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Time Dependent Solution Of $M^{[X]}/G/1$ Queueing Model With second optional service, Bernoulli k-optional Vacation And Balking

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Abstract

This paper investigate a queueing model where in customers arrive in batches according to compound Poisson process with rate λ and are served one by one in FIFO basis. All incoming customers demand the first 'essential' service wherein only some of them demand the second 'optional' service. The service times of both essential and optional services follow arbitrary(general)distribution with different vacation policies. In addition to this, due to the annoyance of seeing long queue in the system the customer may decide not to join the queue, called balking. Such a customer behavior is considered in both busy time and vacation time of the system. For this model, We obtain the time dependent solution and the corresponding steady state solutions. Also, we derive the performance measures, the mean queue size and the average waiting time explicitly.

Key Words : Batch Arrival, Single server, Essential, Optional, Balking, Bernoulli vacation, Transient state solution, Steady state Analysis

AMS Subject Classification : 60K25,60K30

1.Introduction

In queueing theory literature,the research study on queueing systems with server vacation has become an indispensable and interesting area . The utilization of idle time for doing some kind of secondary jobs is concerned with server vacation. Due to its significant impact in real situations, vacation queueing models has been modeled effectively in various situations such as production, banking service, communication systems, and computer networks etc. Many authors paid their interest in studying queueing models with various vacation policies including single and multiple vacation policies. Batch arrival queue with server vacations was investigated by Yechiali(1975). An excellent comprehensive studies on vacation models can be found in Takagi(1990) and Doshi(1986)research papers. One of the classical vacation model in queueing literature is Bernoulli scheduled server vacation. Keilson and Servi(1987) introduced and studied vacation scheme with Bernoulli schedule discipline. Madan(2001) discussed many queueing models with Bernoulli scheduled server vacation vacation. Baba(1986) employed the supplementary variable technique for deriving the transform solutions of waiting time for batch arrival with vacations.

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The research study of Queueing models with second optional service has a prominent place in queueing theory. The server performs first essential service to all arriving customers and after completing the first essential service, second optional service will be provided to some customers those who demand a second optional service. Madan(2000) has first introduced the concept of second optional service of an M/G/1 queueing system in which he has analyzed the time-dependent as well as the steady state behavior of the model by using supplementary variable technique. Medhi(2001)proposed an M/G/1 queueing model with second optional channel who developed the explicit expressions for the mean queue length and mean waiting time. Later Madhan(2002) studied second optional service by incorporating Bernoulli schedule server vacations. Gaudham Choudhury(2003) analyzed some aspects of M/G/1 queueing system with second optional service and obtained the steady state queue size distribution at the stationary point of time for general second optional service. A batch arrival with two phase service model with re-service for each phase of the service has been analyzed by Madan et al.(2004). Wang (2004) studied an M/G/1 queueing system with second optional service and server breakdowns based on supplementary variable technique. Kalyanaraman et al.(2008) studied additional optional batch service with vacation for single server queue.

In many real life situations, the arriving customers may be discouraged due to long queue, and decide not to join the queue and leave the system at once.This behavior of customers is referred as balking. Sometimes customers get impatient after joining the queue and leave the system without getting service. This behavior of customer recognizes as reneging.In the last few years, we see studies on queues with balking and reneging gaining significant importance.Queuing system with impatient customers is a familiar phenomenon we come across in many real life situations. Customers may be discouraged to join a queue due to long waiting time or service times or for other constraints and may leave the queue without joining and this is known as balking. We see applications of queue with balking in emergency services in hospitals dealing serious patients, communication systems, production and inventory system and many more. A queue with balking was initially studied by Haight (1957). Since then, extensive amount of work has been done on queuing systems related to impatient customers. Queues with balking has been studied by authors like Altman and Yechiali (2006), Ancker et al. (1963), Choudhury and Medhi (2011), in the last few years.

In this paper we consider queueing system such that the customers are arriving in batches according to Poisson stream. The server provides a first essential service to all incoming customers and a second optional service will be provided to only some of them those who demand it. Both the essential and optional service times are assumed to follow general distribution. Upon completion of a service, the server may remain in the system to serve the next customer (if any), with probability β_0 or he may proceed on the k^{th} vacation scheme with probability $\beta_k(1 \leq k \leq M)$ and $\sum_{k=1}^{M} = 1$. The vacation times are also assumed to be general whereas the repair time is exponentially distributed. Whenever the system meets a break down, it enters in to a repair process and the customer whose service is interrupted goes back to the head of the queue. Customers arrive in batches to the system and are served on a first come-first served basis. Also assuming that the batch arrival units may decide not to join the system (balks) by estimating the duration of waiting time for a service to get completed or by witnessing the long length of the queue and considering the Bernoulli vacation such that,the server may opt for vacation with probability p or with probability 1-p, remain stay back in the system to provide the service for the next customer.

The rest of the paper has been organized as follows: in section 2, the mathematical description of our model has been found, in section 3, the transient solution of the system has been derived, in section 4, the steady state analysis has been discussed.

2. Mathematical description of the queueing model

To describe the required queueing model, we assume the following.

∘ Let $\lambda c_i dt; i = 1, 2, 3, ...$ be the first order probability of arrival of 'i' customers in batches in the system during a short period of time(t,t+dt) where $0 \leq c_i \leq 1; \sum_{i=1}^{\infty} = 1, \lambda > 0$ is the mean arrival rate of batches.
∘ The single server provides the first essential service to all arriving customers. Let $B_1(v)$ and $b_1(v)$ be the distribution function and the density function of the first service times respectively.

∘ As soon as the first service of a customer is complete, the customer may opt for the second service with probability r, in which case his second service will immediately commence or else with probability 1-r he may opt to leave the system, in which case another customer at the head of the queue (if any) is taken up for his first essential service. The second service time is assumed to be general with the distribution function $B_2(v)$ and the density function $b_2(v)$.

∘ There is a single server which provides service following a general(arbitrary) distribution with distribution function $B_i(v)$and density function b(v). Let $\mu_i(x)dx$ be the conditional probability density function of service completionof i^{th} service during the interval (x,x+dx] given that the elapsed time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)} \tag{1}$$

and therefore

$$b_i(v) = \mu_i(v) e^{-\int_0^v \mu_i(x) dx} \tag{2}$$

∘ As soon as a service is completed, the server may take a vacation of random length with probability p or he may stay in the system providing service with probability (1-p).

∘ The vacation time of the server follows a general (arbitrary) distribution with distribution function V(s)and the density function v(s). Let $\nu_k(x)$ dx be the conditional probability of a completion of a vacation during the interval (x,x+dx] given that the elapsed vacation time is x so that

$$\nu_k(x) = \frac{v_k(x)}{1 - V_k(x)}; k = 1, 2, 3, ...M \tag{3}$$

and therefore

$$v_k(s) = \nu_k(s) e^{-\int_0^s \nu_k(x) dx}; k = 1, 2, 3, ...M \tag{4}$$

∘ We assume that $(1 - a_1)(0 \leq a_1 \leq 1)$ is the probability that an arriving customer balks during the period when the server is busy and $(1 - a_2)(0 \leq a_2 \leq 1)$ is the probability that an arriving customer balks during the period when the server is on vacation.

◦ The customers are served according to the first come -first served queue discipline.

◦ Various stochastic processes involved in the queueing system are assumed to be independent of each other.

3.Definitions and Equations governing the system

We let,

(i)$P_n^{(i)}(x,t)$ = probability that at time 't' the server is active providing i^{th} service and there are 'n' ($n \geq 1$) customers in the queue including the one being served and the elapsed service time for this customer is x. Consequently $P_n^i(t)$ denotes the probability that at time 't' there are 'n' customers in the queue excluding the one customer in i^{th} service irrespective of the value of x.

(ii)$V_n^{(k)}(x,t)$ = probability that at time 't', the server is on k^{th} vacation with elapsed vacation time x, and there are 'n' ($n \geq 0$) customers waiting in the queue for service. Consequently $V_n^{(k)}(t)$ denotes the probability that at time 't' there are 'n' customers in the queue and the server is on k^{th} vacation irrespective of the value of x.

(iii)Q(t)= probability that at time 't' there are no customers in the system and the server is idle but available in the system .

Transient solution of the queueing model

The model is then, governed by the following set of differential-difference equations.

$$\frac{\partial}{\partial t}P_n^{(1)}(x,t)+\frac{\partial}{\partial x}P_n^{(1)}(x,t)+(\lambda+\mu_1(x))P_n^{(1)}(x,t) = \lambda(1-a_1)P_n^{(1)}(x,t)+a_1\lambda\sum_{i=1}^{n}c_iP_{n-i}^{(1)}(x,t); \quad n \geq 1 \tag{5}$$

$$\frac{\partial}{\partial t}P_n^{(2)}(x,t)+\frac{\partial}{\partial x}P_n^{(2)}(x,t)+(\lambda+\mu_2(x))P_n^{(2)}(x,t) = \lambda(1-a_1)P_n^{(1)}(x,t)+a_1\lambda\sum_{i=1}^{n}c_iP_{n-i}^{(2)}(x,t); \quad n \geq 1 \tag{6}$$

$$\frac{\partial}{\partial t}V_n^{(k)}(x,t)+\frac{\partial}{\partial x}V_n^{(k)}(x,t)+(\lambda+\nu_k(x))V_n^{(k)}(x,t) = \lambda(1-a_2)V_n^{(k)}(x,t)+a_2\lambda\sum_{i=1}^{n}c_iV_{n-i}^{(k)}(x,t); n \geq 1 \tag{7}$$

$$\frac{\partial}{\partial t}V_0^{(k)}(x,t)+\frac{\partial}{\partial x}V_0^{(k)}(x,t)+(\lambda+\nu_k(x))V_0^{(k)}(x,t) = \lambda(1-a_2)V_0^{(k)}(x,t); k = 1,2,3,...M \tag{8}$$

$$\frac{d}{dt}Q(t) = \beta_0\left\{(1-r)\int_0^\infty P_1^{(1)}(x,t)\mu_1(x)dx + \int_0^\infty P_1^{(2)}(x,t)\mu_2(x)dx\right\} + \sum_{k=1}^{M}\int_0^\infty V_0^{(k)}(x,t)\nu_k(x)dx - \lambda Q(t) + \lambda(1-a_1)Q(t); k = 1,2,3,...M \tag{9}$$

The above equations are to be solved subject to the boundary conditions given below at x = 0

$$P_n^{(1)}(0,t) = \beta_0 \left\{ (1-r) \int_0^\infty P_{n+1}^{(1)}(x,t)\mu_1(x)dx + \int_0^\infty P_{n+1}^{(2)}(x,t)\mu_2(x)dx \right\}$$
$$+ \sum_{k=1}^{M} \int_0^\infty V_n^{(k)}(x,t)\nu(x)dx + \lambda a_1 c_n Q(t); k = 1,2,3,...M; n \geq 1 \quad (10)$$

$$P_n^{(2)}(0,t) = r \int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx \quad (11)$$

$$V_n^{(k)}(0,t) = \beta_k \left\{ (1-r) \int_0^\infty P_{n+1}^{(1)}(x,t)\mu_1(x)dx + \int_0^\infty P_{n+1}^{(2)}(x,t)\mu_2(x)dx \right\}; k = 1,2,3,...M; n \geq 0 \quad (12)$$

Assuming there are no customers in the system initially so that the server is idle.

$$V_0^{(k)}(0) = 0; V_n(0) = 0; Q(0) = 1; P_n(0) = 0, n = 1,2,3,...; k = 1,2,3,...M \quad (13)$$

4.The time dependent solution
Generating functions of the queue length

Define Laplace transform

$$\overline{f}(s) = \int_0^\infty f(t)e^{-st}dt \quad (14)$$

Taking Laplace transforms of equations (5) to (9)

$$\frac{\partial}{\partial x}\bar{P}_n^{(1)}(x,s) + (s + \lambda a_1 + \mu_1(x))\bar{P}_n^{(1)}(x,s) = a_1\lambda \sum_{i=1}^{n} c_i \bar{P}_{n-i}^{(1)}(x,s) n \geq 1 \quad (15)$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(2)}(x,s) + (s + \lambda a_1 + \mu_1(x))\bar{P}_n^{(2)}(x,s) = a_1\lambda \sum_{i=1}^{n} c_i \bar{P}_{n-i}^{(2)}(x,s) n \geq 1 \quad (16)$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(k)}(x,s) + (s + \lambda a_2 + \nu_k(x))\bar{V}_n^{(k)}(x,s) = a_2\lambda \sum_{i=1}^{n} c_i \bar{V}_{n-i}^{(k)}(x,s); k = 1,2,3,...M; n \geq 1 \quad (17)$$

$$\frac{\partial}{\partial x}\bar{V}_0^{(k)}(x,s) + (s + \lambda a_2 + \nu_k(x))\bar{V_0}^{(k)}(x,s) = 0; k = 1,2,3,...M \quad (18)$$

$$(s + \lambda a_1)\bar{Q}(s) = 1 + \beta_0 \left\{ (1-r) \int_0^\infty \bar{P}_1^{(1)}(x,s)\mu_1(x)dx + \int_0^\infty \bar{P}_1^{(2)}(x,s)\mu_2(x)dx \right\}$$
$$+ \sum_{k=1}^{M} \int_0^\infty \bar{V}_0^{(k)}(x,s)\nu_k(x)dx; k = 1,2,3,...M \quad (19)$$

for boundary conditions

$$\bar{P}_n^{(1)}(0,s) = \lambda a_1 c_n \bar{Q}(s) + \beta_0 \left\{ (1-r) \int_0^\infty \bar{P}_{n+1}^{(1)}(x,s)\mu_1(x)dx + \int_0^\infty \bar{P}_{n+1}^{(2)}(x,s)\mu_2(x)dx \right\}$$
$$+ \sum_{k=1}^{M} \int_0^\infty \bar{V}_n^{(k)}(x,s)\nu_k(x)dx; k = 1,2,3,...M \quad (20)$$

$$\bar{P}_n^{(2)}(0,s) = r \int_0^\infty \bar{P}_n^{(1)}(x,s)\mu_1(x)dx \quad (21)$$

$$\bar{V_n}^{(k)}(0,s) = \beta_k \left\{(1-r)\int_0^\infty \bar{P}_{n+1}^{(1)}(x,s)\mu_1(x)dx + \int_0^\infty \bar{P}_{n+1}^{(2)}(x,s)\mu_2(x)dx\right\}; k = 1,2,3,...M \tag{22}$$

We define the probability generating functions

$$P_q^{(i)}(x,z,t) = \sum_{n=0}^{\infty} z^n P_n^{(i)}(x,t); i = 1,2 \tag{23}$$

$$P_q^{(i)}(z,t) = \sum_{n=0}^{\infty} z^n P_n^{(i)}(t); i = 1,2 \tag{24}$$

$$V_q^{(k)}(x,z,t) = \sum_{n=0}^{\infty} z^n V_n^{(k)}(x,t); k = 1,2,3,...M \tag{25}$$

$$V_q^{(k)}(z,t) = \sum_{n=0}^{\infty} z^n V_n^{(k)}(t); k = 1,2,3,...M \tag{26}$$

$$C(z) = \sum_{n=1}^{\infty} c_n z^n \tag{27}$$

which are convergent inside the circle given by $|z| \leq 1$.

Now multiplying equation (15) and (16) by z^n respectively and take summation over all possible 'n'

$$\frac{\partial}{\partial x}\bar{P}_q^{(1)}(x,z,s) + (s + \lambda a_1(1 - C(z)) + \mu_1(x))\bar{P}_q^{(1)}(x,z,s) = 0 \tag{28}$$

$$\frac{\partial}{\partial x}\bar{P}_q^{(2)}(x,z,s) + (s + \lambda a_1(1 - C(z)) + \mu_2(x))\bar{P}_q^{(2)}(x,z,s) = 0 \tag{29}$$

Multiplying equation (17) by z^n and adding (18), we have

$$\frac{\partial}{\partial x}\bar{V_q}^{(k)}(x,z,s) + (s + \lambda a_2(1 - C(z)) + \nu(x))\bar{V_q}^{(k)}(x,z,s) = 0; k = 1,2,3,...M \tag{30}$$

for the boundary conditions, multiplying (20) by appropriate powers of z and taking summation over all possible values of 'n'

$$z\bar{P}_q^{(1)}(0,z,s) = (\beta_0(1-r)\int_0^\infty \bar{P}_q^{(1)}(x,z,s)\mu_1(x)dx + (1-p)\int_0^\infty \bar{P}_q^{(2)}(x,z,s)\mu_2(x)dx$$
$$+\sum_{k=1}^{M} z\int_0^\infty \bar{V}_q^{(k)}(x,z,s)\nu_k(x)dx + (1 - s\bar{Q}(s))z + \lambda a_1 z(C(z)-1)\bar{Q}(s) \tag{31}$$

$$\bar{P}_q^2(0,z,s) = r\int_0^\infty \bar{P}_q^{(1)}(x,z,s)\mu_1(x)dx \tag{32}$$

Multiplying equation (22) by appropriate powers of z and taking summation over 'n'

$$z\bar{V}_q^{(k)}(0,z,s) = \beta_k\left\{(1-r)\int_0^\infty \bar{P}_q^{(1)}(x,z,s)\mu_1(x)dx + \int_0^\infty \bar{P}_q^{(2)}(x,z,s)\mu_2(x)dx\right\} \tag{33}$$

solving equation (28), we get

$$\bar{P}_q^{(1)}(x,z,s) = \bar{P}_q^{(1)}(0,z,s)e^{-(s+a_1\lambda(1-C(z)))x-\int_0^x \mu_1(t)dt} \tag{34}$$

Similarly solving equation (29)and (30) respectively, we get

$$\bar{P}_q^{(2)}(x,z,s) = \bar{P}_q^{(2)}(0,z,s)e^{-(s+a_1\lambda(1-C(z)))x-\int_0^x \mu_2(t)dt} \tag{35}$$

$$\bar{V_q}^{(k)}(x,z,s) = \bar{V_q}^{(k)}(0,z,s)e^{-(s+a_2\lambda(1-C(z)))x-\int_0^x \nu_k(t)dt} \tag{36}$$

Again integrate equation (34)and (35) respectively, by parts with respect to x yields

$$\bar{P}_q^{(1)}(z,s) = \bar{P}_q^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1(s+a_1\lambda(1-C(z)))}{s+a_1\lambda(1-C(z))}\right] \tag{37}$$

$$\bar{P}_q^{(2)}(z,s) = \bar{P}_q^{(2)}(0,z,s)\left[\frac{1-\bar{B}_2(s+a_1\lambda(1-C(z)))}{s+a_1\lambda(1-C(z))}\right] \tag{38}$$

where

$$\bar{B}_1(s+a_1\lambda(1-C(z))) = \int_0^\infty e^{-(s+a_1\lambda(1-C(z)))x}dB_1(x) \tag{39}$$

$$\bar{B}_2(s+a_1\lambda(1-C(z))) = \int_0^\infty e^{-(s+a_1\lambda(1-C(z)))x}dB_2(x) \tag{40}$$

are Laplace - Stieltjes transforms of first essential service time $B_1(x)$ and second optional service time $B_2(x)$ respectively.

Again integrating equation (36) by parts with respect to x yields

$$\bar{V_q}^{(k)}(z,s) = \bar{V_q}^{(k)}(0,z,s)\left[\frac{1-\bar{V}_k(s+a_2\lambda(1-C(z)))}{s+a_2\lambda(1-C(z))}\right] \tag{41}$$

where

$$\bar{V}_k(s+a_2\lambda(1-C(z))) = \int_0^\infty e^{-(s+a_2\lambda(1-C(z)))x}dV(x) \tag{42}$$

is Laplace - Stieltjes transform of the k^{th} vacation time $V_k(x)$.

Now multiplying both sides of equation (34) by $\mu_1(x)$ and integrating over x, we get

$$\int_0^\infty \bar{P}_q^{(1)}(x,z,s)\mu_1(x)dx = \bar{P}_q^{(1)}(0,z,s)\bar{B}_1(s+a_1\lambda(1-C(z))) \tag{43}$$

multiplying both sides of equation (35) by $\mu_2(x)$ and integrating over x, we get

$$\int_0^\infty \bar{P}_q^{(2)}(x,z,s)\mu_2(x)dx = \bar{P}_q^{(2)}(0,z,s)\bar{B}_2(s+a_1\lambda(1-C(z))) \tag{44}$$

Now multiplying both sides of equation (36) by $\nu_k(x)$ and integrating over x, we get

$$\int_0^\infty \bar{V_q}^{(k)}(x,z,s)\nu(x)dx = \bar{V_q}^{(k)}(0,z,s)\bar{V}_k(s+a_2\lambda(1-C(z))) \tag{45}$$

Using (43),(44) and (45) in (31)

$$z\bar{P}_q^{(1)}(0,z,s) = [(1-s\bar{Q}(s))z + \lambda a_1 z(C(z)-1)\bar{Q}(s)] + \beta_0(1-r)\bar{P}_q^{(1)}(0,z,s)$$
$$\bar{B}_1(s+a_1\lambda(1-C(z))) + \beta_0\bar{P}_q^{(2)}(0,z,s)\bar{B}_2(s+a_1\lambda(1-C(z)))$$
$$+z\sum_{k=1}^{M}\bar{V}_q^{(k)}(0,z,s)\bar{V}_k(s+a_2\lambda(1-C(z))) \tag{46}$$

$$\bar{P}_q^{(2)}(0,z,s) = r\bar{P}_q^{(1)}(0,z,s)\bar{B}_1(s+a_1\lambda(1-C(z))) \tag{47}$$

Using (43) and (44) in (33)

$$z\bar{V}_q(0,z,s) = \beta_k\bar{P}_q^{(1)}(0,z,s)\bar{B}_1(s+a_1\lambda(1-C(z)))[(1-r)+r\bar{B}_2(s+a_1\lambda(1-C(z)))] \tag{48}$$

Solving (46) we get

$$\bar{P}_q^{(1)}(0,z,s) = \frac{[(1-s\bar{Q}(s))z + \lambda a_1 z(C(z)-1)\bar{Q}(s)]}{Dr} \tag{49}$$

where Dr is given by

$$\text{Dr} = z - \Big\{[\beta_0 + \textstyle\sum_{k=1}^{M}\beta_k\bar{V}_k(s+a_2\lambda(1-C(z)))]$$
$$(1-r)\bar{B}_1(s+a_1\lambda(1-C(z))) + r\bar{B}_1(s+a_1\lambda(1-C(z)))\bar{B}_2(s+a_1\lambda(1-C(z)))\Big\} \tag{50}$$

$$\bar{P}_q^{(2)}(0,z,s) = r\frac{[(1-s\bar{Q}(s))z + \lambda a_1 z(C(z)-1)\bar{Q}(s)]\bar{B}_1(s+a_1\lambda(1-C(z)))}{Dr} \tag{51}$$

Using (49) in (48) we get

$$\bar{V}_q(0,z,s) = \frac{\beta_k[(1-s\bar{Q}(s)) + \lambda a_1(C(z)-1)\bar{Q}(s)](1-r)\bar{B}_1(s+a_1\lambda(1-C(z))) + r\bar{B}_1(s+a_1\lambda(1-C(z)))\bar{B}_2(s+a_1\lambda(1-C(z)))}{Dr} \tag{52}$$

Substituting (49) and (51) in (37) and (38) respectively.

$$\bar{P}_q^{(1)}(z,s) = \frac{[(1-s\bar{Q}(s))z + \lambda a_1 z(C(z)-1)\bar{Q}(s)]\left[\frac{1-\bar{B}_1(s+a_1\lambda(1-C(z)))}{s+a_1\lambda(1-C(z))}\right]}{Dr} \tag{53}$$

$$\bar{P}_q^{(2)}(z,s) = \frac{r[(1-s\bar{Q}(s))z + \lambda a_1 z(C(z)-1)\bar{Q}(s)]\bar{B}_1(s+a_1\lambda(1-C(z)))\left[\frac{1-\bar{B}_2(s+a_1\lambda(1-C(z)))}{s+a_1\lambda(1-C(z))}\right]}{Dr} \tag{54}$$

Also sub equation (52) in equation (41)

$$\bar{V}_q^{(k)}(z,s) = \frac{\beta_k[(s\bar{Q}(s)-1) + \lambda a_1(1-C(z))\bar{Q}(s)][(1-r)\bar{B}_1(s+a_1\lambda(1-C(z))) + r\bar{B}_1(s+a_1\lambda(1-C(z)))\bar{B}_2(s+a_1\lambda(1-C(z)))]\left[\frac{1-\bar{V}_k(s+a_2\lambda(1-C(z)))}{s+a_2\lambda(1-C(z))}\right]}{Dr} \tag{55}$$

where Dr is given by in the above.

5.The Steady State Analysis

In this section we shall derive the steady state probability distribution for our queueing model. To define the steady state probabilities, suppress the argument't' where ever it appears in the time dependent analysis.

By using well known Tauberian property

$$Lt_{s\to 0}s\bar{f}(s) = Lt_{t\to\infty}f(t) \tag{56}$$

$$P_q^{(1)}(z) = \frac{z[1-\bar{B}_1(a_1\lambda(1-C(z)))]Q}{D(z)} \tag{57}$$

$$P_q^{(2)}(z) = \frac{zr\bar{B}_1(a_1\lambda(1-C(z)))[1-\bar{B}_2(a_1\lambda(1-C(z)))]Q}{D(z)} \tag{58}$$

$$V_q^{(k)}(z) = \frac{\beta_k(\frac{a_1}{a_2})[(1-r)\bar{B}_1(a_1\lambda(1-C(z))) + r\bar{B}_1(a_1\lambda(1-C(z)))\bar{B}_2(a_1\lambda(1-C(z)))]Q[1-\bar{V}_k(a_2\lambda(1-C(z)))]}{D(z)} \tag{59}$$

where D(z) is given by

$$z-[\beta_0+\sum_{k=1}^{M}\beta_k\bar{V}_k(a_2\lambda(1-C(z)))](1-r)\bar{B}_1(a_1\lambda(1-C(z))) + r\bar{B}_1(a_1\lambda(1-C(z)))\bar{B}_2(a_1\lambda(1-C(z))) \tag{60}$$

Let $W_q(z)$ denotes the probability generating function of queue size irrespective of the state of the system.

$$W_q(z) = P_q^{(1)}(z) + P_q^{(2)}(z) + \sum_{k=1}^{M} V_q^{(k)}(z) \tag{61}$$

In order to find the unknown Q , using the normalizing condition

$$W_q(1) + Q = 1 \tag{62}$$

$$W_q(1) = \frac{\lambda a_1 E(I)Q[E(S_1) + rE(S_2) + \sum_{k=1}^{M}\beta_k E(V_k)]}{dr} \tag{63}$$

$$dr = 1 - \lambda a_1[E(S_1) + rE(S_2)] - \lambda a_2 \sum_{k=1}^{M}\beta_k E(V_k) \tag{64}$$

$$Q = \frac{1-\lambda a_1 E(I)[E(S_1)+rE(S_2)] - \lambda a_2 E(I)\sum_{k=1}^{M}E(V_k)}{1+\lambda(a_1-a_2)E(I)\sum_{k=1}^{M}E(V_k)} \tag{65}$$

$$\rho = 1 - Q \tag{66}$$

where $\rho < 1$ is the stability condition under which the steady state exists, equation(65) gives the probability that the server is idle. Substitute Q from equation(65) in equation (61), $W_q(z)$ have been completely and explicitly determined which is the the probability generating function of the queue size

The Average Queue Size

Let L_q denote the mean number of customers in the queue under the steady state, then

$L_q = \frac{d}{dz} W_q(z) \mid_{z=1}$

since this formula gives 0/0 form, then we write $W_q(z) = \frac{N(z)}{D(z)}$

where N(z) and D(z) are the numerator and denominator of the right hand side of equation (61) respectively, then we use

$$L_q = \frac{D'(1)N''(1) - N'(1)D''(1)}{2[D'(1)]^2} \tag{67}$$

where

$$N^{'}(1) = Q\lambda a_1 E(I) \left\{ [E(S_1) + rE(S_2)] + \sum_{k=1}^{M} \beta_k E(V_k) \right\} \tag{68}$$

$$N^{''}(1) = Q \Big\{ (\lambda a_1 E(I))^2 [E(S_1)^2 + rE(S_2)^2] + 2rE(S_1)E(S_2) + \textstyle\sum_{k=1}^{M} \beta_k E(V_k) E(S_1) + rE(S_2)$$
$$(\lambda E(I) a_1)^2 [E(S_1^2) + rE(S_2^2) + rE(S_1)E(S_2)] + (\lambda E(I)) {a_2}^2 \textstyle\sum_{k=1}^{M} \beta_k E(V_k^2)$$
$$+\lambda a_1 E(I(I-1))[E(S_1) + rE(S_2) + \sum_{k=1}^{M} \beta_k E(V_k)] \Big\} \tag{69}$$

$$D^{'}(1) = \lambda a_1 E(I)[E(S_1) + rE(S_2)] + \lambda a_2 E(I) \sum_{k=1}^{M} \beta_k E(V_k) - 1 \tag{70}$$

$$D^{''}(1) = (\lambda a_1 E(I))^2 \left[E(S_1^2) + r(E(S_2)^2) \right]$$
$$+\lambda a_1 E(I(I-1))[E(S_1) + rE(S_2)] + \lambda a_2 E(I(I-1)) \textstyle\sum_{k=1}^{M} \beta_k E(V_k)$$
$$+2(\lambda a_1 E(I))^2 rE(S_1)E(S_2)(\lambda E(I))(a_2)^2 \textstyle\sum_{k=1}^{M} \beta_k E(V_k^2)$$
$$+2\lambda^2 a_1 a_2 (E(I)^2)[E(S_1) + rE(S_2)] \sum_{k=1}^{M} \beta_k E(V_k) \tag{71}$$

where $E(V^2)$ is the second moment of the vacation time and Q has been found in equation (65). Then if we substitute the values of $N'(1), N''(1), D'(1)$ and $D''(1)$ from equations (68), (69), (70) and (71) in to (67) equation we obtain L_q in a closed form.

Mean waiting time of a customer could be found, as follows

$$W_q = \frac{L_q}{\lambda} \tag{72}$$

by using Little's formula.

Conclusion

This paper clearly analyzes the transient solution of batch arrival single server with second optional service such that first essential service for all incoming customer where as few of them require a second optional service and multi optional vacation. Also balking, a customer impatience is discussed. This queueing model can be utilized in large scale manufacturing industries and communication networks.

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