ISSN 2250-3153

A DIFFERENTIAL GEOMETRIC APPROACH TO FLUID MECHANICS

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Abstract- The paper presents a formulation of some of the most basic entities and equations of fluid mechanics, the continuity equation and the Navier-Stokes equations in a modern differential geometric language using calculus of differential forms on manifold (exterior calculus).

Index Terms- differential geometry, manifolds, exterior calculus, fluid mechanics, Navier-Stokes equations

I. Introduction

Modern differential geometry is a powerful mathematical tool and pervades many branches of physics. Physical theories are often naturally and concisely expressed in terms of differential geometric concepts, using exterior calculus, the calculus of differential forms on manifolds, this calculus of forms provides an intrinsic, coordinate-free approach particularly relevant to concisely describe a multitude of physical models. Even more importantly, equations stated in modern differential geometric terms are not bound to the use of special classes of coordinate systems, such as Cartesian or orthonormal ones. It provides useful and powerful concepts such as differential forms and operators for working with forms and with them a generalization of vector calculus expressions like rotation, gradient and divergence and of various integral theorems like Gauss' divergence theorem or Stokes integral theorem on planes to just one integral theorem. Exterior calculus can thereby be understood as a generalization of vector calculus. In contrast to it, however, exterior calculus is defined on arbitrary manifolds. Exterior calculus is a concise formalism to express differential and integral equation on smooth and curved spaces in a consistent manner, while revealing the geometrical invariants at play.

Modern differential geometry often allows for a clearer, more geometric approach to study physical problems than vector calculus.. In general, it is desirable to express equations in a coordinate and metric free manner, using only differential forms and operators for working with forms, thus simplifying coordinate system changes.

The goal of this work is to give a geometric framework which allows us to describe the fluid systems. Writing equations of fluid mechanics in terms of differential forms enables one to clearly see the geometric features of the fluid field. In this paper we present the modern language of differential geometry to study the fluid mechanics.

2. Differential geometry

This section aims to introduce the basics of modern differential geometry.

2.1. Differentiable manifolds

Manifolds will be our models for space because they offer the most generic coordinate-free model for space.

Definition 2.1. Let M be a set of elements called points. We say that M is a n-dimensional *smooth manifold* if there exists an *atlas*, that is a collection of pairs (U_i, ϕ_i) (called "*charts*") such that:

1. Each U_i is a subset of M and the U_i cover M.
2. Each ϕ_i is a bijection of U_i onto an open subset V_i of $\mathbb{R}^n$. And for any i, j, $\phi_i(U_i \cap U_j)$ is open in $\mathbb{R}^n$.
3. For any i, j, the map $\phi_j\phi_i^{-1}: \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$ is a C^∞ isomorphism.

Definition 2.2. A map $f: M \longrightarrow N$ between smooth manifolds is called *differentiable map* if it is continuous and for each point $p \in M$ there is a chart (U, φ) on M with $p \in U$ and a chart (V, ψ) on M with $f(p) \in V$ such that the composite $\Phi = \psi f \varphi^{-1}$ is differentiable. The map f is called a *diffeomorphism* if it is smooth and bijective with a smooth inverse. Two manifolds are called *diffeomorphic* if there exists a diffeomorphism between them. We denote by Diff(M, N) the set of diffeomorphisms from M to N. The set Diff(M, N) is a group under composition of mappings, called group of diffeomorphisms.

Definition 2.3. Let $\phi: M \longrightarrow N$ be differentiable map from a manifold M to a manifold N, for any $f \in C^\infty(N, \mathbb{R})$. The map $\phi^* f$ ($\phi^*: C^\infty(N, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R})$) given by $(\phi^* f)(x) = (f \circ \phi)(x);\ x \in M$, is called the *dual map* of ϕ (or the pull-back of f by ϕ).

Definition 2.4. Let M be a differentiable manifold, p_0 be a point of M and I be an open interval in $\mathbb{R}$ containing 0. A (differentiable) curve passing through p_0 in M is a differentiable mapping $\sigma: I \longrightarrow M; t \longmapsto \sigma(t)$ such that $\sigma(0) = p_0$. A tangent vector at x is an equivalence class of differentiable curves having the same tangency at x. Let $\sigma: I \longrightarrow M$ be a differentiable curve on a manifold M, with $\sigma(o) = x \in M$, and let D be the set of functions of M which are differentiable at x, the tangent vector to the curve σ at x is map: $\sigma': D \longrightarrow \mathbb{R}$, given by $\sigma'(o)_\phi = \frac{d}{dt}(\phi o \sigma)\big|_{t=0}$; $\phi \in D$, for every chart ϕ containing x. The tangent space to M at the point $x \in M$ written $T_x M$ is the set of equivalence classes of curves at x.

Definition 2.5. Let $\phi: M \longrightarrow N$ be a differentiable map between smooth manifolds M, N. The *differential* (or the *push forward*) of ϕ at $x \in M$ is the linear map $\phi_* = d\phi_x: T_x(M) \longrightarrow T_{\phi(x)}(N)$, defined for $f \in C^\infty(N, \mathbb{R})$ and $X \in T_x(M)$ by $d\phi_x(X)(f) = X_x(\phi^*(f)) = X_x(f \circ \phi)$.

A very useful geometrical construct technically known as the tangent bundle of a differentiable manifold M. This is what the physicist calls velocity space.

Definition 2.6. The *tangent bundle* of a manifold M is the (disjoint) union of the tangent spaces. The tangent bundle *projection* is the (C^∞) surjective mapping $\pi_M: TM \longrightarrow M: (x, X_x) \mapsto x$, given by $\pi_M(x, X_x) = x$; $\forall (x, X_x) \in TM$. A C^k*cross- section* of tangent bundle TM is a mapping s of class C^k of M into TM such that the composition of s with the tangent bundle projection is the identity on M. $s: M \longrightarrow TM$; $x \mapsto X_x \in T_x M$, which are called differentiable vector fields, with $s \circ \pi = Id_M$. Thus, *a vector field* on a manifold M is a mapping $X: M \longrightarrow TM;\ z \mapsto X(z) = (z, X_z);\ X(z) \in T_z M$, $\forall z \in M$, which assigns to each point $z \in M$ a pair composed of a point z and a tangent vector at this point z, such that $\pi \circ X = id|_M: M \longrightarrow M;\ z \mapsto \pi(X(z)) = z$, The set of all vector fields on M denoted by $\mathfrak{X}(M)$. The set of all smooth scalar fields is denoted by $\mathcal{F}(M)$. A *time-dependent* vector field is a differentiable map: $X: \mathbb{R} \times M \longrightarrow TM;\ (t, x) \mapsto X_t(x) = X(t, x)$, $X_t \in \mathfrak{X}(M)$ and $X(t, x) \in T_x M$.

Definition 2. 7. An *integral curve* of vector field X on a manifold M is a differentiable map $c: I \longrightarrow M; t \mapsto c(t)$, where I is an interval in $\mathbb{R}$, such that $c'(t) = X(c(t))$ for all $t \in I$.

Definition 2. 8. A *one-parameter group of diffeomorphisms* of a manifold M is a smooth map ϕ from $\mathbb{R} \times M$ onto M, $\phi: \mathbb{R} \times M \longrightarrow M;\ (t, x) \mapsto \phi_t(x)$, such that $\phi_t: M \longrightarrow M$, $(: x \mapsto \phi_t(x);\ \forall t \in \mathbb{R})$ is a diffeomorphism, ϕ_0 is identity on M and $\phi_{s+t} = \phi_s \circ \phi_t$. Any one parameter group of diffeomorphism $\{\phi_t\}$ on M induce a vector field $X \in \mathfrak{X}(M)$.

Definition 2.9. Let M be a n-dimensional manifold. The dual space $T_x^* M$ to the tangent space $T_x M$ at the point x of a manifold M is called the *cotangent space*, and it's element called *cotangent vectors* (*covectors*) based at x. The cotangent bundle of a manifold M is the (disjoint) union of the cotangent spaces. The tangent bundle *projection* is the (C^∞) surjective mapping $\pi^*: T^*M \longrightarrow M;\ (x, \omega_x) \mapsto x$, given by $\pi^*(x, \omega_x) = x;\ \forall (x, \omega_x) \in T^*M$. A C^k *cross-section* of cotangent bundle T^*M is a mapping s of class C^k of M into T^*M such that the composition of s with the cotangent bundle projection is the identity on M. $s: M \longrightarrow T^*M;\ x \mapsto \omega_x \in T_x^* M$, which are called a field of covector (or differential forms of degree 1), with $\pi^* \circ s = id|_M$. 1-forms are important geometric objects. They can represent, among other things, force, momentum, and phase velocity.

Definition 2.10. Let M be a differentiable manifold and TM its tangent bundle. We call $T_s^r(M) = T_s^r(TM)$ the *Tensor bundle* of type (r, s) over M, which is the collection of all tensors of type (r, s) at all points of M.

Definition 2.11. A *tensor field* T of type (r, s) on a manifold M is a differentiable cross-section of the tensor bundle. $T: M \longrightarrow T_s^r(M);\ x \mapsto T(x) \in T_s^r(x)$. A tensor field T is *symmetric* (resp. *skew-symmetric*) if $T(x)$ is symmetric (resp. skew-symmetric), for every $x \in M$.

2.2. Differential forms and exterior calculus on a manifold

Many important objects in continuum mechanics are made to be integrated. mass density, internal energy density, body forces, surface tractions, and heat influx are just a few important examples, are differential forms. Differential forms provides the language for expressing the equations of mathematical physics in a coordinate-free form, one of the fundamental principles of relativity. In this paper, differential forms are used to model fluid system.

Definition 2.12. Let M be a differentiable manifold, and the spaces TM, T^*M denote the tangent and cotangent bundle of M respectively. The pth *exterior bundle* over M is

$$\Lambda^p(M) = \Lambda^p(T^*M) := \{(x, \omega_x) | x \in M, \omega_x \in \Lambda^p(T_x^*(M))\}.$$

Definition 2.13. Let M be a differentiable manifold. A *differential form* of degree p (or p-form) on M is a differentiable cross section: $\omega: M \longrightarrow \Lambda^p(T^*M);\ x \mapsto \omega(x) = (x, \omega_x)$. In other words differential p-form is skew-symmetric p-linear mapping.

Definition 2.14. For each $p \geq 0$, $q \geq 0$ with $p + q \leq n$ and for any p-form $\omega \in \Omega^p(M)$, q-form $\theta \in \Omega^q(M)$ there is an operation called *wedge product*:, $\wedge: \Omega^p(M) \times \Omega^q(M) \longrightarrow \Omega^{p+q}(M);\ (\omega, \theta) \mapsto \omega \wedge \theta$.

Definition 2.15. The *exterior derivative* $\boldsymbol{d}$ of a differential p-form on a manifold M is a mapping: $\boldsymbol{d}: \Omega^p(M) \longrightarrow \Omega^{p+1}(M);\ \omega \mapsto \boldsymbol{d}\omega$, where $\boldsymbol{d}\omega$ is a differential $(p+1)$-form defined over the same manifold M. The exterior derivative satisfies the following properties:

i. If $f \in \Omega^0(M)$, then $\boldsymbol{d}(f) = df$, $df(X) = X(f)$ for $X \in \mathfrak{X}(M)$.
ii. $\forall \omega \in \Omega^p(M)$, $\forall \theta \in \Omega^q(M): \boldsymbol{d}(\omega \wedge \theta) = \boldsymbol{d}\omega \wedge \theta + (-1)^p \omega \wedge \boldsymbol{d}\theta$.
iii. For every form $\omega \in \Omega^p(M)$, $\boldsymbol{d}(\boldsymbol{d}\omega) = 0$.

Definition 2.16. Let M and M' be two differentiable manifolds and $\phi: M \longrightarrow M'$ be differentiable mapping. the *pullback* $\phi^*: \Omega^p(M') \longrightarrow \Omega^p(M)$ is a map define as follows: For any p-form ω on M', the pullback $\phi^*\omega$ of ω by ϕ is a p-form on M given by: $(\phi^*\omega)(x, X_1, \dots, X_p) = \omega(\phi(x), \phi_* X_1, \dots, \phi_* X_p)$; $X_1, \dots, X_p \in T_x(\mathcal{M})$. The pullback has the following properties:

i. The pullback ϕ^* is a linear, $\forall \omega, \theta \in \Omega^p(M): \phi^*(\omega + \theta) = \phi^*\omega + \phi^*\theta$.
ii. $\forall \omega \in \Omega^p(M)$, $\forall \theta \in \Omega^q(M): \phi^*(\omega \wedge \theta) = \phi^*\omega \wedge \phi^*\theta$.
iii. $\phi^*(f\omega) = (f \circ \phi)\phi^*\omega, f \in C^\infty(V)$.
iv. Pullback commutes with exterior derivative, $d(\phi^*\omega) = \phi^*(d\omega)$.

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Definition 2.17. Let $X_1, \dots, X_{p-1} \in \mathfrak{X}(M)$ denote a set of vector fields on M and $\omega \in \Omega^p(M)$ be any p-form on M. The *contraction* of ω by the vector field X is an operator $i_X: \mathfrak{X}(M) \times \Omega^p(M) \longrightarrow \Omega^{p-1}(M)$; $(X, \omega) \longmapsto i_X\omega$, it maps a p-form into a $(p-1)$-form, defined by: $(i_X\omega)(X_1, \dots, X_{p-1}) = \omega(X, X_1, \dots, X_{p-1})$.

The arguably most important derivative on manifolds is the Lie derivative. It describes how a geometric object (a vector field, a tensor, a form) changes when it is dragged along some vector field X.

Definition 2.18. Consider a 1-parameter group $\{\phi_t\}$ of diffeomorphism of differentiable manifold M, if $\omega \in \Omega^p(M)$ is any p-form. The *Lie derivative* $L_X\omega$ of ω with respect to the vector field $X \in \mathfrak{X}(M)$ is the differential operator $L: \mathfrak{X}(M) \times \Omega^p(M) \longrightarrow \Omega^p(M)$; $(X, \omega) \longmapsto L_X\omega$, defined by $(L_X\omega) = \frac{d}{dt}\phi_t^*\omega\Big|_{t=0}$. It has the following properties:

i. $\forall \omega \in \Omega^p(M), \forall \theta \in \Omega^q(M): L_X(\omega \wedge \theta) = L_X\omega \wedge \theta + \omega \wedge L_X\theta$
ii. $\frac{d}{dt}\phi_t^*\omega\Big| = \phi_t^* L_X\omega = L_X(\phi_t^*\omega)$.
iii. f $f \in \Omega^0(M)$, then $L_X f = Xf = \frac{\partial}{\partial t} f(\phi_t)\Big|_{t=0}$.
iv. $\forall \omega \in \Omega^p(M), \forall X \in \mathfrak{X}(M): L_X\omega = i_X d\omega + d i_X\omega,\ \forall X \in \mathfrak{X}(M)$.
v. The Lie derivative of constant function is equal to zero.

The concept of invariance along the flow of a vector field will therefore be of considerable importance in the following.

Definition 2.19. Let M be a manifold and $X \in \mathfrak{X}(M)$ a vector field on M. A differential form $\omega \in \Omega^p(M)$ is *invariant* under the flow of X when $L_X\omega = 0$.

Definition 2.20. A *volume form* on an n-manifold M is an n-form $\omega \in \Omega^n(M)$ such that $\omega(x) \neq 0$; $\forall x \in M$. For a volume form ω, $\boldsymbol{d}\omega = 0$. A manifold M is called *orientable* if and only if there is volume form on it.

In the following, we provide two important theorems about the integration of differential forms on manifolds.

Theorem 2.1 (Stokes' Theorem). Let ω be an $(n-1)$-form on an orientable n-manifold M. Stokes' theorem states that the integral of an $(n-1)$-form ω over the boundary ∂M of M equals the integral of the exterior derivative of this form $\boldsymbol{d}\omega$ over M.

$$\int_{\partial M} \omega = \int_M \boldsymbol{d}\omega. \tag{2.2.1}$$

Another simple but, nevertheless, very useful result of integral of differential forms concerns behavior of integrals with respect to mappings of manifolds.

Theorem 2.2 (Change of Variables Theorem). Suppose M and N are oriented n-manifolds and $\phi: M \to N$ is an orientation-preserving diffeomorphism. If $\omega \in \Omega^n(N)$ has compact support, then $\phi^*\omega$ has compact support and

$$\int_N \omega = \int_M \phi^*\omega. \tag{2.2.2}$$

Remark 2.1. With slight abuse of notation, one often writes for the change of variables formula

$$\int_{\phi(M)} \omega = \int_M \phi^*\omega. \tag{2.2.3}$$

3. THE GEOMETRIC SETUP

With the differential geometrical tools we have developed, now we give a geometric models of basic kinematics used in modeling the fluid flow.

3.1. The fluid space

The first step is to model the fluid region. In an attempt to give an invariant formulation of fluid mechanics, the placement of a fluid in space, the space the fluid flows in is assumed to be a smooth manifold M. A fluid particle is a point in the manifold. Points in a domain $D \subset M$ represent the geometric positions of material particles, these points are denoted by $x \in M$ and called the particle labels.

3.2. Fluid motions

The next step is the geometric notion of the fluid motion. Consider fluid moving in M, our manifold M whose points are supposed to represent the fluid particles. As t increase, call $\phi_t(x)$ the curve followed by the fluid particle, which is initially at $x \in M$. For fixed t, each ϕ_t will be a diffeomorphism of M. Thus the fluid motion is a smooth one-parameter family of diffeomorphisms $\phi_t: M \longrightarrow M$; with $\phi_0 = Id$.

Suppose M is a differentiable n-manifold and $t \longrightarrow \phi(t)$ is a one-parameter family of diffeomorophisms of M. Intuitively, M is the space in which the fluid moves. For each value of t, define a vector field $X_t \in \mathfrak{X}(M)$ as follows:

For $x \in M$ (x be a point in M), $X_t(x)$ is the tangent vector to the curve $u \longrightarrow \phi_u \phi_t^{-1}(x)$ at $u = t$, physically, X_t is the velocity field corresponding to the fluid motion defined by ϕ_t, that is, $X_t(x)$ is the velocity vector of the particle that, at time t, is at the point x.

$$X(\phi_t(x), t) \coloneqq \frac{\partial \phi_t}{\partial t}(x). \tag{3.2.1}$$

Geometrically, the velocity vector describes the path the particle would have taken if it had continued on with uniform motion. For $f \in F(M)$, X_t in terms of its action on functions on M is:

$$X_t(f) = (\phi_t^{-1})^* \frac{\partial}{\partial t}\big(\phi_t^*(f)\big); \text{ for } f \in F(M). \tag{3.2.2}$$

Consider fluid flow on a manifold M described by a time-dependent vector field X. A flow is called *steady* (or *stationary*) if its vector field satisfies

$$\frac{\partial X}{\partial t} = 0, \tag{3.2.3}$$

i.e. X is constant in time. This condition means that the "shape" of the fluid flow is not changing. Even if each particle is moving under the flow, the global configuration of the fluid does not change.

Given a fluid flow with velocity (vector) field $X(x, t)$, a *streamline* at a fixed time t is an integral curve of X; that is, if $c(s)$ is a streamline parameterized by s at the instant t, then $c(s)$ satisfies

$$\frac{dc}{ds} = X(c(s), t), \qquad t \text{ fixed.} \tag{3.2.4}$$

On the other hand, a trajectory is the curve traced out by a particle as time progresses, that is, is a solution of the differential equation

$$\frac{dc}{dt} = X(c(t), t), \tag{3.2.5}$$

with given initial conditions. If X is independent of t then, streamlines and trajectories coincide.

4. Geometric formulation of fluid equations

In this section, we present some of the basic equations of fluid mechanics in invariant form by using tools and language of modern differential geometry.

4.1. Continuity equation

Let $D \subset M$ be a sub region of M (M denotes a differentiable n-manifold). Consider a fluid moving in a domain $D \subset M$ and suppose ω is a fixed-volume element differential form on M. Then, the total mass of the fluid in the region D at time t is

$$m(D, t) = \int_D \rho_t \,\omega, \tag{4.1.1}$$

where ρ_t is the function $\rho_t : x \longmapsto \rho(x, t)$ on M, describes the mass-density of the fluid at time t.

One of the most important properties of the fluid dynamics is the principle of mass conservation, which states that the total mass of the fluid, which at time $t = 0$ occupied a nice region D, remains unchanged after time t.

Thus, the total mass of the fluid at time $t = 0$ occupied a region D is maintained with time, i.e.

$$\int_{\phi_t(D)} \rho_t \,\omega = \int_D \rho_0 \,\omega, \tag{4.1.2}$$

where ϕ_t is the one-parameter family of diffeomorophisms.

Equation $(4.1.2)$ is the integral invariant for conservation of mass. By the change of variable formula (theorem 2.2), the left hand side of this relation is equal to

$$\int_D \phi_t^*(\rho_t \,\omega) = \int_D \rho_0 \,\omega. \tag{4.1.3}$$

Since the right-hand side of $(4.1.3)$ is independent of t, we can differentiate $(4.1.3)$ with respect to t we get

$$\frac{\partial}{\partial t}\int_D \phi_t^*(\rho_t \,\omega) = \int_D \frac{\partial}{\partial t}\big(\phi_t^*(\rho_t \,\omega)\big) = 0. \tag{4.1.4}$$

From definition and properties of Lie derivative we have

$$\frac{\partial}{\partial t}\big(\phi_t^*(\rho_t \,\omega)\big) = \phi_t^*\big(L_{X_t}(\rho_t \,\omega)\big) + \phi_t^*\Big(\frac{\partial}{\partial t}(\rho_t)\omega\Big),$$

then $(4.1.4)$ takes the form

$$\int_D \Big[\phi_t^*\Big(L_{X_t}(\rho_t \,\omega) + \frac{\partial}{\partial t}(\rho_t)\omega\Big)\Big] = 0, \tag{4.1.5}$$

by change of variable theorem (theorem 2.2), which equals

$$\int_{\phi_t(D)} \Big[L_{X_t}(\rho_t \,\omega) + \frac{\partial}{\partial t}(\rho_t)\omega\Big] = 0; \qquad \forall D. \tag{4.1.6}$$

Since D is an arbitrary open set, this can be true only if the integrand is zero ; that is

$$L_{X_t}(\rho_t \,\omega) + \frac{\partial}{\partial t}(\rho_t)\omega = 0. \tag{4.1.7}$$

This is the equation of continuity in invariant form. This equation allows interpreting $\rho_t \,\omega$ as a density form on the fluid that is dynamically conserved along the flow $\{\phi_t\}$.

If the fluid is incompressible (i.e. ρ is a constant function), then the continuity equation takes the following form

$$L_X(\rho\omega) = 0. \quad (4.1.8)$$

From properties of the Lie derivative (property(i)) this reduces to

$$(L_X\rho)\omega + \rho L_X\omega = 0. \quad (4.1.9)$$

Since ρ is a constant function, (the Lie derivative of constant function is equal to zero) the first term on the left-hand side of equation $(4.1.9)$ is vanish, then equation $(4.1.9)$ take the following form

$$L_X\omega = 0. \quad (4.1.10)$$

This is the geometric form of continuity equation for incompressible fluid. Hence this equation implies that X preserves the volume form i.e. $\phi_t^*(\omega) = \omega$.

4.2. Equation of motion

The most general equation for description of fluid phenomena is the Navier-Stokes equation, which as special case, comprises Euler's equation of motion. Let M be a differentiable n-manifold, θ be a differential 1-form on M, ω be a volume-element differential form on M and D be a domain in M. Consider a fluid moving in a domain $D \subset M$. For any continuum there are two types of forces acting on a piece of material:

- First there are external or body forces:

  Suppose that F is a vector field representing the volume forces on a domain D, then the θ-component of the volume forces acting on the domain D are

  $$\text{Force}_{\text{vol}} = \int_D \rho\theta(F)\omega, \quad (4.2.1)$$

  where ρ is a function on M giving the density of mass.

- Second there are forces of stress (discipline of continuum mechanics can also encounter forces that come from the region surrounding a bit of fluid, expressed by the stress tensor):

  Let T be a tensor field on M representing this stress. At $x \in M$, T is a skew-symmetric tensor field on M define in the following form:

  $$\theta^{n-1}(X_1, X_2, \dots, X_{n-1}) = \theta\big(T(X_1, X_2, \dots, X_{n-1})\big),\ X_i \in T_x(M), \quad (4.2.2)$$

  then we define the stress tensor of the fluid inside $D \subset M$ in terms of the multilinear map:

  $$T = T(X_1, X_2, \dots, X_{n-1}) : T_x(M) \times \dots \times T_x(M) \longrightarrow T_x(M). \quad (4.2.3)$$

  For $x \in M$, $X_1, X_2, \dots, X_{n-1} \in T_x(M)$, $\theta\big(T(X_1, X_2, \dots, X_{n-1})\big)$ is an $(n-1)$ covector on M. Then $\theta(T)$ defining for each θ an $(n-1)$-differential form on M. If D is a domain in M with boundary ∂D, then the total θ-component of the stress force is

  $$\text{Force}_{\text{str}} = \int_{\partial D} \theta(T). \quad (4.2.4)$$

  By applying Stokes' theorem (theorem 2.1), to $\int_{\partial D} \theta(T)$ we get

  $$\text{Force}_{\text{str}} = \int_{\partial D} \theta(T) = \int_D d\theta(T). \quad (4.2.5)$$

From $(4.2.1)$ and $(4.2.5)$ the θ-component of the force acting on the domain D at fixed time is

$$\text{Force}_{\text{tot}} = \int_D [\rho\theta(F)\omega + d\theta(T)] = \int_{\phi_t(D)} \Big[\rho_t\theta(F_t)\omega + d\Big(\theta\big(T(z_t)(i_z\omega)\big)\Big)\Big], \quad (4.2.6)$$

where ϕ_t is the one-parameter family of diffeomorophisms.

Suppose a group of particles making up the fluid starts out at $t = 0$ to occupy the domain D. At time t, they will be in domain $\phi_t(D)$, their θ-component of total momentum will be

$$\text{Momentum}_{\text{tot}} = \int_{\phi_t(D)} \theta(X_t)\,\rho_t\,\omega, \quad (4.2.7)$$

by the change of variables theorem (theorem 2.2), which is equal to

$$\text{Momentum}_{\text{tot}} = \int_D \phi_t^*(\theta(X_t)\rho_t\omega). \quad (4.2.8)$$

The rate of change of momentum

$$\frac{d}{dt}\int_D \phi_t^*(\theta(X_t)\rho_t\omega) = \int_D \frac{d}{dt}[\phi_t^*(\theta(X_t)\rho_t\omega)]$$
$$= \int_D \phi_t^*\big(X_t(\theta(X_t)\rho_t\omega)\big) + \int_D \phi_t^*\left(\theta\left(\frac{\partial X_t}{\partial t}\right)\rho_t\omega + \theta(X_t)\frac{\partial\rho_t}{\partial t}\omega\right), \quad (4.2.9)$$

by the change of variables theorem (theorem 2.2) we get

$$\frac{d}{dt}\int_D \phi_t^*(\theta(X_t)\rho_t\omega) = \int_{\phi_t(D)} \left[X_t\left(\theta(X_t)\rho_t\omega + \theta\left(\frac{\partial X_t}{\partial t}\right)\rho_t\omega\right) + \theta(X_t)\frac{\partial\rho_t}{\partial t}\omega\right]. \quad (4.2.10)$$

By applying Newton's law of motion (principle of balance of momentum): the rate of momentum of a portion of the fluid equals the total force applied to it. By equating $(4.2.6)$ to the expression $(4.2.10)$ for the force acting on this bunch we get

$$\int_{\phi_t(D)} \left[\rho_t \theta(F_t)\omega + \boldsymbol{d}\left(\theta\left(T(z_t)(i_z\omega)\right)\right)\right] = \int_{\phi_t(D)} \left[X_t\left(\theta(X_t)\rho_t\omega + \theta\left(\frac{\partial X_t}{\partial t}\right)\rho_t\omega\right) + \theta(X_t)\frac{\partial \rho_t}{\partial t}\omega\right]. \quad (4.2.11)$$

Since this relation is to hold for all domain D, we have

$$\rho_t\theta(F_t)\omega + \boldsymbol{d}\theta(T_t) = X_t(\theta(X_t)\rho_t\omega) + \theta\left(\frac{\partial X_t}{\partial t}\right)\rho_t\omega + \theta(X_t)\frac{\partial \rho_t}{\partial t}\omega. \quad (4.2.12)$$

This is the geometric version of Navier-Stokes equation of fluid motion, which is coordinate-free.

Finally, we notice that the fluid mechanics can be formulated in a geometric language. The fluid space is smooth manifold, the (dynamics) fluid motions is one-parameter family of diffeomorphisms and equations of fluid mechanics written in terms of differential forms are invariant under diffeomorphisms.

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