Abstract- Representation theory (G-module theory) has had its origin in the 20th century. In the 19th century, groups were generally regarded as subsets of some permutation set, or of the set GL(V) of automorphisms of a vector space V, closed under composition and inverse. Here we consider \( P_r \), the periodic arithmetical functions (mod r), define a fuzzy G-module on it and verify the quasi injective property of its summands.

Index Terms- Injectivity, quasi injectivity, fuzzy set, fuzzy G-module, fuzzy injectivity and quasi fuzzy injectivity.

I. INTRODUCTION

In representation theory, we consider the embedding of a finite group into a linear group. Here we consider those finite groups, which can be embedded in a finite linear group. The fuzzy set theory was introduced by L.A. Zadeh[2] in 1965. Rosenfeld[4] started fuzzification of algebraic structures. As a continuation of these works the concept fuzzy finite G-module was introduced and analysed by us in [5].

In this paper, we discuss injectivity and quasi injectivity of fuzzy G-modules. We introduce the G-module \( P_r \) of periodic arithmetic functions mod r and discuss quasi injectivity in relation to it.

1. Preliminaries

1.1. Definition [1]. Let G be a finite group, M be a vector space over K (a subfield of \( \mathbb{C} \)) and GL(M) be the group of all linear isomorphisms from M onto itself. A linear representation of G with representation space M is a homomorphism \( T: G \rightarrow \text{GL}(M) \).

1.2. Example. Let F be a field, K be an extension field of F, and \( a \in K \). Let \( M = F(a) \), the field obtained by adjoining \( a \) to F.

(i.e) \( M = \{ b_0 + b_1a + b_2a^2 + \ldots | b_i \in F \} \)

Let \( G = (a) \), the cyclic group generated by \( a \). For \( j \in \mathbb{Z} \), define \( T_j : M \rightarrow M \) by

\[
T_j (\sum b_i a^i) = \sum b_i a^{i+j}
\]

Then \( T_j \) is an isomorphism of M onto itself. Also the map \( T : G \rightarrow \text{GL}(M) \) defined by

\[
T(a^i) = T_j, \quad \forall j \in \mathbb{Z}
\]

is a homomorphism and hence a linear representation of G.

1.3. Definition [1]. Let G be a finite group. A vector space M over a field K is called a G-module if for every \( g \in G \) and \( m \in M \), there exist a product (called the action of G on M)

\( m \cdot g = m, \forall m \in M (1_G \text{ being the identity element in G}) \)

\( m \cdot (g \cdot h) = (m \cdot g) \cdot h, \forall m \in M, g, h \in G \); and

\( (iii) (k_1m_1+k_2m_2) \cdot g = k_1(m_1 \cdot g) + k_2(m_2 \cdot g), \forall k_1, k_2 \in K; m_1, m_2 \in M; g \in G \)

1.4. Example. Let \( G = \{1,-1\} \) and \( M=Q(\sqrt{2}) \). Then M is a vector space over Q, and under the usual addition and multiplication of the elements of \( M \), we can show that, M is a G-module.

1.5. Definition [6]. An arithmetical function is a complex-valued function defined on the set of positive integers. For a positive integer r, an arithmetical function f is said to be periodic (mod r) if \( f(n+r) = f(n) \) for all \( n \in \mathbb{N} \)

1.6. Proposition. Let \( P_r \) denote the set of all periodic arithmetical functions (mod r). Then \( P_r \) is a complex vector space. Also \( P_r \) is isomorphic to \( \mathbb{C}^r \), the r-dimensional complex space

Proof: Given \( P_r = \{ \text{functions} f: \mathbb{N} \rightarrow \mathbb{C}; f(n+r) = f(n) \text{ for all } n \in \mathbb{N} \} \)

Define the operations addition and scalar multiplication in \( P_r \) by

\[
\langle f+g \rangle(n) = f(n) + g(n), n \in \mathbb{N}
\]

\[
\langle cf \rangle(n) = c f(n), c \in \mathbb{C}, n \in \mathbb{N}
\]

Then \( P_r \) is a complex vector space. It is an r-dimensional space and is isomorphic to \( \mathbb{C}^r \). The set \( \{ \alpha_k = \frac{r}{2\pi i kn} | k = 1, 2, \ldots, r \} \) is a basis of \( P_r \), where \( \alpha_k \in P_r \) is defined by

\[
\alpha_k(n) = \exp\left(2\pi i kn/r\right)
\]

1.7. Remark. Let \( G = \{1,-1\} \) or \( G = \{1,1,i,-i\}. \) Then the vector space \( P_r \) is a G-module.

1.8. Definition[3] Let \( M_1, M_2, \ldots, M_n \) be vectorspaces over a field K. Then the set \( \{ m_1+m_2+\ldots+m_n : m_i \in M_i \} \) becomes a vectorspace over K under the operations

\[
(m_1+m_2+\ldots+m_n) + (m'_1+m'_2+\ldots+m'_n) = (m_1+m'_1)+ (m_2+m'_2)+\ldots+(m_n+m'_n)
\]

\[
\alpha(m_1+m_2+\ldots+m_n)=\alpha m_1+\alpha m_2+\ldots+\alpha m_n; \alpha \in K, m_i \in M_i
\]

It is called the direct sum of the vector spaces \( M_1,M_2,\ldots, M_n \). It is denoted by \( \bigoplus_{i=1}^n M_i \).

1.9. Example. The set \( Q(\sqrt{2},\sqrt{3}) \) is the field obtained by adjoining the real numbers \( \sqrt{2}, \sqrt{3} \) to \( Q \). Then we have \( Q(\sqrt{2},\sqrt{3}) \)

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is a vector space over \( \mathbb{Q} \) and the set \( \{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\} \) is a basis for \( \mathbb{Q}(\sqrt{2}, \sqrt{3}) \) over \( \mathbb{Q} \). Let \( M_1 = \mathbb{Q}, M_2 = \mathbb{Q}(\sqrt{2}), M_3 = \mathbb{Q}(\sqrt{3}) \) and \( M_4 = \mathbb{Q}(\sqrt{6}) \). Then \( \mathbb{Q}(\sqrt{2}, \sqrt{3}) = \bigoplus_{i=1}^{4} M_i \).

1.10. Remark. The G-module \( M = \mathbb{P} \) can be expressed as the direct sum of r G-submodules as follows:

We have observed that \( \mathbb{P} \) is a complex vector space. It is an r-dimensional space and is isomorphic to \( C^r \). The set \( \{e_k = \frac{1}{r} e_j : k = 1, 2, \ldots, r\} \) is a basis of \( \mathbb{P} \), where \( e_k \in \mathbb{P} \) is defined by \( e_k(n) = \exp\left(\frac{2\pi i km}{r}\right) \). Then \( \bigoplus_{i=1}^{r} M_i \), where \( M_i = \mathbb{C}e_k \).

1.11. Definition[1]. A G-module \( M \) is injective if for any G-module \( M^* \) and any G-submodule \( N \) of \( M^* \), every homomorphism from \( N \) into \( M \) can be extended to a homomorphism from \( M^* \) into \( M \).

1.12. Example. Let \( G = \{1,-1,i,-i\} \) and \( M = (\mathbb{Q} > \mathbb{Q})(\sqrt{2}, \sqrt{3}) \) over \( \mathbb{Q} \). Let \( M \) be a G-module. Then \( M^* = \mathbb{Q}^r \), where \( r \) is the dimension of \( \mathbb{Q} \) over \( \mathbb{Q} \).

1.13. Definition[1]. Let \( M \) and \( M^* \) be G-modules. Then \( M \) is \( M^*-\text{injective} \) if for every G-submodule \( N \) of \( M^* \), any homomorphism \( \varphi : N \rightarrow M \) can be extended to a homomorphism \( \psi : M^* \rightarrow M \).

1.14. Proposition[5]. Let \( M = M_1 \bigoplus M_2 \), where \( M_1 \) and \( M_2 \) are G-submodules of \( M \). Then \( M \) is injective if and only if \( M_1 \) and \( M_2 \) are both injective.

Proof: Let \( M \) be injective. Let \( M^* \) be a G-module and \( N \) be any G-submodule of \( M^* \) and let \( \eta : N \rightarrow M_1 \) be a homomorphism. Since \( M \) is injective, there exists a homomorphism \( \eta_1 : M^* \rightarrow M \). Let \( \pi : M \rightarrow M_1 \) be the projection map. Then \( \eta_1 = \pi \eta : M^* \rightarrow M_1 \) is an extension of \( \eta \). Therefore \( M_1 \) is injective. Similarly we can show that \( M_2 \) is injective.

Conversely suppose \( M_1 \) and \( M_2 \) are injective. Let \( M^* \) be a G-module and \( N \) be any G-submodule of \( M^* \) and let \( \eta : N \rightarrow M \) be a homomorphism. Let \( \pi_1 \) and \( \pi_2 \) be the projections of \( M_1 \) and \( M_2 \) respectively. Since \( M_1 \) and \( M_2 \) are injective, the mappings \( \pi_1 \eta : N \rightarrow M_1 \) and \( \pi_2 \eta : N \rightarrow M_2 \) can be extended to homomorphisms \( \eta_1 : M^* \rightarrow M_1 \) and \( \eta_2 : M^* \rightarrow M_2 \) respectively. Define \( \eta_3 : M^* \rightarrow M \) by \( \eta_3(m) = \eta_1(m) + \eta_2(m) \). Then \( \eta_3 \) is a homomorphism. For every \( m \in M \), \( \eta_3(m) = \eta_1(m) + \eta_2(m) = \pi_1(m) \pi_2(m) \). Therefore \( \eta_3 \) extends \( \eta \) and hence \( M \) is injective.

1.15. Proposition[5]. Let \( M \) and \( M^* \) be G-modules such that \( M \) is \( M^*-\text{injective} \). If \( N^* \) is a G-submodule of \( M^* \), then \( M \) is \( N^*\)-injective and \( M \) is \( M^*/N^*\)-injective.

Proof: Since \( N^* \subseteq M^* \) and \( M \) is \( M^*-\text{injective} \), it is obvious that \( M \) is \( N^*\)-injective.

Let \( X^*/N^* \) be a G-submodule of \( M^*/N^* \) and \( \varphi : X^*/N^* \rightarrow M \) be a homomorphism. Let \( \pi : M^* \rightarrow M^*/N^* \) be the canonical map and \( \pi_1 = \pi \circ \varphi \). Then \( \varphi \circ \pi_1 : X^* \rightarrow M \) is a homomorphism. Since \( M \) is \( M^*-\text{injective} \), \( \exists \) an extension \( \vartheta : M \rightarrow M \) of \( \varphi \circ \pi_1 \). Then \( \theta(N^*) = \varphi \circ \vartheta(N^*) = \varphi (\pi_1(N^*)) = \varphi (0) = 0 \). Therefore \( \ker \vartheta \) is a G-submodule of \( \ker \theta \). Then \( \exists \) a map \( \psi : M^*/N^* \rightarrow M \) such that \( \psi \circ \varphi = \theta \). Also for any \( x \in X \), \( \psi (x+N^*) = \varphi (\pi_1(x)) = \varphi (\varphi \circ \pi_1(x)) = \varphi (0+N^*) = 0+N^* \).

Therefore \( \psi \) extends \( \varphi \). Hence \( M \) is \( M^*/N^*\)-injective.

1.16. Example. Let \( M^* = \mathbb{R}^n \). This is n-dimensional vector space over \( \mathbb{R} \). Let \( \{a_1, a_2, \ldots, a_n\} \) be a basis for \( M^* \). Then \( M^* = \mathbb{R}a_1 \bigoplus \mathbb{R}a_2 \bigoplus \cdots \bigoplus \mathbb{R}a_n \).

Let \( M = \mathbb{R} \) and \( G \) be any finite multiplicative subgroup of \( \mathbb{R} \). Then both \( M \) and \( M^* \) are G-modules. Let \( N \) be any G-submodule of \( M^* \) and \( \varphi : N \rightarrow M \) be a homomorphism.

(i). If \( N = \{0\} \), then \( \varphi = 0 \), then \( \varphi = 0 : M^* \rightarrow M \) is a homomorphism.

(ii). If \( N = \mathbb{R}a_i \), \( 1 \leq i \leq n \).

Then \( \psi : M^* \rightarrow M \) defined by \( \psi(c_1a_1 + \cdots + c_ka_k) = \varphi(c_1a_1) \) is a homomorphism which extends \( \varphi \).

(iii). \( N = \bigoplus_{j=1}^{k} \mathbb{R}a_j \). Then \( \psi : M^* \rightarrow M \) defined by \( \psi(c_1a_1 + \cdots + c_ka_k) = \varphi(c_1a_1 + \cdots + c_ka_k) \) extends \( \varphi \). Therefore \( M \) is \( M^*-\text{injective} \).

1.17. Definition[1]. A G-module \( M \) is quasi-injective if \( M \) is M-injective.
1.18. Example. Let \( S = \{1, \omega, 2\omega\} \), where \( \omega \) is a complex cube root of unity and \( G = S_3 \), the symmetric group of degree three. Let \( M = \text{span}(S) \) over \( R = \{ a + \beta \omega + \gamma \omega^2 : a, \beta, \gamma \in R \} \). Then \( M \) is a vector space over \( R \). For each \( x \in G \), define \( T_x : M \rightarrow M \) by

\[
T_x (a + \beta \omega + \gamma \omega^2) = a x(1) + \beta x(\omega) + \gamma x(\omega^2)
\]

Then \( T_x \) is an isomorphism of \( M \) onto itself. Also the map \( T : G \rightarrow \text{GL}(M) \) defined by

\[
T(x) = T_x, \quad \forall x \in G,
\]

is a representation of \( G \), and hence \( M \) is a \( G \)-module. Also the only \( G \)-submodules of \( M \) are \( M \) and \( \{0\} \). We will show that \( M \) is \( M \)-injective. Let \( N \) be any \( G \)-submodule of \( M \). Then \( N = \{0\} \) or \( N = M \). Define \( \phi : N \rightarrow M \) to be any homomorphism.

Case (i). \( N = \{0\} \): Then the map \( \psi : M \rightarrow M \) defined by \( \psi(x) = 0 \), \( \forall x \in M \) extends \( \phi \).

Case (ii). \( N = M \): In this case, \( \phi \) is a homomorphism from \( M \) into itself; and hence \( \psi = \phi \) is the required extension.

Thus, in both cases, \( \phi : N \rightarrow M \) can be extended to a homomorphism \( \psi : M \rightarrow M \). Therefore \( M \) is \( M \)-injective; and hence quasi-injective.

II. FUZZY G-MODULE INJECTIVITY

2.1. Definition (Fuzzy set) [2]. The characteristic function of a crisp set (classical set or non-fuzzy sets) assigns a value of either 1 or 0 to each individual element in the universal set, thereby discriminating between members and non-members of the crisp sets under consideration. This function can be generalised in such a way that the values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements in the set in question. Larger values denote the higher degrees of the set membership. Such a function is called a membership function, and the set defined by it a fuzzy set.

The most commonly used range of values of membership functions is the unit interval \([0,1]\), i.e. a fuzzy set \( \mu \) on the set \( X \) is a function \( \mu : X \rightarrow [0,1] \).

2.2. Definition [5]. Let \( G \) be a finite group and \( M \) be a \( G \)-module over \( K \), which is a subfield of complex numbers. Then a fuzzy \( G \)-module on \( M \) is a fuzzy subset \( \mu \) of \( M \) such that

(i) \( \mu(ax+by) \geq \mu(x) \land \mu(y), \forall a, b \in K \) and \( x, y \in M \)

and

(ii) \( \mu(gm) \geq \mu(m), \forall g \in G, m \in M \). Where \( \Lambda \) is the minimum [infimum] operator.

2.3. Example. Let \( G = \{1, -1, i, -i\} \). Then \( M = C \), the field of complex numbers is a \( G \)-module over itself. Define \( \mu : M \rightarrow [0,1] \) by

\[
\mu(x+iy) = 1, \quad \text{if } x = y = 0
\]

\[
= \frac{1}{2}, \quad \text{if } x \neq 0, y = 0
\]

\[
= \frac{1}{4}, \quad \text{if } y \neq 0
\]

Then \( \mu \) is a fuzzy \( G \)-module on \( M \).

2.4. Theorem. Let \( M \) be the \( G \)-module \( B_r \). Then there exist a fuzzy \( G \)-module on \( M \).

Proof: Here \( M \) is an \( r \)-dimensional \( G \)-module over \( K = C \).

Let \( \mathbf{B} = \{x_k = r^{-k} e_k : k = 1, 2 \ldots r\} \), a basis for \( M \). Define \( \nu : M \rightarrow [0,1] \) by

\[
\nu (c_1 a_1 + c_2 a_2 + \ldots + c_r a_r) = 1, \quad \text{if } c_i = 0 \quad \text{for all } i
\]

\[
= \frac{1}{2}, \quad \text{if } c_1 \neq 0, c_2 = c_3 = \ldots = c_r = 0
\]

\[
= \frac{1}{4}, \quad \text{if } c_1 \neq 0, c_2 = c_3 = \ldots = c_r = 0
\]

..............

\[
= \frac{1}{r-1}, \quad \text{if } c_2 \neq 0, c_3 = c_4 = \ldots = c_r = 0
\]

\[
= \frac{1}{r}, \quad \text{if } c_1 \neq 0, c_r = 0
\]

\[
= \frac{1}{r+1}, \quad \text{if } c_1 = 0
\]

Then \( \nu \) is a fuzzy \( G \)-module on \( M \).

2.5. Proposition [5]. For any fuzzy \( G \)-module \( \mu \) on a \( G \)-module \( M \) and for each \( k \in (0,1] \), \( \mu_k : M \rightarrow [0,1] \) defined by \( \mu_k(x) = k \mu(x), \forall x \in M \) is also a fuzzy \( G \)-module on \( M \).

Proof: Let \( M \) be the \( G \)-module \( B_r \). Then from theorem 2.4, there exists a fuzzy \( G \)-module \( \nu \) on \( M \). Let \( k \in (0,1] \), then from the above proposition we have \( \nu_k \) defined by

\[
\nu_k(x) = k \nu(x), \forall x \in M.
\]

is a fuzzy \( G \)-module on \( M \). In the definition of fuzzy \( G \)-module \( \nu \) in the theorem 1.15, replace 1 in the numerator by \( k \). Then \( \nu_k \) is a fuzzy \( G \)-module on \( M \) for each \( k \in (0,1] \).

2.6. Proposition. For any positive integer \( r \), there exists infinite number of fuzzy \( G \)-modules on the \( G \)-module \( B_r \).

Proof: Let \( M \) be the \( G \)-module \( B_r \). Then from theorem 2.4, there exists a fuzzy \( G \)-module \( \nu \) on \( M \). Let \( k \in (0,1] \), then from the above proposition we have \( \nu_k \) defined by

\[
\nu_k(x) = k \nu(x), \forall x \in M
\]

is a fuzzy \( G \)-module on \( M \). Let \( \mu \) be any fuzzy \( G \)-module on \( M \) and \( \nu \) be any fuzzy \( G \)-module on \( M \). Then \( \mu \) is \( \nu \)-injective if

(i) \( \mu \) is \( \nu \)-injective.

(ii) \( \nu(m) \leq \mu(\nu(m)) \), \( \forall \mu \in \text{Hom}(M*,M) \) and \( \forall m \in M \).

Where \( \text{Hom}(M*,M) \) is the set of all \( G \)-module homomorphism’s from \( M^* \) to \( M \).

2.8. Example. Let \( G = \{1, i, -i\} \), \( M = C \) and \( M^* = Q \). Then \( M \) and \( M^* \) are \( G \)-modules over \( Q \). Define \( \mu : M \rightarrow [0,1] \) and \( \nu : M^* \rightarrow [0,1] \) by

\[
\mu(x) = 1, \quad \text{if } x = 0
\]

\[
= \frac{1}{2}, \quad \text{if } x \in Q, 1 - \{0\}
\]

\[
= \frac{1}{4}, \quad \text{if } x \in Q, C \cdot Q, 1 - \{0\}
\]

and

\[
\nu(x) = \frac{1}{4}, \quad \text{if } x = 0
\]

\[
= \frac{1}{5}, \quad \text{if } x \neq 0
\]

Then \( \mu \) and \( \nu \) are fuzzy \( G \)-modules on \( M \) and \( M^* \) respectively. Let \( X \) be any \( G \)-submodule of \( M^* \). Then either \( X = \{0\} \) or \( X = M^* \). Define \( \phi : X \rightarrow M \) to be any homomorphism.

Case (i). \( \phi = 0 \), then \( \phi = \phi \) extends \( \phi \).

Case (ii). \( \phi \in M^* \), then \( \phi = \phi \) extends \( \phi \).

Therefore \( M \) is \( M^* \)-injective. Also it follows from the definitions of \( \mu \) and \( \nu \) that

\[
\nu(m) \leq \mu(\nu(m)), \quad \forall \mu \in \text{Hom}(M*,M) \text{ and } \forall m \in M^*.
\]

Therefore \( \mu \) is \( \nu \)-injective.
2.9. Definition [5]. Let $M$ be a G-module and $\mu$ be a fuzzy $G$-module on $M$. Then $\mu$ is quasi-injective if

(i) $M$ is quasi-injective

(ii) $\mu(m) \leq \mu(\psi(m))$, $\forall \psi \in \text{Hom}(M, M)$ and $m \in M$.

2.10. Remark. Let $M$ be a quasi-injective $G$-module. Then

the functions $\mu : M \rightarrow [0,1]$ defined by $\mu(x) = t$, $\forall x \in M$ and

$\mu(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0, \text{where ‘$t$’ is a fixed element in } [0,1] \end{cases}$

are quasi-injective fuzzy $G$-modules on $M$.

2.11. Example. The $G$-module $M$ in example 1.16 is quasi-injective. On this $M$, if we define a function $\mu$ as in remark 2.10, then $\mu$ is quasi-injective.

2.12. Proposition [5]. Let $M$ be a $G$-module over $K$ and $M = \bigoplus_{i=1}^{n} M_i$, where $M_i$'s are $G$-submodules of $M$. If $v_i$ $(1 \leq i \leq n)$ are fuzzy $G$-submodules on $M_i$, then $v : M \rightarrow [0,1]$ defined by

$v(m) = \bigwedge \{ v_i(m_i) : i=1,2,...,n \}$, where $m = \sum_{i=1}^{n} m_i \in M$

is a fuzzy $G$-module on $M$.

Proof: Since each $v_i$ is a fuzzy $G$-module on $M_i$, for every $x, y \in M_i, g \in G$ & $a, b \in K$, we have $v_i(ax+by) \geq v_i(x) \Lambda v_i(y)$ and $v_i(gx) \geq v_i(x)$ Let $x = \sum_{i=1}^{n} m_i \in M$ and $a, b \in K$, then $v(ax+by) = v(\sum (a m_i + b m_i)) = \bigwedge \{ v_i(\sum (a m_i + b m_i)) : i=1,2,...,n \} \geq v_i(\sum (a m_i + b m_i)) \Lambda v_i(x)$

Also for $g \in G$ and $x = \sum_{i=1}^{n} m_i \in M$, 

$v(gx) = v(\sum g m_i) = \bigwedge \{ v_i(\sum g m_i) : i=1,2,...,n \} = v_i(g m_i)$, for some $j \geq v_i(m_i) \Lambda v_i(x)$

Therefore $v$ is a fuzzy $G$-module on $M$.

2.13. Remark. In the above proposition, if $v_i(0)$ are all equal then we have $v(0) = \bigwedge \{ v_i(0) : i=1,2,...,n \} = v_i(0)$, for all $i$.

2.14. Definition [5]. The fuzzy $G$-module $v$ on $M = \bigoplus_{i=1}^{n} M_i$, in the proposition 2.12 with $v(0) = v_i(0)$ for all $i$, is called the direct sum of the fuzzy $G$-modules $v_i$ and is denoted by $v = \bigoplus_{i=1}^{n} v_i$.

2.15. Example. Let $G = \{1, -1\}$ and $M = \mathbb{C}$ over $R$. Then $M$ is a $G$-module. We have $M = M_1 \oplus M_2$, where $M_1 = \mathbb{R}, M_2 = i\mathbb{R}$. Define $v : M \rightarrow [0,1]$ by

$v(x+iy) = 1$, if $x = y = 0$

$v = \frac{1}{2}$, if $x = 0, y \neq 0$.

Then $v$ is a fuzzy $G$-module on $M$. Also the mappings $v_1 : M_1 \rightarrow [0,1]$ defined by $v_1(x) = 0$, if $x = 0$

and $v_2 : M_2 \rightarrow [0,1]$ defined by $v_2(y) = 0$, if $y = 0$

are fuzzy $G$-modules on $M_1$ and $M_2$ respectively and $v = v_1 \oplus v_2$.

2.16. Theorem [5]. Let $M$ be a $G$-module such that $M = \bigoplus_{i=1}^{n} M_i$, where $M_i$'s are $G$-submodules of $M$. Let $v_i$'s be fuzzy $G$-modules on $M_i$ and let $v = \bigoplus_{i=1}^{n} v_i$. Let $\mu$ be any fuzzy $G$-module on $M$. Then $\mu$ is $v$-injective if and only if $\mu$ is $v_i$-injective, for all $i$.

Proof: $\Rightarrow$ Assume $\mu$ is $v$-injective. Then

(i) $M = \bigoplus_{i=1}^{n} M_i$-injective and

(ii) $\forall m \in M$ defined by $\psi = \bigoplus_{i=1}^{n} \psi_i$ for all $\psi \in \text{Hom}(M, M)$.

To prove that $\mu$ is $v_i$-injective, for $1 \leq i \leq n$. (i.e, to prove (a) $M$ is $M_i$-injective and (b) $v_i(m) \leq \psi(m_i)$, for all $\psi \in \text{Hom}(M_i, M_i)$)

Proof of (a): Since $M_i$ is a G-submodule of $M$, from proposition 1.15, it follows that $M$ is $M_i$-injective.

Proof of (b): Let $\psi \in \text{Hom}(M_i, M_i)$ and let $m_i \in M_i$, so $m_i = \sum_{i=1}^{n} m_i + \sum_{i=1}^{n} m_i$.

Then $v(m_i) = v(\sum_{i=1}^{n} m_i + \sum_{i=1}^{n} m_i) = v_i(0) \Lambda v_2(0) \Lambda ... \Lambda v_i(0)$

Thus $v_i(m_i) \leq \psi(m_i)$, for all $\psi \in \text{Hom}(M_i, M_i)$.

Therefore $\mu$ is $v_i$-injective for all $i$ $(1 \leq i \leq n)$.

$\Rightarrow$ Assume $\mu$ is $v_i$-injective for all $i$ $(1 \leq i \leq n)$.

To prove $\mu$ is $v$-injective, (i.e, to prove (c) $M$ is $M_i$-injective and (d) $\forall m \in M_i$ defined by $\psi(m_i) \leq \mu(\psi(m_i))$, for all $\psi \in \text{Hom}(M_i, M_i)$).

Proof of (c): Let $N$ be a $G$-submodule of $M$ and $\varphi : N \rightarrow M$ be a homomorphism. Then we have three cases:

(1) $N$ is a G-submodule of $M_i$ for some $i$

(2) $N = M_i$, for some $i$

(3) $N = \bigoplus_{i=1}^{n} M_i$, where $m \leq n$

Case (1). $N$ is a $G$-submodule of $M_i$, for some $i$.

Since $M_i$ is $M_i$-injective, $\exists$ an extension $\psi : M_i \rightarrow M$ of $\varphi$. Then $\eta : M \rightarrow M$ defined by $\eta(m) = \psi(m_i)$, where $m = \sum_{i=1}^{n} m_i \in M$ is a homomorphism and $\eta|_{M_i} = \psi$. So $\eta|M = \psi|_M = \varphi$, and therefore $\varphi$ extends $\varphi$.

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Case (2). \( N = M_i \), for some \( i \): The function \( \eta \) obtained as in case (1) with \( \psi = \varphi \) is an extension \( \text{Case (3)}. \) If \( N = M_i \), where \( m \leq n \): Then the mapping \( \eta : M \rightarrow M \) defined by
\[
\eta(m) = \varphi \left( \sum_{i=1}^{n} m_i \right), \quad \text{where} \quad m = \sum_{i=1}^{n} m_i \in M \text{ is a homomorphism and } \eta \text{ extends } \varphi.
\]
Thus in all the cases, \( \eta : M \rightarrow M \) extends \( \varphi \); and hence \( M \) is \( M \)-injective.

**Proof of (d):** Let \( \psi \in \text{Hom}(M, M) \) and \( m \in M \). Then
\[
v(m) = \sum_{i=1}^{n} m_i, \quad \text{where} \quad m_i \in M_i, \text{ for each } i
\]
\[
\vdash v(m) = \varphi \left( \sum_{i=1}^{n} m_i \right) = \sum_{i=1}^{n} \varphi(m_i).
\]
Since \( \mu \) is \( \nu \)-injective, then the corresponding \( \nu = \nu_i \)-injective for every \( i \) and \( j \in \{1,2\} \) such that \( v = v_i(v) \cdot v_2 \) and if \( v \) is \( \nu \)-injective, then \( v_i \) is \( v_i \)-injective for \( i, j \in \{1,2\} \).

**Proof:** Assume that \( M = M_1 \oplus M_2 \) is quasi-injective. Then by proposition 1.15, \( M \) is \( M_i \)-injective for \( i, j \in \{1,2\} \). Also it follows from proposition 1.14, \( M_i \) is \( M_i \)-injective for \( i, j \in \{1,2\} \). This proves the first part of the theorem.

Now assume that \( v \) is \( \nu \)-injective. Then (i) \( M \) is \( \nu \)-injective and (ii) \( v(m) \leq v(\nu(m)) \) for all \( \psi \in \text{Hom}(M,M) \).

**First to prove \( v \) is \( v_i \)-injective (i.e., to prove (a) \( M_i \) is \( M_i \)-injective and (b) \( v_2 \) is \( v_2 \)-injective for all \( \psi \in \text{Hom}(M_i,M_2) \) and \( m \in M_2 \)).**

**Proof of (a):** From (i), we have \( M \) is \( M \)-injective. Hence it follows from the first part of the theorem that \( M_i \) is \( M_i \)-injective.

**Proof of (b):** Let \( \psi \in \text{Hom}(M_2, M_i) \). Consider the inclusion homomorphism \( \varphi : M_i \rightarrow M_1 \oplus M_2 = M \). Then
\[
\psi = \psi_1 \circ \psi_2 : M_i \rightarrow M_1 \oplus M_2 = M \text{ is a homomorphism. Since } M \text{ is } M_i \text{-injective, } \exists \text{ an extension } \psi : M \rightarrow \psi_1 \text{ such that } \psi_1|_{M_2} = \psi_1.
\]
Since \( \varphi \in \text{Hom}(M, M) \), from (ii), \( v(m) \leq \psi(v(m)) \) for all \( m \in M \).

Therefore, \( v_2(M_2) \leq v_1(0) \cup v_2(M_2) \) \( \cup v_2(M_2) \).
ν (c₁α₁+c₂α₂+...+cᵣαᵣ) = 1, if cᵢ=0 for all i
=½, if c₁≠0, c₂=c₃=......=cᵣ=0
=½, if c₂≠0, c₃=c₄=......=cᵣ=0
=¼, if c₃≠0, c₄=c₅=......=cᵣ=0
..............................................
..............................................
=1/r, if cᵣ-1≠0, cᵣ=0
=1/r-1, if cᵣ≠0, cᵣ-1=0

is a fuzzy G-module on M =ₚ₋. Then for each i, the function νᵢ : Mᵢ → [0,1] defined by
νᵢ (cᵢαᵢ) = 1, if cᵢ=0
= 1/i + 1, if cᵢ≠0
are fuzzy G-modules on Mᵢ and from the preceding theorem νᵢ is νᵢ-injective for every i and j. So νᵢ’s are quasi injective for every i.

III. Conclusion

We have discussed injectivity and quasi injectivity of fuzzy G-modules, in some detail, and have constructed some structure revealing examples. We have also introduced a new G-module ₚ₋ of periodic analytic functions mod r and proved that it is the direct sum of r specific G-submodules (remark 1.9). It is proved that ₚ₋ has infinitely many fuzzy G-modules on it. In theorem 2.19, we have proved that, if ₚ₋ is quasi injective, then each Mᵢ is Mᵢ injective, where Mᵢ and Mᵢ are summands in ₚ₋.

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