On πp- Compact spaces and πp- connectedness

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Abstract: In this paper we introduce a new notion called πp-compact spaces and πp- connectedness on generalized topological space. Some properties and characterizations of such spaces are investigated.

Mathematics subject classification: 54A05, 54D15
Index Terms: πp- compact space, πp- connected, strongly πp- continuous, πp(μ)-seperated.

1. Introduction

The theory of generalized topological space (GTS), were introduced by Á.Császár [2, 4, 6], is one of the most important developments of general topology in recent years. It is well known that the concept of compactness and connectedness plays an important role in generalized topological space. The notion of μ-compactness in generalized topological space was introduced by Jyothis Thomas and Sunil Jacob John[7]. In [3] Á.Császár has also introduced the concept of γ-compact in generalized topological space. R.X.Shen [8] has introduced the notions of α-connected, σ-connected, π-connected, β-connected and also studied the preservation of connectedness under the basic operators in generalized topological spaces. The purpose of this paper is to define πp- compact spaces and πp-connectedness on generalized topological space and obtain their properties and characterizations.

2. Preliminaries

We recall some basic concepts and results.

Let X be a nonempty set and let exp(X) be the power set of X. μ⊂exp(X) is called a generalized topology [4](briefly, GT) on X, if ∅ ∈μ and unions of elements of μ belong to μ. The pair (X, μ) is called a generalized topological space (briefly, GTS). The elements of μ are called μ-open [2] subsets of X and the complements are called μ-closed sets. If (X, μ) is a GTS and A⊆X, then the interior of (denoted by $i_\mu(A)$) is the union of all G⊆A, G∈μ and the closure of A (denoted by $c_\mu(A)$) is the intersection of all μ-closed sets containing A. Note that $c_\mu(A) = X-i_\mu(X-S)$ and $i_\mu(A) = X - c_\mu(X-A)$ [3].

Definition 2.1[2] Let (X, μ_x) be a generalized topological space and A⊆ X. Then A is said to be

(i) μ- semi open if $A\subseteq c_\mu(i_\mu(A))$.
(ii) μ- pre open if $A\subseteq i_\mu(c_\mu(A))$.
(iii) μ-α-open if $A\subseteq i_\mu(c_\mu(i_\mu(A)))$.
(iv) μ-β-open if $A\subseteq c_\mu(i_\mu(c_\mu(A)))$.
(v) μ-r-open [9] if $A = i_\mu(c_\mu(A))$
(vi) μ-rα-open [1] if there is a μ-r-open set U such that $U\subset A\subset c_\alpha(U)$.

Definition 2.2 [1] Let (X, μ_x) be a generalized topological space and A⊆ X. Then A is said to be μ-πrα closed set if $c_\pi(A) \subseteq U$ whenever A⊆U and U is μ-rα-open set. The complement of μ-πrα closed set is said to be μ-πrα open set.

The complement of μ-semi open (μ-pre open, μ-α-open, μ-β-open, μ-r-open, μ-rα-open) set is called μ- semi closed (μ-pre closed, μ-α- closed, μ-β- closed, μ-r- closed, μ-rα-closed) set.

Let us denote the class of all μ-semi open sets, μ-pre open sets, μ-α-open sets, μ-β-open sets, and μ-πrα open sets on X by σ(μ) (σ for short), π(μ) (π for short), α(μ) (α for short), β(μ) (β for short) and πp(μ) (πp for short)respectively. Let μ be a generalized topology on a non empty set X and S⊆X. The μ-α-closure (resp. μ-semi closure, μ-pre closure, μ-β-closure, μ-πrα-closure) of a subset S of X denoted by $c_\alpha(S)$ (resp. $c_\sigma(S)$, $c_\pi(S)$, $c_\beta(S)$, $c_{\pi p}(S)$) is the intersection of μ-α-closed(resp. μ- semi closed, μ- pre closed, μ-β-closed, μ-πrα closed) sets including S. The μ-α-interior (resp. μ-semi interior, μ-pre interior, μ-β-interior, μ-πrα-interior) of a subset S of X denoted by $i_\alpha(S)$ (resp. $i_\sigma(S)$, $i_\pi(S)$, $i_\beta(S)$, $i_{\pi p}(S)$) is the union of μ-α-open (resp. μ- semi open, μ- pre open, μ-β-open, μ-πrα open) sets contained in S.

Definition 2.3 [1] A function f between the generalized topological spaces (X, μ_x) and (Y, μ_y) is called

(i) (μ_x, μ_y)- πrα – continuous function if $f^{-1}(A)$∈μ-πrα (X, μ_x) for each A ∈(Y, μ_y).
(ii) (μ_x, μ_y)- πrα – irresolute function if $f^{-1}(A)$∈μ-πrα (X, μ_x) for each A ∈ μ-πrα(Y, μ_y).

3. πp- compact spaces

Definition 3.1 A generalized topological space (X, μ_x) is called πp- compact if each cover of X composed of elements of μ-πrα open sets admits a finite sub cover.

Definition 3.2 Let (X, μ_x) be a generalized topological space then

(i) a collection {A_λ; λ∈ Λ} of μ-πrα open sets of X is called πp- open cover of a subset B of X if B ⊂ ∪{A_λ; λ∈ Λ}holds.

ISSN 2250-3153

(ii) a subset B of generalized topological space X is called πp- compact relative to X if for every collection $\{A_\lambda; \lambda\in \Lambda\}$of μ-πrα open subsets of X such that $B \subseteq \cup\{A_\lambda; \lambda\in \Lambda\}$,there exist a finite subset Λ_0 of Λ such that $B \subseteq \cup\{A_\lambda; \lambda\in \Lambda_0\}$.

(iii) a subset B of generalized topological space X is said to be πp- compact if B is πp- compact as a generalized subspace of X.

Notice that if (X, μ_x) is a generalized topological space and A⊆X then $\mu_A = \{U\cap A; U\in\mu_x\}$ is a generalized topology on A. (A, μ_A) is called a generalized subspace of (X, μ_x)

Remark 3.3 If X is finite then (X, μ_x) is πp- compact for any generalized topology μ_x on X.

Theorem 3.4 Let (X, μ_x) be a generalized topological space then every μ-πrα closed subset of πp- compact space X is πp- compact relative to X.

Proof: Let A be μ-πrα closed subset of X then X\A is μ-πrα open. Let $\{A_\lambda; \lambda\in \Lambda\}$be a cover of A by μ-πrα open subsets of X then $\{A_\lambda; \lambda\in \Lambda\} \cup(X\backslash A)$ is an πp- open cover of X. By hypothesis, X is πp- compact. Then it has a finite sub cover of X, say $(A_1\cup A_2\cup\text{------}\cup A_n)\cup(X\backslash A)$. But A and X\A are disjoint, hence $A\subset (A_1\cup A_2\cup\text{------}\cup A_n)$. So$\{A_\lambda; \lambda\in \Lambda\}$contains a finite sub cover for A. Therefore A is πp- compact relative to X.

Theorem 3.5 Let (X, μ_x) and (Y, μ_y) be two GTS's and f: X →Y be a bijective map.

(i) If X is πp- compact and f is (μ_x, μ_y)-πrα continuous then Y is μ- compact.

(ii) If f is (μ_x, μ_y) –πrα irresolute and X is πp- compact then so is Y.

Proof: (i) Let f be an (μ_x, μ_y)-πrα continuous bijective map and X be an πp-compact space. Let $\{A_\lambda/ \lambda\in \Lambda\}$ be an μ- open cover for Y, then $\{f^{-1}(A_\lambda)/ \lambda\in \Lambda\}$ is an πp- open cover of X. Since X is πp-compact it has a finite sub cover say $\{f^{-1}(A_1), f^{-1}(A_2), f^{-1}(A_3),\ldots\ldots\ldots, f^{-1}(A_n)\}$. But f is bijective, so $\{A_1, A_2,\ldots\ldots.., A_n\}$is a finite sub cover of Y. Hence Y is μ- compact.

(ii) Let $\{A_\lambda/ \lambda\in \Lambda\}$ be any collection of μ-πrα open subsets of Y. Since f is (μ_x, μ_y) –πrα irresolute bijective map, then $\{f^{-1}(A_\lambda)/ \lambda\in \Lambda\}$is a πp- open cover of X. Since X is πp- compact and f is bijective map, there exists a finite πp- open sub cover of Y. Hence (Y, μ_y) is πp- compact.

Definition 3.6 A function f between the GTS's (X, μ_x) and (Y, μ_y) is called strongly πp- continuous function if inverse image of every μ-πrα open set in Y is μ- open in X.

Theorem 3.7 Let (X, μ_x) and (Y, μ_y) be two GTS's and f: X →Y be a strongly πp- continuous onto map. If (X, μ_x) is μ- compact then so is (Y, μ_y).

Proof: Let $\{A_\lambda /\lambda\in \Lambda\}$ be a πp- open cover of Y.

Since f is strongly πp- continuous then $\{f^{-1}(A_\lambda)/\lambda\in \Lambda\}$ is an μ- open cover of X. Since X is μ-compact, it has a finite sub cover say$\{ f^{-1}(A_1), f^{-1}(A_2),\ldots\ldots, f^{-1}(A_n)\}$. Therefore $\{A_1, A_2,\ldots\ldots, A_n\}$ is a finite πp- open cover of Y. Hence Y is πp- compact.

Remark 3.8 Since μ- open implies μ-πrα open, it follows that πp-compact implies μ- compact.

Theorem 3.9 A generalized topological space X is πp- compact if and only if every family of μ-πrα closed sets in X with empty intersection has a finite subfamily with empty intersection.

Proof: Suppose X is πp- compact and $\{A_\alpha; \alpha\in \nabla\}$is a family of μ-πrα closed sets in X such that $\cap\{A_\alpha; \alpha\in \nabla\}= \emptyset$. Then $\cup\{X\backslash A_\alpha; \alpha\in \nabla\}$ is a πp- open cover for X. Since X is πp- compact this cover has a finite sub cover, say $\{X\backslash A_{\alpha 1}, X\backslash A_{\alpha 2},\ldots\ldots\ldots., X\backslash A_{\alpha n}\}$for X. That is $X= \cup\{X\backslash A_{\alpha i}: i = 1,2,\ldots\ldots.,n\}$. This implies that$\bigcap_{i=1}^{n} A_{\alpha i} =\emptyset$.

Conversely, suppose that every family of μ-πrα closed sets in X which has empty intersection has a finite subfamily with empty intersection. Let $\{B_\alpha; \alpha\in \nabla\}$be a πp- open cover for X. Then $\cup\{B_\alpha; \alpha\in \nabla\}= X$. Taking the complements we get $\cap\{X\backslash B_\alpha; \alpha\in \nabla\}= \emptyset$. Since X\Bα is μ-πrα closed for each α∈ ∇, by the assumption, there is a finite subfamily $\{X\backslash B_{\alpha 1}, X\backslash B_{\alpha 2},\ldots\ldots\ldots., X\backslash B_{\alpha n}\}$with empty intersection. That is $\bigcap_{i=1}^{n} B_{\alpha i} =\emptyset$. Taking the compliments on both sides we get$\bigcup_{i=1}^{n} B_{\alpha i} = X$. Hence X is πp- compact.

4.πp- connected spaces.

Definition 4.1 A generalized topological space (X, μ) is said to be connected [8] (called γ- connected in [5]) if there are no non empty disjoint sets U, V ∈μ such that U∪V = X.

Definition 4.2 [8] A generalized topological space (X, μ) is called α-connected (resp. σ-connected, π-connected, β-connected) if (X, α(μ))(resp. (X, σ(μ)), (X, π(μ)),(X, β(μ))) is connected.

Definition 4.3 A generalized topological space (X, μ_x) is called πp-connected (μ-πrα connected [1]) if (X, πp(μ)) is connected.

It is easy to see from the definition that

		πp- connected		
		⇓		
β-connected	⇒	π-connected		
⇓		⇓		
σ-connected	⇒	α-connected	⇒	connected

Example 4.4 Let X= {a, b, c, d} and μ= {∅, X, {a},{c}, {a, c}, {a, b, c}}. Then the GTS (X, μ) is connected and β-connected but not πp- connected.

Example 4.5 Let X= {a, b, c} and μ= {∅, X, {a}, {b},{a,b}}. Then the GTS (X, μ) is π- connected but not πp- connected.

Example 4.6 Let X= {a, b, c, d} and μ= {∅, X, {a, b, c}, {b, c, d}, {a, c, d}, {a, b, d}}. Then the GTS (X, μ) is σ-connected but not πp- connected.

Example 4.7 Let X= {a, b, c, d} and μ = {∅, {a}, {a,b}, {b,c},{a, b, c}}. Then the GTS (X, μ) is πp- connected but not σ-connected and β-connected.

Definition 4.8 [5] Let (X, μ_x) be a generalized topological space. Given U, V⊆X, let us say that U and V are α(μ)-separated if $c_\alpha(U)\cap V= c_\alpha(V)\cap U =\emptyset$.

Definition 4.9 Let (X, μ_x) be a generalized topological space. The subsets U and V are πp(μ)- separated if X=U∪V and $c_{\pi p}(U)\cap V= c_{\pi p}(V)\cap U =\emptyset$.

Lemma 4.10 For U, V ⊆ X, the following statements are equivalent: (i) U and V are πp(μ)- separated.

(ii)There are μ-πrα closed sets F_U and F_V such that $U\subseteq F_U\subseteq X-V$ and $V\subseteq F_V\subseteq X-U$.

(iii)There are μ-πrα open sets G_U and G_V such that $U\subseteq G_U\subseteq X-V$ and $V\subseteq G_V\subseteq X-U$.

Proof: (i) ⇒(ii) Let $F_U =c_{\pi p}(U)$, $F_V= c_{\pi p}(V)$. Since U and V are πp(μ)- separated then F_U and F_V are μ-πrα closed sets such that $U\subseteq F_U\subseteq X-V$ and $V\subseteq F_V\subseteq X-U$.

(ii)⇒(iii) Let F_U and F_V are μ-πrα closed sets such that $U\subseteq F_U\subseteq X-V$ and $V\subseteq F_V\subseteq X-U$. Then $G_U =X-F_V$ and $G_V= X-F_U$ are μ-πrα open sets such that $U\subseteq G_U\subseteq X-V$ and $V\subseteq G_V\subseteq X-U$.

(iii)⇒(ii)Let $F_U= X-G_V$, $F_V= X-G_U$, the implication is obvious.

(ii)⇒(i) Clearly $c_{\pi p}(U)\subseteq F_U$, $c_{\pi p}\subseteq F_V$, the implication is obvious.

Lemma 4.11 Let (X, μ_x) and (Y, μ_y) be generalized topological spaces. If f: X→Y is (μ_x, μ_y)-πrα irresolute and U_y and V_y are πp(μ)-seperated, then $f^{-1}(U_y)$ and $f^{-1}(V_y)$ are πp(μ)-separated.

Proof: Let U_y and V_y are πp(μ)- separated. Then by Lemma 3.9, there exist μ-πrα open sets G_{Uy},G_{Vy} in Y such that $U_y\subseteq G_{Uy}\subseteq Y-V_y$ and $V_y\subseteq G_{Vy}\subseteq Y-U_y$. Then $f^{-1}(U_y)\subseteq f^{-1}(G_{Uy})\subseteq X-f^{-1}(V_y)$ and $f^{-1}(V_y)\subseteq f^{-1}(G_{Vy})\subseteq X-f^{-1}(U_y)$. Since $f^{-1}(G_{Uy})$ and $f^{-1}(G_{Vy})$ are μ-πrα open sets, and by Lemma 4.10, $f^{-1}(U_y)$ and $f^{-1}(V_y)$ are πp(μ)-seperated.

Lemma 4.12 If S is πp- connected subsets of a generalized topological space (X, μ) such that S⊆U∪V where U,V are πp(μ)- separated sets then either S⊆U or S⊆V.

Proof: Since U and V are both πp- separated sets, it follows that X=U∪V and $U\cap c_{\pi p}(V)$ and $V\cap c_{\pi p}(U) =\emptyset$, U, V are μ-πrα closed sets in X. For any subset S of X, we have S= (S∩U) ∪(S∩V).Clearly S∩U and S∩V are πp(μ)-seperated. Since S is πp- connected hence atleast one of them S∩U and S∩V should be empty. Hence either S⊆U or S⊆V.

Theorem 4.13 Let A be πp- connected subspace of generalized topological space X. If $A\subset B\subset c_{\pi p}(A)$ then B is also πp-connected.

Proof: Let A be πp-connected and let $A\subset B\subset c_{\pi p}(A)$.Suppose that B=U∪V is a πp(μ)-separation of B by μ-πrα open set. Then by Lemma 4.12, above A must lie entirely in U or in V. Suppose that A⊂U, then $c_{\pi p}(A)\subseteq c_{\pi p}(U)$. Since $c_{\pi p}(U)$ and V are disjoint, B cannot intersect V. So V= ∅, which is contradicts to the fact that V is non empty subset of B. Hence B is πp-connected.

Theorem 4.14 Let (X, μ_x) and (Y, μ_y) be generalized topological spaces and let f: X→Y be a (μ_x, μ_y)-πrα irresolute function. If (X, μ_x) is πp- connected then so is (Y, μ_y).

Proof: Suppose $Y = U_y\cup V_y$ with πp(μ)- separated sets U_y, V_y of Y. Then $f^{-1}(Y) = f^{-1}(U_y)\cup f^{-1}(V_y)$.By Lemma 4.11, $f^{-1}(U_y)$, $f^{-1}(V_y)$ are πp(μ)-seperated. By Lemma 4.12, either $f^{-1}(Y)\subseteq f^{-1}(U_y)$ or $f^{-1}(Y) \subseteq f^{-1}(V_y)$. So $Y\subseteq U_y$ or $Y\subseteq V_y$ and $V_y = \emptyset$ or $U_y = \emptyset$. Hence (Y, μ_y) is πp- connected.

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