i.e. A fuzzy set μ on the set X is a function μ : X→[0,1].

2.2. Definition [5]. Let G be a finite group and M be a G-module over K, which is a subfield of complex numbers. Then a ***fuzzy G-module*** on M is a fuzzy subset μ of M such that

(i) μ (ax+by) ≥ μ (x) ^ μ (y),∀ a, b ε K and x, y ε M
and (ii) μ (gm) ≥ μ(m), ∀ g ε G, m ε M. Where Λ is the minimum[infimum] operator.

2.3. Example. Let G = {1,-1, i, -i}. Then M = **C**, the field of complex numbers is a G-module over itself. Define μ : M →[0,1] by

μ (x+iy) =1, if x = y =0
=½, if x ≠0, y =0
=¼, if y≠0

Then μ is a fuzzy G-module on M.

2.4. Theorem. Let M be the G-module P_r . Then there exist a fuzzy G-module on M.

Proof: Here M is an r-dimensional G-module over K= **C**. Let **B** = {$\alpha_k = r^{\frac{-1}{2}} \varepsilon_k$: k = 1, 2… r}, a basis for M.
Define ν : M → [0,1] by

ν ($c_1\alpha_1+c_2\alpha_2+..+c_r\alpha_r$) = 1, if c_i=0 for all i
=½, if c_1≠0, c_2=c_3=.....=c_r=0
=⅓, if c_2≠0, c_3=c_4=.....=c_r=0
=¼, if c_3≠0, c_4=c_5=.....=c_r=0
....
....
=1/r-1, if c_{r-2}≠0, c_{r-1}=c_r=0
=1/r, if c_{r-1}≠0, c_r=0
=1/r+1, if c_r≠0

Then ν is a fuzzy G-module on M ■

2.5. Proposition[5]. For any fuzzy G-module μ on a G-module M and for each k ε (0,1], μ_k : M → [0,1] defined by μ_k(x) = k.μ(x), ∀x ε M is also a fuzzy G-module on M ■

2.6. Proposition. For any positive integer r, there exits infinite number of fuzzy G-modules on the G-module P_r.

Proof: Let M be the G-module P_r. Then from theorem 2.4, there exists a fuzzy G-module ν on M. Let k ε (0,1], then from the above proposition we have ν_k defined by

ν_k(x) = k.ν(x), ∀x ε M

is a fuzzy G-module on M. In the definition of fuzzy G-module ν in the theorem 1.15, replace 1 in the numerator by k. Then ν_k is a fuzzy G-module on M for each k ε (0,1] ■

2.7. Definition[5]. Let M and M* be G-modules. Let μ be any fuzzy G-module on M and ν be any fuzzy G-module on M*. Then **μ is ν-injective** if
(i) M is M*-injective.
(ii) ν (m) ≤ μ(ψ(m)) , ∀ ψ ε Hom(M*,M) and ∀ m ε M*. Where Hom(M*,M) is the set of all G-module homomorphism's from M* to M.

2.8. Example. Let G = (i), M = **C** and M* = **Q**(i). Then M and M* are G-modules over **Q**. Define μ : M → [0,1] and ν : M* → [0,1] by

μ(x) = 1, if x=0
= ½, if x ε **Q** (i) – {0}
= ¼, if x ε **C**- **Q** (i)

and

ν(x) = ¼, if x=0
= ⅕, if x≠0

Then μ and ν are fuzzy G-modules on M and M* respectively. Let X be any G-submodule of M*. Then either X = {0} or X= M* . Let φ : X → M be any homomorphism.

Case(i) If X = {0} : then φ = 0, so ψ =0 : M* → M extends φ .

*Case(ii) If X= M**: then ψ = φ extends φ .

Therefore M is M*-injective. Also it follows from the definitions of μ and ν that

ν(m) ≤ μ(ψ(m)), ∀ ψ ε Hom(M*,M) and ∀ m ε M*.

Therefore μ is ν-injective .

2.9. Definition[5]. Let M be a G-module and μ be a fuzzy G-module on M. Then μ is ***quasi-injective*** if

(i) M is quasi-injective

(ii) $\mu(m) \leq \mu(\psi(m))$, $\forall$ $\psi \in Hom(M,M)$ and $m \in M$.

2.10. Remark. Let M be a quasi-injective G-module. Then the functions $\mu: M \to [0,1]$ defined by (i) $\mu(x) = t$, $\forall x \in M$ and

(ii) $\mu(x) = 1$, if $x = 0$

$= t$, if $x \neq 0$, where 't' is a fixed element in [0,1], are quasi-injective fuzzy G-modules on M.

2.11. Example. The G-module M in example 1.16 is quasi-injective. On this M, if we define a function μ as in remark 2.10, then μ is quasi-injective .

2.12. Proposition[5]. Let M be a G-module over K and $M = \overset{n}{\underset{i=1}{\oplus}} M_i$, where M_i's are G-submodules of M .If ν_i $(1 \leq i \leq n)$ are fuzzy G-submodules on M_i, then $\nu : M \to [0,1]$ defined by

$\nu(m) = \Lambda \{ \nu_i(m_i) : i=1,2,...,n\}$, where $m = \sum_{i=1}^{n} m_i \in M$ is a fuzzy G-module on M.

Proof: Since each ν_i is a fuzzy G-module on M_i, for every $x,y \in M_i$, $g \in G$ & $a,b \in K$, we have

$\nu_i(ax+by) \geq \nu_i(x) \Lambda \nu_i(y)$ and $\nu_i(gx) \geq \nu_i(x)$

Let $x = \sum m_i$, $y = \sum m_i^1 \in M$ and $a,b \in K$, then

$\nu(ax + by) = \nu(\sum(a m_i + b m_i^1))$

$= \Lambda \{\nu_i(a mi + b m_i^1) : i=1,2,....,n\}$

$= \nu_i(a m_i + b m_i^1)$, where $1 \leq j \leq n$

$\geq \nu_i(m_i) \Lambda \nu_i(m_i^1)$

$\geq \nu(x) \Lambda \nu(y)$

Also for $g \in G$ and $x = \sum m_i \in M$,

$\nu(gx) = \nu(\sum g m_i) = \Lambda \{\nu_i(\sum g m_i) : i=1,2,....,n\}$

$= \nu_j(g m_j)$, fore some j

$\geq \nu_j(m_j)$

$\geq \nu(x)$

Therefore ν is a fuzzy G-module on M ■

2.13. Remark. In the above proposition, if $\nu_i(0)$ are all equal then we have $\nu(0) = \Lambda \{\nu_i(0) : i=1,2,....,n\} = \nu_i(0)$, for all i .

2.14. Definition[5]. The fuzzy G-module ν on $M = \overset{n}{\underset{i=1}{\oplus}} M_i$, in the proposition 2.12 with $\nu(0) = \nu_i(0)$ for all i, is called the ***direct sum*** of the fuzzy G-modules ν_i and is denoted by $\nu = \overset{n}{\underset{i=1}{\oplus}} \nu_i$.

2.15. Example. Let $G = \{1,-1\}$ and M= **C** over **R**. Then M is a G-module. We have $M = M_1 \oplus M_2$,

where M_1=**R**, M_2= i**R**. Define $\nu : M \to [0,1]$ by

$\nu(x+iy) = 1$, if $x = y = 0$

$= ½$, if $x \neq 0, y=0$

$= ⅓$, $y \neq 0$

Then ν is a fuzzy G-module on M. Also the mappings $\nu_1 : M_1 \to [0,1]$ defined by

$\nu_1(x)=0$, if x=0

$=½$, if $x \neq 0$

and $\nu_2 : M2 \to [0,1]$ defined by

$\nu_2(y)=0$, if y=0

$=⅓$, if $y \neq 0$

are fuzzy G-modules on M_1 and M_2 respectively and $\nu = \nu_1 \oplus \nu_2$ ■

2.16. Theorem[5]. Let M be a G-module such that $M = \overset{n}{\underset{i=1}{\oplus}} M_i$, where M_i's are G-sub modules of M. Let ν_i's be fuzzy G-modules on M_i and let $\nu = \overset{n}{\underset{i=1}{\oplus}} \nu_i$. Let μ be any fuzzy G-module on M .Then μ is ν-injective if and only if μ is ν_i-injective, for all i.

Proof: ($\Rightarrow$) Assume μ is ν-injective. Then

(i) M is $M = \overset{n}{\underset{i=1}{\oplus}} M_i$-injective and

(ii) $\nu(m) \leq \mu(\psi(m))$, for all $\psi \in Hom(M,M)$.

To prove that μ is ν_i-injective, for $1 \leq i \leq n$. (i.e to prove (a) M is M_i-injective and (b) $\nu_i(m_i) \leq \mu(\psi(m_i))$, for all $\psi \in Hom(M_i,M)$)

*Proof of (a)***:** Since M_i is a G-submodule of M, from proposition 1.15, it follows that M is M_i-injective.

Proof of (b) **:** Let $\psi \in Hom(M_i, M)$ and let $m_i \in M_i$, so $m_i = 0+0+.....0+m_i+0+...+0$. Then $\nu(m_i) = \nu(0+0+.....0+m_i+0+...+0)$

$= \nu_1(0) \Lambda \nu_2(0) \Lambda..... \Lambda \nu_i(m_i) \Lambda ...... \Lambda \nu_n(0)$

$= \nu_i(mi)$

Since M is M-injective, $\ni$ an extention $\varphi : M \to M$ of ψ ; and hence for each $m_i \in M_i$,

$\nu_i(m_i) = \nu(m_i)$

$\leq \mu(\varphi(m_i))$ [by (ii)]

$\leq \mu(\psi(m_i))$

Thus $\nu_i(m_i) \leq \mu(\psi(m_i))$, for all $\psi \in Hom(M_i,M)$

Therefore μ is ν_i-injective for all i $(1 \leq i \leq n)$.

($\Leftarrow$)Assum μ is ν_i-injective for all i $(1 \leq i \leq n)$.

To prove μ is ν-injective. ((i.e, to prove (c) M is M-injective and (d) $\nu(m) \leq \mu(\psi(m))$, for all $\psi \in Hom(M,M)$).

*Proof of (c)***:** Let N be a G-submodule of M and $\varphi : N \to M = \overset{n}{\underset{i=1}{\oplus}} M_i$ be a homomorphism .Then we have three cases ;

(1) N is a G-submodule of M_i for some i

(2) $N = M_i$, for some i

(3) $N = \overset{m}{\underset{i=1}{\oplus}} M_i$, where $m \leq n$

Case(1). N is a G-submodule of M_i , for some i : Since M is M_i-injective, $\ni$ an extension $\psi : M_i \to M$ of φ . Then $\eta : M \to M$ defined by $\eta(m) = \psi(m_i)$, where $m = \sum_{i=1}^{n} m_i \in M$ is a homomorphism and $\eta|_{M_i} = \psi$. So $\eta|_N = \psi|_N = \varphi$, and therefore η extends φ

Case(2). $N = M_i$, for some i : The function η obtained as in case (1) with $\psi = \varphi$ is an extension *Case(3). If $N = \overset{m}{\underset{i=1}{\oplus}} M_i$, where $m \le n$:* Then the mapping $\eta : M \to M$ defined by $\eta(m) = \varphi\ (\sum_{i=1}^{n} m_i)$, where $m = \sum_{i=1}^{n} m_i \varepsilon M$ is a homomorphism and η extends φ.

Thus in all the cases, $\eta : M \to M$ extends φ ; and hence M is M-injective.

Proof of (d) : Let $\psi\ \varepsilon$ Hom (M, M) and $m\ \varepsilon$ M. Then $m = \sum_{i=1}^{n} m_i$, where $m_i\ \varepsilon\ M_i$, for each i

$\therefore\ \ \nu(m) = \nu(\sum_{i=1}^{n} m_i)$

$= \Lambda\ \{\nu_i(m_i) : i = 1,2,...,n\}$

$\le \nu_i(m_i)$, for all i (1)

Since μ is ν_i- injective for every i,

$\nu_i(m_i) \le \mu(\psi_i(m_i))$, where $\psi_i = \psi|_{M_i}$ (2)

$\therefore\ \ \nu_i(m_i) \le \mu(\psi(m_i))$ for all i (3)

From (1) and (3), $\nu(m) \le \nu_i(m_i) \le \mu(\psi(m_i))$, for all i

$\therefore\ \ \nu(m) \le \Lambda\ \{\mu(\psi(m_i)) : i=1,2,....,n\}$

$\le \mu(\psi(m_1) + \psi(m_2) + \text{--------} + \psi(m_n))$, since μ is a fuzzy G-module

$\le \mu(\psi(m_1 + m_2 + \text{---------} + m_n))$

$\le \mu(\psi(m))$, since $m = \sum_{i=1}^{n} m_i$

Thus, $\nu(m) \le \mu(\psi(m))$ for all $\psi\ \varepsilon$ Hom(M,M). Hence μ is ν-injective ■

2.17. Theorem. Let M_1 and M_2 be G-submodules of a G-module M such that $M = M_1 \oplus M_2$. If M is quasi-injective, then M_i is M_j-injective for $i,j\ \varepsilon\{1,2\}$. Further if ν_i's are fuzzy G-modules on M_i (i =1,2) such that $\nu = \nu_1 \oplus \nu_2$ and if ν is quasi-injective, then ν_i is ν_i-injective for $i,j\ \varepsilon\ \{1,2\}$.

Proof: Assume that $M = M_1 \oplus M_2$ is quasi-injective. Then by proposition 1.15, M is M_j-injective for j =1,2. Also it follows from proposition 1.14, M_i is M_j-injective for $i,j\ \varepsilon$ {1,2}. This proves the first part of the theorem.

Now assume that ν is quasi-injective. Then (i) M is M-injective and (ii) $\nu(m) \le \nu(\psi(m))$ for all $\psi\ \varepsilon$ Hom(M,M).

First to prove ν_1 is ν_2-injective (i.e, to prove (a) M_1 is M_2-injective and (b) $\nu_2(m2) \le \nu_1(\psi(m2))$ for all $\psi\ \varepsilon$ Hom (M_2,M_1) and $m_2\ \varepsilon\ M_2$).

Proof of (a) : From (i), we have M is M –injective. Hence it follows from the first part of the theorem that M_1 is M_2-injective.

Proof of (b) : Let $\psi\ \varepsilon$ Hom (M_2, M_1). Consider the inclusion homomorphism $\varphi : M_1 \to M_1 \oplus M_2 = M$. Then $\psi^1 = \varphi \bullet \psi : M_2 \to M_1 \oplus M_2 = M$ is a homomorphism. Since M is M-injective, $\exists$ an extension $\varphi^1 : M \to M$ of ψ^1, so that $\varphi^1|_{M2} = \psi^1$. (1)

Since $\varphi^1\ \varepsilon$ Hom (M,M) , from (ii), $\nu(m) \le \nu(\varphi^1(m))$ for all $m\ \varepsilon$ M (2)

Since $M = M_1 \oplus M_2$, if $m_2\ \varepsilon\ M_2$, then $m_2 = 0 + m_2\ \varepsilon\ M_1 \oplus M_2 = M$

$\therefore$ From (2), we get

$\nu(m2) \le \nu(\varphi^1(m2))$ (3)

Also, $\nu(m_2) = \nu(0 + m_2)$

$= \nu_1(0)\ \Lambda\ \nu_2(m_2)$

$= \nu_2(m_2)$ (4)

From (1),

$\varphi^1(m_2) = \psi^1(m_2) = \varphi\ (\psi(m_2)) = \psi(m_2)$.

Therefore, $\nu(\varphi^1(m_2)) = \nu(\psi(m_2))$

$= \nu(\psi(m_2) + 0)$

$= \nu_1(\psi(m_2))\ \Lambda\ \nu_2(0)$

$= \nu_1(\psi(m_2))$ (5)

From (3), (4) and (5),

$\nu_2(m_2) \le \nu_2(\psi(m_2))$, for all $\psi\ \varepsilon$ Hom(M_2,M_1)

Therefore ν_1 is ν_2-injective.

Similarly we can show that ν_2 is ν_1 -injective .

Now to prove ν_1 is ν_1-injective.

From (i) we have, M is M-injective. Hence, from the first part of this theorem, we get, M_1 is M_1-injective. Now, let $\psi\ \varepsilon$ Hom (M_1,M_1) and let $\varphi : M_1 \to M$ be the inclusion homomorphism. Then $\varphi \bullet \psi : M_1 \to M$ is a homomorphism. Since M is M-injective, $\exists$ an extension $\varphi'' : M \to M$ of $\varphi \bullet \psi$, so that $\varphi''|_{M1} = \varphi \bullet \psi$. Since $\varphi''\ \varepsilon$ Hom(M,M) , from (ii), we get

$\nu(m) \le \nu(\varphi''(m))$, $\forall m\ \varepsilon$ M

$\therefore\ \nu(m_1) \le \nu(\varphi''(m_1))$, $\forall\ m_1\ \varepsilon\ M_1$ (6)

If $m_1\ \varepsilon\ M_1$, then we have

$\nu(m_1) = \nu(m_1 + 0)$

$= \nu_1(m_1)\ \Lambda\ \nu_2(0)$

$= \nu_1(m_1)$ (7)

Also, $\varphi''(m_1) = (\varphi \bullet \psi)(m_1) = \varphi(\psi(m_1)) = \psi(m_1)\ \varepsilon\ M_1$

$\therefore\ \ \nu(\varphi''(m_1)) = \nu(\psi(m_1))$

$= \nu(\psi(m_1) + 0)$

$= \nu_1(\psi(m_1))\ \Lambda\ \nu_1(0)$

$= \nu_1(\psi(m_1))$ (8)

From (6), (7) and (8), we get

$\nu_1(m_1) \le \nu_1(\psi(m_1))$, for all $\psi\ \varepsilon$ Hom(M_1,M_1) .

Therefore ν_1 is ν_1 -injective. Similarly we can show that ν_2 is ν_2-injective.

This completes the proof ■

2.18. Corollory. Let $M = \overset{n}{\underset{i=1}{\oplus}} M_i$ be a G-module, where M_i's are G-submodules of M. If M is quasi-injective, then M_i is M_j-injective for $i, j\ \varepsilon\ \{1,2,.,n\}$. Also if ν_i's are fuzzy G-modules on M_i's such that $\nu = \overset{n}{\underset{i=1}{\oplus}} \nu_i$ and if ν is quasi-injective, then ν_i is ν_i-injective for every i and j ■

2.19. Theorem. If the G-module $M = P_r = \overset{r}{\underset{i=1}{\oplus}} M_i$ is quasi-injective, then M_i is M_j-injective for $i,j\ \varepsilon\ \{1,2,.,r\}$. Also any fuzzy G-module ν on P_r is quasi injective, then the corresponding summands ν_i's are also quasi injective for every i ■

Proof: The first part is clear from the preceding corollary. We have the set **B** = $\{\alpha_k = r^{\frac{-1}{2}} \varepsilon_k$: k = 1, 2… r}, a basis for M. Then from the theorem 2.4, the function $\nu : M \to [0,1]$ defined by

$\nu(c_1\alpha_1+c_2\alpha_2+..+c_r\alpha_r) = 1$, if $c_i=0$ for all i
$=\frac{1}{2}$, if $c_1\neq 0$, $c_2=c_3=.....=c_r=0$
$=\frac{1}{3}$, if $c_2\neq 0$, $c_3=c_4=.....=c_r=0$
$=\frac{1}{4}$, if $c_3\neq 0$, $c_4=c_5=.....=c_r=0$
....
....
$=1/r-1$, if $c_{r-2}\neq 0$, $c_{r-1}=c_r=0$
$=1/r$, if $c_{r-1}\neq 0$, $c_r=0$
$=1/r+1$, if $c_r\neq 0$

is a fuzzy G-module on M $=P_r$ .Then for each i, the function ν_i : $M_i \rightarrow [0,1]$ defined by

$\nu_i(c_i\alpha_i) = 1$, if $c_i=0$
$= 1/i+1$, if $c_i \neq 0$

are fuzzy G-modules on M_i and from the preceding theorem ν_i is ν_i-injective for every i and j. So ν_i's are quasi injective for every i. ■

III. Conclusion

We have discussed injectivity and quasi injectivity of fuzzy G-modules, in some detail, and have constructed some structure revealing examples. We have also introduced a new G-module P_r of periodic analytic functions mod r and proved that it is the direct sum of r specific G-submodules (remark 1.9). It is proved that P_r has infinitely many fuzzy G-modules on it. In theorem 2.19, we have proved that, if P_r is quasi injective, then each M_i is M_j injective, where M_i and M_j are summands in P_r.

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