A Study on Properties of Regularity for Fractional Difference Equations in Stochastic Processes

M. Reni Sagayaraj[1], P. Manoharan[2]

[1,2]Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur, India

Abstract- In this paper, we present some results for vector-valued fractional difference equations. We are successful to completely characterize the maximal regularity of solutions for the problem in Lebesgue vector-valued spaces defined on the set Z+. Our approach use as main ingredients Blunck's operator valued multiplier theorem, and the introduction of a special sequence of bounded operators, that we called α-resolvent families, which will play a central role in the representation of the solution of the problem by means of a kind of discrete variation of parameters formula.

Key Words: Difference equation, linear differential equation, existence for nonlinear difference equation.

I. Introduction

First studies on time differences of fractional order are due to Kutter. [14] Diaz and Osler introduced in 1974 a discrete fractional difference operator defined as an infinite series. Grey and Zhang [13] developed a fractional calculus for the discrete nabla (backward) difference operator. At the same time, Miller and Ross defined a fractional sum via the solution of a linear difference equation. More recently, Atici and Eloe introduced the Riemann-Liouville like fractional difference by using the definition of fractional sum of Miller and Ross, and developed some of its properties that allow to obtain solutions of certain fractional difference equations. Ferreira introduced the concept of left and right fractional sum/difference and started a fractional discrete-time theory of the calculus of variations. Holm [15,16] further developed and applied the tools of discrete fractional calculus to the arena of fractional difference equations. See also the recent paper for related work. Concerning qualitative properties, Goodrich in a series of papers studied existence of positive solutions and geometrical properties. On the other hand, the theory of discrete fractional equations is also a promising tool for several biological and physical applications where the memory effect appears. For instance, applications to concrete models have been analyzed recently by Atici and Seng¨ul in. In spite of the significant increase of research in this area, there are still many open questions regarding fractional difference equations. In particular, the study of regularity properties on vector-valued Lebesgue spaces l p remains an open problem. The maximal regularity property is key to handle vector-valued nonlinear difference equations by operator theoretical methods, because it is a prior and essential step to the use of fixed point arguments, see the monograph . However, the literature on the subject is scarce, and no attempt to study maximal regularity of fractional difference equations has been done.

II. Regularity of Fractional Differential Equations model

Concepts are needed for the application of Blunck's theorem to fractional difference equations analyzed here in the context of Banach spaces. We also establish the definition of the fractional difference operator that we will use and that seems to be more convenient for our purposes. In this line of ideas, we note the recent paper, where it was proved that many concepts of fractional differences currently used in the literature are simply related by translation. [17]We remark that our definition is at the basis of this equivalence. Introduces a new concept that we called α-resolvent sequences, denoted by S_α(n), as a necessary tool for the study of l_p-maximal regularity. We show how this tool help us to prove an explicit representation of the solution for the fractional difference equation with initial value u (0) = x.

$$u(n) = S_\alpha(n)x + (S_\alpha * f)(n-1), \qquad n \in \mathbb{N}$$

On the other hand, the notion of α-resolvent families has own interest because it should correspond to the vector-valued concept of Mittag-Leffler operator sequence. In this context, gives a interesting characterization, showing that α-resolvent families must have the form

$$S_\alpha(n) = \sum_{j=0}^{n} \frac{\Gamma(\mathrm{n-j+(j+1)\alpha})}{\Gamma(\mathrm{n-j+1})\Gamma(\mathrm{j\alpha+\alpha})} T^j, \qquad n \in \mathbb{Z}_+$$

the main result of this paper, that shows a characterization of l p-maximal regularity of the Equation solely in terms of the data of the problem. In other words, we prove that if a vector valued sequence $f \in l_p$ is defined on a U M D - space X then the solution u of exists and is such that u, $\Delta^\alpha u \in l_p$ if and only if the set

$$\{z^{1-\alpha}(z-1)^\alpha (z-1)^\alpha - T)^{-1} : |z| = 1, z \neq 1\}$$

is R-bounded. Finally, a simpler criteria in case of Hilbert spaces is given.

III. Nonlinear Fractional Difference Equation model

The existence of solutions for nonlinear fractional difference equations

$$\Delta_*^\alpha x(t) = f\big(t+\alpha-1, x(t+\alpha-1)\big),\ \ t \in \mathbb{N}_{1-\alpha},\ \ 0<\alpha\le 1,$$
$$x(0) = x_0$$

where Δ_*^α is a Caputo like discrete fractional difference, f : [0, +∞)] × X → X is continuous in t and X. (X, ‖ · ‖) is a real Banach space with the norm ‖x‖ = sup{‖x (t)‖, t ∈ N}, $\mathbb{N}_{1-\alpha}$ ={1 − α, 2 − α, . . .}.

IV. IV. Lemmas and Difference Operators

We start with some necessary definitions from discrete fractional calculus theory and preliminary results so that this paper is self-contained.

Let ν > 0. The νth fractional sum f is defined by

$$\Delta^{-v} f(t,a) = \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t-s-1)^{(v-1)} f(s) \tag{2.1}$$

Here f is defined for $s = a$ a mod and $\Delta^{-v} f$ is defined for $t = (a+v)$ mod in particular, Δ^{-v} maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+v}$ where $\mathbb{N}_1 = \{t, t+1, t+2, \ldots\}$. In addition, $t^{(v)} = \Gamma(t+1)/\Gamma(t-v+1)$. Atici and Eloe pointed out that this definition of the νth fractional sum is the development of the theory of the fractional calculus on time scales.

Let $\mu > 0$ and $m-1<\mu<m$, where m denotes a positive integer, m = [μ],[.] ceiling of number. Set ν = $m-\mu$. The μ th fractional Caputo like difference is defined as

V. $$\Delta_*^\mu f(t) = \Delta^{-v}(\Delta^m f(t)) = \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t-s-1)^{(v-1)} (\Delta^m f)(s),\ \forall t \in N_{a+v} \tag{2.2}$$

Here Δ^m is the mth order forward difference operator

$$(\Delta^m f)(s) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} f(s+k) \tag{2.3}$$

For μ > 0, μ non integer, $m = [\mu]$, $v = m-\mu$, it holds

$$f(t) = \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \Delta^k f(a) + \frac{1}{\Gamma(\mu)} \sum_{s=a+v}^{t-\mu} (t-s-1)^{(\mu-1)} \Delta_*^\mu f(s), \tag{2.4}$$

Where f is defined on $\mathbb{N}_a$ $with\ \alpha \in N$

In particular, when 0 < μ < 1 and a=0, we have

$$f(t) = f(0) + \frac{1}{\Gamma(\mu)} \sum_{s=1-\mu}^{t-\mu} (t-s-1)^{(\mu-1)} \Delta_*^\mu f(s) \tag{2.5}$$

Lemma 4.1. A solution $x(t): N \to X$ is a solution of the IVP if and only if x(t) is a solution of the following fractional Taylor's difference formula:

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=1-a}^{t-a} (t-s-1)^{(\mu-1)} f\big(s+\alpha+1, x(s+\alpha-1)\big),$$
$$0<\alpha\le 1, x(0) = x_0 \tag{2.6}$$

Proof. Suppose that x(t) for t ∈ N is a solution, that is $\Delta_*^a x(t) = f\big(t+\alpha+1, x(t+\alpha-1)\big)$ for t ∈ $\mathbb{N}_{1-\alpha}$, then we can obtain according to Theorem 2.3.

Conversely, we assume that x(t) is a solution of (2.6), then

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \sum_{s=1-a}^{t-a} (t-s-1)^{(\mu-1)} f\big(s+\alpha+1, x(s+\alpha-1)\big) \tag{2.7}$$

On the other hand, Theorem 2.3 yields that

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \sum_{s=1-a}^{t-a} (t-s-1)^{(a-1)} \Delta_*^a x(s) \tag{2.8}$$

Comparing with the above two equations, it is obtained that

$$\frac{1}{\Gamma(\alpha)} \sum_{s=1-a}^{t-a} (t-s-1)^{(a-1)} \left[\Delta_*^a x(s) - f\big(s+a-1, x(s+a-1)\big)\right] = 0 \tag{2.9}$$

Let t=1,2,….., respectively, we have that $\Delta_*^a x(s) - f\big(s+a-1, x(s+a-1)\big)$ for $t \in \mathbb{N}_{1-a,}$ which implies that x(t) is a solution.

Lemma 4.2. One has

$$\sum_{s=1-a}^{t-a} (t-s-1)^{(a-1)} = \frac{\Gamma(t+a)}{a\Gamma(t)} \tag{2.10}$$

Proof. For $x > k, x, k \epsilon R, k > -1, x > -1$, we have

$$\frac{\Gamma(x+1)}{\Gamma(k+1)\Gamma(x-k+1)} = \frac{\Gamma(x+2)}{\Gamma(k+1)\Gamma(x-k+1)} = \frac{\Gamma(x+3)}{\Gamma(k+1)\Gamma(x-k+1)} \quad (2.11)$$

That is,

$$\frac{\Gamma(x+1)}{\Gamma(x-k+1)} = \frac{1}{k+1}\left[\frac{\Gamma(x+2)}{\Gamma(x-k+1)} - \frac{\Gamma(x+1)}{\Gamma(x-k)}\right] \quad (2.12)$$

Then

$$\sum_{s=1-a}^{t-a}(t-s-1)^{(a-1)} = \sum_{s=1-a}^{t-a}\frac{\Gamma(t-2)}{\Gamma(t-s-a+1)}$$
$$= \sum_{s=1-a}^{t-a-1}\frac{\Gamma(t-2)}{\Gamma(t-s-a+1)} + \Gamma(a)$$
$$= \sum_{s=1-a}^{t-a-1}\frac{1}{\alpha} + \left[\frac{\Gamma(t-s+1)}{\Gamma(t-s-\alpha+1)} - \frac{\Gamma(t-2)}{\Gamma(t-s-\alpha)}\right] + \Gamma(a) \quad (2.13)$$
$$= \frac{\Gamma(t+a)}{\alpha\Gamma(t)}$$

Let $v \neq 1$ and assume $\mu + v + 1$ is not a non positive integer. Then

$$\Delta^{-v}t^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+v+1)}t^{(\mu+v)} \quad (2.14)$$

In particular, $\Delta^{-v}\, a = a\Delta^{-v}(t+\alpha-1)^{(0)} = (a/\Gamma(v+1))(t+\alpha-1)^{(v)}$, where a is a constant. The following fixed point theorems will be used in the text. This difference makes that fractional difference equation with the Caputo like difference operator is more similar to classical integer-order difference equation.

VI. Application

It is interesting to note that in case α = 1 we have from the above theorem

$$S_1(n) = \sum_{j=0}^{n}\frac{\Gamma(n+1)}{\Gamma(n-j+1)\Gamma(j+1)}T^j = \sum_{j=0}^{n}\frac{n!}{\Gamma(n-j)!\, j!}T^j = \sum_{j=0}^{n}\binom{n}{j}T^j = (I+T)^n,$$

We will need the following lemma that is of independent interest.

Lemma 5.1

Let $0 < \alpha < 1$, $a: \mathbb{N}_0 \to \mathbb{C}$ and $S: \mathbb{N}_0 \to X$ be given. Then $\Delta^{\alpha}(\alpha * S)(n) = (\alpha * |\Delta^{\alpha}S(n) + S(0)\alpha(n+1), n \epsilon \mathbb{N}_0$

Holds.

Proof. By definition, we have

$$\Delta^{\alpha}(\alpha * S)(n) = \Delta \circ \Delta^{-(1-\alpha)}(\alpha * S)(n)$$
$$= \Delta^{-(1-\alpha)}(\alpha * S)(n+1) - \Delta^{-(1-\alpha)}(\alpha * S)(n)$$
$$= (k^{-(1-\alpha)} * \alpha * S)(n+1) - (k^{-(1-\alpha)} * \alpha * S)(n)$$
$$= \sum_{j=0}^{n+1}\left(k^{-(1-\alpha)} * S\right)(n+1-j)a(j) - \sum_{j=0}^{n}\left(k^{-(1-\alpha)} * S\right)(n-j)a(j)$$
$$= \sum_{j=0}^{n}[(k^{1-\alpha} * S)(n+1-j) - (k^{1-\alpha} * S)(n-j)]a(j) + (k^{1-\alpha} * S)(0)a(n+1)$$
$$= \sum_{j=0}^{n}\Delta(k^{1-\alpha} * S)(n-j)a(j) + k^{1-\alpha}(0)S(0)a(n+1)$$
$$= \sum_{j=0}^{n}\Delta^{\alpha}S(n-j)a(j) + S(0)a(n+1)$$

Proving the lemma Now we are ready to prove the second main result of this section Let $0 < \alpha < 1$, and $f: \mathbb{N} \to X$ be given. The unique solution of with initial condition u(0)=x can be represented by $u(n) = S_{\alpha}(n)u|(0) + (S_{\alpha} * f)(n-1), n \epsilon \mathbb{N}$

Proof: Let $n \in$ be given. We have $\Delta^{\alpha}u(n) = \Delta^{\alpha}S_{\alpha}(n)u(0) + \Delta^{\alpha}(S_{\alpha} * f)(n-1).$ Where by definition

$$\Delta^{\alpha}S_{\alpha}(n) = \Delta^{\alpha}k^{\alpha}(n) + T\Delta^{\alpha}(k^{\alpha} * S_{\alpha})(n-1).$$

Note that $\Delta^{\alpha}k^{\alpha}(n) = \Delta \circ \Delta^{-(1-\alpha)}k^{\alpha}(n) = \Delta(k^{1-\alpha} * k^{\alpha})(n) = \Delta^{1}k^{\alpha}(n) = 0,$ therefore using Lemma 3.6 we obtain

$$\Delta^{\alpha} S_{\alpha}(j+1) = T\Delta^{\alpha} = \Delta(k^{\alpha} * S_{\alpha})(j) + k^{\alpha}(0) T\, S_{\alpha}(j+1) = T\, S_{\alpha}(j+1),$$

For all $j \in \mathbb{N}_0$. Hence $\Delta^{\alpha} S_{\alpha}(n) = T\, S_{\alpha}(n)$ and from (3.4) we have

$\Delta^{\alpha} u(n) = TS_{\alpha}(n)u(0) + \Delta^{\alpha}(S_{\alpha} * f)(n-1).$

Now, again by Lemma we get For all $j \in \mathbb{N}$. Therefore from we conclude that

$$\begin{aligned} \Delta^{\alpha} u(n) &= T\, S_{\alpha}(n)u(0) + \Delta^{\alpha}(S_{\alpha} * f)(n-1) + f(n) \\ &= T\, S_{\alpha}(n)u(0) + T\,(S_{\alpha} * f)(n-1) + f(n) \\ &= T\,[S_{\alpha}(n)u(0) + (S_{\alpha} * f)(n-1)] + f(n) \\ &= T\, u(n) + f(n). \end{aligned}$$

Conclusion:

In this paper, we considered the initial value of fractional difference equation and Non linear fractional difference equation with the caputo link difference operators. We obtained the sufficient condition for existence of all solutions regularity of fractional differential equation model and non-linear fractional difference equation model of the system of difference equation is investigated.

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AUTHORS

First Author – M. Reni Sagayaraj, Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur, India

Second Author – P. Manoharan,, Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur, India

Correspondence Author – M. Reni Sagayaraj, Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur, India, Email: reni.sagaya@gmial.com