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New Bilateral Generating Functions Pertaining to I-Function

V.B.L. Chaurasi[1], R.C. Meghwal[2]

[1] Department of Mathematics, University of Rajasthan, Jaipur-302004 (Rajasthan), India
[2] Department of Mathematics, Government Post-Graduate College, Neemuch-458441 (MP) India

***Abstract*- In** this work we derive the bilateral generating relations, pertaining to the product of V.P. Saxena's I-function [3] and the multivariable H-function of Srivastava and Panda [6]. By suitably specializing the various parameters involved, this formula would yield the corresponding bilateral generating functions for a variety of simpler special functions.
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Index Terms- I-function; Fox H-function; H-function of several complex variables; Lauricella function.

I. Introduction

Let $\Delta(\xi,\eta)$ and $\nabla(\xi,\eta)$ stands for the ξ–parameter sequence $\frac{\eta}{\xi}.\frac{\eta+1}{\xi},...,\frac{\eta+\xi+1}{\xi}$ and $1-\frac{\eta}{\xi},1-\frac{\eta+1}{\xi},...,1-\frac{\eta+\xi+1}{\xi}$ respectively for an arbitrary complex number and for all integer $\xi \geq 1$.

1.1 I-function

The I-function, which is more general than the Fox's H-function, defined by V.P. Saxena [3], by means of the following Mellin-Barnes type contour integral.

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$$I(z) = I^{M,N}_{P_i,Q_i:R}[z] \, I^{M,N}_{P_i,Q_i:R}\left[z \,\middle|\, \begin{matrix} (a_j,\alpha_j)_{1,N},(a_{j_i},\alpha_{j_i})_{N+1,P_i} \\ (b_j,\beta_j)_{1,M},(b_{j_i},\beta_{j_i})_{M+1,Q_i} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)} \int_L \theta(s) z^s \, ds, \qquad \text{…(1.1)}$$

where

$$\theta(s) = \frac{\prod_{j=1}^{M} \Gamma(b_j - \beta_j s) \prod_{j=1}^{N} \Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{R} \left\{ \prod_{j=M+1}^{Q_i} \Gamma(1-b_j+\beta_j s) \prod_{j=N+1}^{P_i} \Gamma(a_{j_i} - \alpha_{j_i} s) \right\}}. \qquad \text{…(1.2)}$$

$P_i, Q_i \, (i=1,..., r)$, M,N are integers satisfying $0 \leq N \leq P_i . 0 \leq M \leq Q_i ; \alpha_j, \beta_j, \alpha_{j_i}, \beta_{j_i}$ are real and positive and $a_j, b_j, a_{j_i}, b_{j_i}$ are complex numbers, L is a suitable contour of the Mellin-Barnes type running from $\gamma - i\alpha$ to $\gamma + i\alpha (\gamma$ is real) in the complex –s plane. Details regarding existence conditions and various parametric restrictions of I-function, we may refer [3]. For R = 1 (1.1) reduce to the Fox's H-function [2].

$$I^{M,N}_{P_i,Q_i:1}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,N};(a_{j_i},\alpha_{j_i})_{N+1,P_i}\\(b_j,\beta_j)_{1,M};(b_{j_i},\beta_{j_i})_{M+1,Q_i}\end{matrix}\right.\right]=H^{M,N}_{P,Q}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,N};(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M};(b_j,\beta_j)_{M+1,Q}\end{matrix}\right.\right] \quad \ldots(1.3)$$

1.2 The Multivariable H-function

The multivariable H-function introduced and studied by Srivastava and Panda [6], occurring in this paper will be defined and represented in the following manner:

$$H^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[b':\phi'];\ldots;[b^{(r)}:\phi^{(r)}];\\ [(c):\psi',\ldots,\psi^{(r)}]:[d',\delta'];\ldots;[d^{(r)}:\delta^{(r)}];\end{matrix}\ x_1;\ldots;x_r\right]$$

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$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\ldots\int_{L_r}R_1(s_1)\ldots R_r(s_r)T(s_1,\ldots,s_r)x_1^{s_1}\ldots x_r^{s_r}\,ds_1\ldots ds_r;\omega=\sqrt{-1}, \quad \ldots(1.4)$$

where

$$R_i(s_i)=\frac{\prod_{j=1}^{u^{(i)}}\Gamma(d_j^{(i)}-\delta_j^{(i)}s_i)\prod_{j=1}^{v^{(i)}}\Gamma(1-b_j^{(i)}+\phi_j^{(i)}s_i)}{\prod_{j=u^{(i)}+1}^{D^{(i)}}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod_{j=v^{(i)}+1}^{B^{(i)}}\Gamma(b_j^{(i)}-\phi_j^{(i)}s_i)},\forall i=1,\ldots,r \quad \ldots(1.5)$$

and

$$T(s_1,\ldots,s_r)=\frac{\prod_{j=1}^{\lambda}\Gamma\left(1-a_j+\sum_{i=1}^{r}\theta_j^{(i)}s_i\right)}{\prod_{j=\lambda+1}^{A}\Gamma\left(a_j-\sum_{i=1}^{r}\theta_j^{(i)}s_i\right)\prod_{j=1}^{C}\Gamma\left(1-c_j+\sum_{i=1}^{r}\psi_j^{(i)}s_i\right)} \quad \ldots(1.6)$$

For more details of the multivariable H-function, we refer to the book Srivastava et al. [5].

II. Main Result

Formula 2.1. The bilateral generating function formula which holds true whenever both of its sides have meanings

$$\sum_{\eta=0}^{\infty}I^{M,N+\xi}_{P_i+\xi,Q_i+\xi:R}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,N},(\nabla(\xi,-\eta),1)_{1,\xi},(a_{j_i},\alpha_{j_i})_{N+1,P_i}\\(b_j,\beta_j)_{1,Q_i},(\nabla(\xi,-\eta),0)_{1,\xi}\end{matrix}\right.\right]$$

$$\times H^{0,\lambda+\xi:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+\xi,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left[y_1,\ldots,y_r\left|\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]_{1,\lambda};(\nabla(\xi,\sigma+\eta):1,\ldots,1)_{1,\xi};[(a):\theta',\ldots,\theta^{(r)}]_{\lambda+1,A}:\\ [(c):\psi',\ldots,\psi^{(r)}]_{1,C};\text{------------};\text{------------}:\end{matrix}\right.\right.$$

$$\left.\begin{matrix}[b':\phi']_{1,B'};\ldots;[b^{(r)}:\phi^{(r)}]_{1,B^{(r)}}\\ [d':\delta']_{1,D'};\ldots;[d^{(r)}:\delta^{(r)}]_{1,D^{(r)}}\end{matrix}\right]\frac{z^{\eta}}{\eta!}$$

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$$= \left(1-\frac{z}{\xi}\right)^{-\sigma} \sum_{i=1}^{R} H^{0,\lambda+\xi:\,(u',v');\ldots;(u^{(r)},v^{(r)});(M,N)}_{A+\xi,C:[B',D'];\ldots;[B^{(r)}:D^{(r)}];[P_i,Q_i]}$$

$$\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]_{1,\lambda};(\nabla(\xi,\sigma):1,\ldots,1)_{1,\xi};[(a):\theta',\ldots,\theta^{(r)}]_{\lambda+1,A}: \\ [(c):\psi',\ldots,\psi^{(r)}]_{1,c};\text{------------};\text{------------}:\end{matrix}\right.$$

$$\left.\begin{matrix}[b':\phi']_{1,B'};\ldots;[b^{(r)}:\phi^{(r)}]_{1,B^{(r)}};(a_j,\alpha_{j_i})_{1,P_i}; \\ [d':\delta']_{1,D'};\ldots;[d^{(r)}:\delta^{(r)}]_{1,D^{(r)}};(b_j,\beta_{j_i})_{1,Q_i};\end{matrix}\; y_1\left(1-\frac{z}{\xi}\right)^{-\xi},\ldots,y_r\left(1-\frac{z}{\xi}\right)^{-\xi}, z\left(1-\frac{z}{\xi}\right)^{-\xi}\right], \qquad \ldots(2.1)$$

where σ is an arbitrary complex number, ξ is an integer ≥ 1.

Proof. To obtain (2.1), express the multivariable H-function occurring left hand side of (2.1) with the help of (1.4) and then interchange the order of summation and integration, we find that left-hand side of (2.1)

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\ldots\int_{L_r} R_1(s_1)\ldots R_r(s_r)\,T(s_1,\ldots,s_r)\prod_{j=1}^{\xi}\Gamma\left(\Delta(\xi,\sigma)+\sum_{i=1}^{r}s_i\right)$$

$$\left\{\sum_{\eta=0}^{\infty}\frac{\left(\sigma+\xi\sum_{i=1}^{r}s_i\right)_\eta}{\eta!}\left(\frac{z}{\xi}\right)^{\eta} I^{M,N+\xi}_{P_i+\xi,Q_i+\xi:R}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,N},(\nabla(\xi,-\eta),1)_{1,\xi},(a_{j_i},\alpha_{j_i})_{N+1,P_i} \\ (b_j,\beta_j)_{1,Q_i},(\nabla(\xi,-\eta),0)_{1,\xi}\end{matrix}\right.\right]\right\}$$

$$y_1^{s_1}\ldots y_r^{s_r}\,ds_1\ldots ds_r \qquad \ldots(2.2)$$

Now applying the following formula:

$$\sum_{\eta=0}^{\infty}\frac{(\sigma)_\eta}{\eta!} I^{M,N+\xi}_{P_i+\xi,Q_i+\xi:R}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,N},(\nabla(\xi,-\eta),1)_{1,\xi},(a_{j_i},\alpha_{j_i})_{N+1,P_i} \\ (b_j,\beta_j)_{1,Q_i},(\nabla(\xi,-\eta),0)_{1,\xi}\end{matrix}\right.\right]t^{\eta}$$

$$= (1-t)^{-\sigma}\, I^{M,N+\xi}_{P_i+\xi,Q_i+\xi:R}\left[z\left(\frac{t}{t-1}\right)^{\xi}\left|\begin{matrix}(a_j,\alpha_j)_{1,N},(\nabla(\xi,\sigma),1)_{1,\xi},(a_j,\alpha_j)_{N+1,P_i} \\ \text{------------},\text{------------},(b_j,\beta_j)_{1,Q_i},(\nabla(\xi,\sigma),0)_{1,\xi}\end{matrix}\right.\right] \qquad \ldots(2.3)$$

New bilateral generating functions pertaining to I-function and then replace the I-function by its Mellin-Barnes contour integral, given by (1.1) and interpret the resulting Mellin-Barnes contour integral in terms of H-function of r-variables by means of (1.4). Now we approach to the right hand side of (2.1), and the final result follows by pondering the principal of analytic continuation.

III. Special Cases

3.1: On putting R = 1, (2.1) reduce to the following known result derived by Chaurasia and Kumawat [1].

$$\sum_{\eta=0}^{\infty} H^{M,N+\xi}_{P+\xi,Q+\xi}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,N},(\nabla(\xi,-\eta),1)_{1,\xi},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,Q},(\nabla(\xi,-\eta),0)_{1,\xi}\end{matrix}\right.\right]$$

$$\times H^{0,\lambda+\xi\,:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+\xi,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left[y_1,\ldots,y_r\left|\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]_{1,\lambda};(\nabla(\xi,\sigma+\eta):1,\ldots,1)_{1,\xi};[(a):\theta',\ldots,\theta^{(r)}]_{\lambda+1,A}: \\ [(c):\psi',\ldots,\psi^{(r)}]_{1,c};\text{------------};\text{------------}:\end{matrix}\right.\right.$$

$$\left.\begin{matrix}[b':\phi']_{1,B'};\ldots;[b^{(r)}:\phi^{(r)}]_{1,B^{(r)}} \\ [d':\delta']_{1,D'};\ldots;[d^{(r)}:\delta^{(r)}]_{1,D^{(r)}}\end{matrix}\right]\frac{z^{\eta}}{\eta!}$$

$$=\left(1-\frac{z}{\xi}\right)^{-\sigma} H^{0,\lambda+\xi:\ (u',v')\ ;\ldots;(u^{(r)},v^{(r)})\ ;(M,N)}_{A+\xi,C:[B',D'];\ldots;[B^{(r)}:D^{(r)}];[P,Q]}$$

$$\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]_{1,\lambda}\ ;(\nabla(\xi,\sigma):1,\ldots,1)_{1,\xi};[(a):\theta',\ldots,\theta^{(r)}]_{\lambda+1,A}: \\ [(c):\psi',\ldots,\psi^{(r)}]_{1,c};\text{--------------};\text{--------------}:\end{matrix}\right.$$

$$\left.\begin{matrix}[b':\phi']_{1,B'};\ldots;[b^{(r)}:\phi^{(r)}]_{1,B^{(r)}};(a_j,\alpha_j)_{1,P}; \\ [d':\delta']_{1,D'};\ldots;[d^{(r)}:\delta^{(r)}]_{1,D^{(r)}};(b_j,\beta_{j_i})_{1,Q};\end{matrix}\ y_1\left(1-\frac{z}{\xi}\right)^{-\xi},\ldots,y_r\left(1-\frac{z}{\xi}\right)^{-\xi},z\left(1-\frac{z}{\xi}\right)^{-\xi}\right]. \qquad \ldots(3.1)$$

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3.2 Again on taking $\lambda = A, U^{(i)} = 1, V^{(i)} = B^{(i)}, D^{(i)} = D^{(i)} + 1 \forall\, i = 1,\ldots, r,$ the result (2.1) reduced in terms of generalized Lauricella function of several complex variable [4]

$$\sum_{\eta=0}^{\infty}\left\{\prod_{j=1}^{\xi}\Gamma(\Delta(\xi,\sigma+\eta))\right\} I^{M,N+\xi}_{P_i+\xi,Q_i+\xi:R}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,N},(\nabla(\xi,-\eta),1)_{1,\xi},(a_{j_i},\alpha_{j_i})_{N+1,P_i} \\ (b_j,\beta_j)_{1,Q_i},(\nabla(\xi,-\eta),0)_{1,\xi}\end{matrix}\right.\right]$$

$$\times F^{A+\xi:B',\ldots,B^{(r)}}_{C:D',\ldots,D^{(r)}}\left[-y_1,\ldots,-y_r\left|\begin{matrix}[(1-a_j:\theta',\ldots,\theta^{(r)}]_{1,A}\ ;(\nabla(\xi,\sigma+\eta):1,\ldots,1)_{1,\xi}; \\ (1-c_j:\psi',\ldots,\psi^{(r)})_{A,C}:\text{--------------};\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(1-b':\phi')_{1,B'};\ldots;(1-b^{(r)}:\phi^{(r)})_{1,B^{(r)}} \\ (1-d':\delta')_{1,D'};\ldots;(1-d^{(r)}:\delta^{(r)})_{1,D^{(r)}}\end{matrix}\right]\frac{z^{\eta}}{\eta!}$$

$$=\left(1-\frac{z}{\xi}\right)^{-\sigma}\sum_{i=1}^{R}\left\{\frac{\prod_{j=1}^{\xi}\Gamma(\Delta(\xi,\sigma))\prod_{j=1}^{P_i}\Gamma(1-a_j)}{\prod_{j=1}^{Q_i}\Gamma(1-b_j)}\right\}$$

$$F^{A+\xi:B',\ldots,B^{(r)},P_i}_{C:D',\ldots,D^{(r)},Q_i}\left[\begin{matrix}[(1-a_j):\theta',\ldots,\theta^{(r)}]_{1,A}\ ;(\nabla(\xi,\sigma):1,\ldots,1)_{1,\xi}: \\ [(1-c_j):\psi',\ldots,\psi^{(r)}]_{1,c};\text{--------------}:\end{matrix}\right.$$

$$\left.\begin{matrix} (1-b':\phi')_{1,B'};\dots;(1-b^{(r)}:\phi^{(r)})_{1,B^{(r)}};(1-a_j,\alpha_j)_{1,P_i}; \\ (1-d':\delta')_{1,D'};\dots;(1-d^{(r)}:\delta^{(r)}]_{1,D^{(r)}};(1-b_j,\beta_j)_{1,Q_i}; \end{matrix}\; -y_1\left(1-\frac{z}{\xi}\right)^{-\xi},\dots,-y_r\left(1-\frac{z}{\xi}\right)^{-\xi},-z\left(1-\frac{z}{\xi}\right)^{-\xi}\right], \qquad \dots(3.2)$$

3.3: If we set R = 1 in (3.2), then it yields the result derived by Chaurasia and Kumawat [1] as :

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$$\sum_{\eta=0}^{\infty}\left\{\prod_{j=1}^{\xi}\Gamma(\Delta(\xi,\sigma\,|\,+\eta))\right\}H^{M,N+\xi}_{P+\xi,Q+\xi}\left[z\,\middle|\,\begin{matrix}(a_j,\alpha_j)_{1,N},(\nabla(\xi,-\eta),1)_{1,\xi},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,Q_1},(\nabla(\xi,-\eta),0)_{1,\xi}\end{matrix}\right]$$

$$\times F^{A+\xi:B',\dots,B^{(r)}}_{C:D',\dots,D^{(r)}}\left[-y_1,\dots,-y_r\,\middle|\,\begin{matrix}[(1-a_j:\theta',\dots,\theta^{(r)}]_{1,A}\;;(\nabla(\xi,\sigma+\eta):1,\dots,1)_{1,\xi};\\(1-c_j:\psi',\dots,\psi^{(r)})_{1,C}\;:\text{-----------------};\end{matrix}\right.$$

$$\left.\begin{matrix}(1-b':\phi')_{1,B'};\dots;(1-b^{(r)}:\phi^{(r)})_{1,B^{(r)}}\\(1-d':\delta')_{1,D'};\dots;(1-d^{(r)}:\delta^{(r)})_{1,D^{(r)}}\end{matrix}\right]\frac{z^{\eta}}{\eta!}$$

$$=\left(1-\frac{z}{\xi}\right)^{-\sigma}\left\{\frac{\prod_{j=1}^{\xi}\Gamma(\Delta(\xi,\sigma))\prod_{j=1}^{P}\Gamma(1-a_j)}{\prod_{j=1}^{Q_i}\Gamma(1-b_j)}\right\}$$

$$F^{A+\xi:B',\dots,B^{(r)},P}_{C:D',\dots,D^{(r)},Q}\left[\begin{matrix}[(1-a_j):\theta',\dots,\theta^{(r)}]_{1,A}\;;(\nabla(\xi,\sigma):1,\dots,1)_{1,\xi}:\\ [(1-c_j):\psi',\dots,\psi^{(r)}]_{1,c};\text{-----------------}:\end{matrix}\right.$$

$$\left.\begin{matrix}(1-b':\phi')_{1,B'};\dots;(1-b^{(r)}:\phi^{(r)})_{1,B^{(r)}};(1-a_j,\alpha_j)_{1,P};\\(1-d':\delta')_{1,D'};\dots;(1-d^{(r)}:\delta^{(r)}]_{1,D^{(r)}};(1-b_j,\beta_j)_{1,Q};\end{matrix}\;-y_1\left(1-\frac{z}{\xi}\right)^{-\xi},\dots,-y_r\left(1-\frac{z}{\xi}\right)^{-\xi},-z\left(1-\frac{z}{\xi}\right)^{-\xi}\right], \qquad \dots(3.3)$$

3.4: The I-function, presented in this paper, is quite basic in nature. Therefore, on specializing the parameters of this function, we may obtain various other result as its special cases.

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AUTHORS

First Author – V.B.L. Chaurasia, Department of Mathematics, University of Rajasthan, Jaipur-302004 (Rajasthan), India, E-mail: drvblc@yahoo.com

Second Author – R.C. Meghwal, Department of Mathematics, Government Post-Graduate College, Neemuch-458441 (MP) India, E-mail: meghwal66@gmail.com