A linear operator of a new class of multivalent harmonic functions

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Abstract- New classes of multivalently functions with a linear operator are introduced . We give sufficient coefficient bounds for $f(z) \in KM_p(K,\beta,q)$ and then we show that these sufficient coefficient conditions are also necessary for $f(z) \in AM_p(K,\beta,q)$. Furthermore, we determine extreme points, convex combination, convolution property and integral operator for these functions. Also we obtain new results in this paper.

Index Terms- Multivalent harmonic functions, Coefficient bounds, Extreme points, Convex combination, Integral operator.

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I. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $D \subset \mathbb{C}$,we can write $f = h + \bar{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f . A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D, see Clunie and Sheil-Small [5].
Denote by $M(p)$ the class of functions $f = h + \bar{g}$ that are harmonic multivalent and sense-preserving in the unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$. The class $M(p)$ was studied by Ahuja and Jahangiri [1] and class $M(p)$ for $p = 1$ was defined and studied by Jahangiri et. al. in [6].
For $f = h + \bar{g} \in M(p)$, we may express the analytic functions h and g as:

$$h(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad , \qquad g(z) = \sum_{n=p}^{\infty} b_n z^n \ , |b_p| < 1 . \tag{1}$$

Let W_p denote the subclass of $M(p)$ consisting of functions $f = h + \bar{g}$, where h and g are given by:

$$h(z) = z^p - \sum_{n=p+1}^{\infty} |a_n| z^n \quad , \qquad g(z) = -\sum_{n=p}^{\infty} |b_n| z^n \ , |b_p| < 1 . \tag{2}$$

Now, we define anew class $KM_p(K,\beta,q)$ of harmonic functions of the form (1) that satisfy the inequality

$$Re\left\{\frac{K\left[\left(D_p(\lambda,\beta,\gamma)f(z)\right)^q + z^2\left(D_p(\lambda,\beta,\gamma)f(z)\right)^{q+2}\right]}{z\left(D_p(\lambda,\beta,\gamma)f(z)\right)^{q+1}}\right\} > \beta, \tag{3}$$

where $0 \leq \beta < \frac{1}{p}, p > q, p \in \mathbb{N} = \{1,2,\ldots\}, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, 0 \leq K \leq 1, \lambda \geq 0, \gamma \geq 0,$

$$f^q(z) = \delta(p,q) z^{p-q} + \sum_{n=1}^{\infty} \delta(n,q) a_n z^{n-q},$$

$$\delta(i,j) = \frac{i!}{(i-j)!} = \begin{cases} 1 & j = 0 \\ i(i-1)\ldots(i-j+1) & j \neq 0 \end{cases},$$

and

$$\left(D_p(\lambda,\beta,\gamma)f(z)\right) = \left(D_p(\lambda,\beta,\gamma)h(z)\right) + \overline{\left(D_p(\lambda,\beta,\gamma)g(z)\right)} . \tag{4}$$

The operator $D_p(\lambda,\beta,\gamma)$ denotes the linear operator introduced in [8] . For h and g given by (1), we obtain

$$D_p(\lambda,\beta,\gamma)h(z) = z^p + \sum_{n=p+1}^{\infty}\left[1 + \frac{(n-p)\lambda}{p+\gamma}\right]^{\eta} a_n z^n, \tag{5}$$

$$D_p(\lambda,\beta,\gamma)g(z) = \sum_{n=p}^{\infty}\left[1 + \frac{(n-p)\lambda}{p+\gamma}\right]^{\eta} b_n z^n, \tag{6}$$

where $p \in \mathbb{N} = \{1,2,\dots\}, \lambda \geq 0, \gamma \geq 0, \eta \geq 0$.
We further denote by $AM_p(K,\beta,\gamma)$ the subclass of $KM_p(K,\beta,\gamma)$ that satisfies the relation

$$AM_p(K,\beta,\gamma) = AM_p \cap KM_p(K,\beta,\gamma). \quad (7)$$

II. Coefficient Inequality

We need the following lemma in our results:
Lemma 1[2]: $\mathrm{Re}\{w(z)\} > \beta \text{ if and only if } |w(z) - (1+\beta)| \leq |w(z) + (1-\beta)|$.
In the following theorem, we find a coefficient inequality for functions in the class $KM_p(K,\beta,\gamma)$.
Theorem 1: Let $f = h + \bar{g}$ (h and g being given by (1)) . If

$$\sum_{n=p+1}^{\infty} \delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|a_n|$$
$$\sum_{n=p}^{\infty} \delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|b_n| \leq \delta(p,q), \quad (8)$$

where $0 \leq \beta < \frac{1}{p}, p > q, p \in \mathbb{N} = \{1,2,\dots\}, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, 0 \leq K \leq 1, \eta \geq 0, \lambda \geq 0, \gamma \geq 0$, then f is harmonic p-valent sense-preserving in U and $f \in KM_p(K,\beta,\gamma)$.
Proof : Let

$$w(z) = \left\{\frac{K\left[\left(D_p(\lambda,\beta,\gamma)f(z)\right)^{q} + z^2\left(D_p(\lambda,\beta,\gamma)f(z)\right)^{q+2}\right]}{z\left(D_p(\lambda,\beta,\gamma)f(z)\right)^{q+1}}\right\}.$$

Using the fact in Lemma (1)
$\mathrm{Re}\{w(z)\} > \beta \text{ if and only if } |w(z) - (1+\beta)| \leq$
$|w(z) + (1-\beta)|$. (9)
Substituting for w and resorting to simple calculations, we find that

$$|w(z) - (1+\beta)| \leq \delta(p,q)\big[(p-q)\big((p-q-1)K-(1+\beta)\big)+K\big]|z|^{p-q}$$
$$+\sum_{n=p+1}^{\infty} \delta(n,q)\big[(n-q)\big((n-q-1)K-(1+\beta)\big)+K\big]|a_n||z|^{n-q}$$
$$+\sum_{n=p}^{\infty} \delta(n,q)\big[(n-q)\big((n-q-1)K-(1+\beta)\big)+K\big]|b_n||z|^{n-q}, \quad (10)$$

and

$$|w(z) + (1-\beta)| \geq \delta(p,q)\big[(p-q)\big((p-q-1)K+(1-\beta)\big)+K\big]|z|^{p-q}$$
$$-\sum_{n=p+1}^{\infty} \delta(n,q)\big[(n-q)\big((n-q-1)K+(1-\beta)\big)+K\big]|a_n||z|^{n-q}$$
$$-\sum_{n=p}^{\infty} \delta(n,q)\big[(n-q)\big((n-q-1)K+(1-\beta)\big)+K\big]|b_n||z|^{n-q}. \quad (11)$$

Evidently (10) and (11) in conjunction with (8) yields

$$|w(z) - (1+\beta)| - |w(z) + (1-\beta)| \leq 0.$$

The harmonic functions

$$f(z) = z^p + \sum_{n=p+1}^{\infty} \frac{x_n}{\delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}} z^n$$
$$+\sum_{n=p}^{\infty} \frac{\overline{y_n}}{\delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}} \overline{z^n}, \quad (12)$$

where

$$\left(\sum_{n=p+1}^{\infty}|x_n| + \sum_{n=p}^{\infty}\overline{|y_n|} = \delta(p,q)\right),$$

show that the coefficients bounds given by (8) is sharp.
The functions of the form (12) are in $KM_p(K,\beta,\gamma)$ because in view of (12) infer that

$$\sum_{n=p+1}^{\infty} \delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|a_n| + \sum_{n=p}^{\infty} \delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|b_n| = \sum_{n=p+1}^{\infty}|x_n| + \sum_{n=p}^{\infty}\overline{|y_n|} = \delta(p,q).$$

The restriction placed in Theorem (1) on the moduli of coefficients of $f = h+\bar{g}$ implies for arbitrary rotation of the coefficients of f, the resulting functions would still be harmonic multivalent and $f \in KM_p(K,\beta,\gamma)$.
The following theorem shows that the condition (8) is also necessary for function f to belong to $AM_p(K,\beta,\gamma)$.

Theorem 2: Let $f = h+\bar{g}$ with h *and* g are given by (2). Then $f \in AM_p(K,\beta,\gamma)$ if and only if

$$\sum_{n=p+1}^{\infty} \delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|a_n| + \sum_{n=p}^{\infty} \delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|b_n| \le \delta(p,q), \tag{13}$$

where $0 \le \beta < \frac{1}{p}, p > q, p \in \mathbb{N} = \{1,2,\dots\}, q \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}, 0 \le K \le 1, \eta \ge 0, \lambda \ge 0, \gamma \ge 0.$

Proof : By noting that $AM_p(K,\beta,\gamma) \subset KM_p(K,\beta,\gamma)$, the sufficiency part of Theorem (2) follows at once from Theorem (1). To prove the necessary part, let us assume that $f \in AM_p(K,\beta,\gamma)$. Using (3), we get

$$Re\left\{\frac{K\left[\left(D_p(\lambda,\beta,\gamma)h(z)\right)^q + \left(\overline{D_p(\lambda,\beta,\gamma)g(z)}\right)^q + z^2\left[\left(D_p(\lambda,\beta,\gamma)h(z)\right)^{q+2} + \left(\overline{D_p(\lambda,\beta,\gamma)g(z)}\right)^{q+2}\right]\right]}{z\left[\left(D_p(\lambda,\beta,\gamma)h(z)\right)^{q+1} + \left(\overline{D_p(\lambda,\beta,\gamma)g(z)}\right)^{q+1}\right]}\right\}$$

$$= Re\left\{\frac{Kt - \sum_{n=p+1}^{\infty} Kc\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|a_n|z^{n-q} - \sum_{n=p}^{\infty} Kc\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|b_n|\bar{z}^{n-q}}{\delta(p,q+1) - \sum_{n=p+1}^{\infty}\delta(n,q+1)\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|a_n|z^{n-q} - \sum_{n=p}^{\infty}\delta(n,q+1)\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|b_n|\bar{z}^{n-q}}\right\} > \alpha,$$

where
$t = \delta(p,q)[1+(p-q)(p-q-1)]$ *and* $c = \delta(n,q)[1+(n-q)(n-q-1)]$.
If we choose z to be real and let $z \to 1^-$, we obtain the condition (13).

III. Extreme Points

Next, we determine the extreme points of the closed convex hull of $AM_p(K,\beta,\gamma)$, denoted by $\overline{AM_p(K,\beta,\gamma)}$.
Theorem 3: $f \in \overline{AM_p(K,\beta,\gamma)}$ if and only if

$$f(z) = \sum_{n=p}^{\infty}(\mu_n h_n + \theta_n g_n), \tag{14}$$

where $z \in U, h_p(z) = z^p$,

$$h_n(z) = z^p - \frac{\delta(p,q)}{\delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}} z^n, \tag{15}$$

$(n = p+1, p+2, \dots)$

$$g_n(z) = z^p - \frac{\delta(p,q)}{\delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}} \bar{z}^n, \tag{16}$$

$(n = p, p+1, \dots)$ and

$$\sum_{n=p}^{\infty}(\mu_n+\theta_n) = 1, \qquad (\mu_n \ge 0, \theta_n \ge 0).$$

In particular, the extreme points of $AM_p(K,\beta,\gamma)$ are $\{h_n\}$ *and* $\{g_n\}$.

Proof : Suppose f is of the from (14). Using (15) and (16), we get

$$f(z)=\sum_{n=p}^{\infty}(\mu_n h_n+\theta_n g_n)$$

$$=\sum_{n=p}^{\infty}(\mu_n+\theta_n)\,z^p-\sum_{n=p+1}^{\infty}\frac{\delta(p,q)}{\delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}}\mu_n z^n$$
$$-\sum_{n=p}^{\infty}\frac{\delta(p,q)}{\delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}}\theta_n\overline{(z)}^n$$
$$=z^p-\sum_{n=p+1}^{\infty}\frac{\delta(p,q)}{\delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}}\mu_n z^n$$
$$-\sum_{n=p}^{\infty}\frac{\delta(p,q)}{\delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}}\theta_n\overline{(z)}^n.$$

Then

$$\sum_{n=p+1}^{\infty}\delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}\frac{\delta(p,q)}{\delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}}\mu_n$$
$$+\sum_{n=p}^{\infty}\delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}\frac{\delta(p,q)}{\delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}}\theta_n$$
$$=\delta(p,q)\left(\sum_{n=p}^{\infty}(\mu_n+\theta_n)-\mu_p\right)=\delta(p,q)\big(1-\mu_p\big)\le\delta(p,q),$$

which implies that $f\in\overline{AM_p(K,\beta,\gamma)}$. Conversely, assume that $f\in\overline{AM_p(K,\beta,\gamma)}$. Putting

$$\mu_n=\frac{\delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}}{\delta(p,q)}|a_n|,$$

$(n=p+1,p+2,\dots)$,

$$\theta_n=\frac{\delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}}{\delta(p,q)}|b_n|,$$

$(n=p,p+1,\dots)$, we get

$$f(z)=\sum_{n=p}^{\infty}(\mu_n h_n+\theta_n g_n).$$

Theorem 4: The class $AM_p(K,\beta,\gamma)$ is a convex set.
Proof : Let the function $f_{n,j}\,(j=1,2)$defined by

$$f_{n,j}(z)=z^p-\sum_{n=p+1}^{\infty}|a_{n,j}|z^n-\overline{\sum_{n=p}^{\infty}|b_{n,j}|z^n}$$

be in the class $AM_p(K,\beta,\gamma)$.
It is sufficient to prove that the function

$$H(z)=\tau f_{n,1}(z)+(1-\tau)f_{n,2}(z),\qquad(0\le\tau<1),$$

is also in the class $AM_p(K,\beta,\gamma)$. Since for $0\le\tau<1$,

$$H(z)=z^p-\sum_{n=p+1}^{\infty}\big(\tau|a_{n,1}|+(1-\tau)|a_{n,2}|\big)z^n-\sum_{n=p}^{\infty}\big(\tau|b_{n,1}|+(1-\tau)|b_{n,2}|\big)\overline{(z)}^n$$

with the aid of Theorem (2), we have

$$\sum_{n=p+1}^{\infty} \delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}\left(\tau|a_{n,1}|+(1-\tau)|a_{n,2}|\right)$$
$$+\sum_{n=p}^{\infty} \delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}\left(\tau|b_{n,1}|+(1-\tau)|b_{n,2}|\right)$$
$$=\tau\left[\sum_{n=p+1}^{\infty} \delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|a_{n,1}|\right.$$
$$\left.+\sum_{n=p}^{\infty} \delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|b_{n,1}|\right]$$
$$+(1-\tau)\left[\sum_{n=p+1}^{\infty} \delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|a_{n,2}|\right.$$
$$\left.+\sum_{n=p}^{\infty} \delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|b_{n,2}|\right]$$
$$\le \tau\delta(p,q)+(1-\tau)\delta(p,q)=\delta(p,q).$$

Hence, $H(z) \in AM_p(K,\beta,\gamma)$.
For harmonic functions

$$f(z)=z^p-\sum_{n=p+1}^{\infty}|a_n|z^n-\sum_{n=p}^{\infty}|b_n|\overline{(z)}^n \tag{17}$$

and

$$F(z)=z^p-\sum_{n=p+1}^{\infty}|A_n|z^n-\sum_{n=p}^{\infty}|B_n|\overline{(z)}^n. \tag{18}$$

We define the convolution of f and F as

$$(f*F)(z)=z^p-\sum_{n=p+1}^{\infty}|a_nA_n|z^n-\sum_{n=p}^{\infty}|b_nB_n|\overline{(z)}^n. \tag{19}$$

IV. Convolution Property

In the following theorem we examine the convolution property of the class $AM_p(K,\beta,\gamma)$.
Theorem 5: If f and F are in the class $AM_p(K,\beta,\gamma)$, then $(f*F)$ also in the class $AM_p(K,\beta,\gamma)$.
Proof : Let f and F of the from (17) and (18) belong to $AM_p(K,\beta,\gamma)$. Then the convolution of f and F is given by (19). Note that $|A_n| \le 1$ and $|B_n| \le 1$, since $F \in AM_p(K,\beta,\gamma)$. Then we can write

$$\sum_{n=p+1}^{\infty} \delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|a_n||A_n|$$
$$+\sum_{n=p}^{\infty} \delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|b_n||B_n|$$
$$\le \sum_{n=p+1}^{\infty} \delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|a_n|$$
$$+\sum_{n=p}^{\infty} \delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|b_n|.$$

The right hand side of the above inequality is bounded by $\delta(p,q)$ because $f \in AM_p(K,\beta,\gamma)$. Therefore$(f*F) \in AM_p(K,\beta,\gamma)$.
5- Integral Operator:
Definition 1[7]: The June-Kim-Srivastava integral operator is defined by

$$\mathcal{J}^{\sigma}h(z)=\frac{(p+1)^{\sigma}}{z\Gamma(\sigma)}\int_0^z\left(log\frac{z}{t}\right)^{\sigma}K(t)dt, \qquad \sigma>0. \tag{20}$$

If

$$h(z)=z^p-\sum_{n=p+1}^{\infty}a_nz^n,$$

then

$$\mathcal{J}^{\sigma}h(z)=z^p-\sum_{n=p+1}^{\infty}\left(\frac{p+1}{n+1}\right)^{\sigma}a_nz^n, \quad (21)$$

also $\mathcal{J}^{\sigma}$ is a linear operator.

Remark 1: If $f(z)=h(z)+\overline{g(z)}$, where

$$h(z)=z^p-\sum_{n=p+1}^{\infty}|a_n|z^n, \qquad g(z)=-\sum_{n=p}^{\infty}|b_n|z^n, |b_p|<1,$$

then

$$\mathcal{J}^{\sigma}f(z)=\mathcal{J}^{\sigma}h(z)+\overline{\mathcal{J}^{\sigma}g(z)}. \quad (22)$$

Theorem 6: If $f\in AM_p(K,\beta,\gamma)$, then $\mathcal{J}^{\sigma}f$ is also in $AM_p(K,\beta,\gamma)$.

Proof : By (21) and (22), we obtain

$$\mathcal{J}^{\sigma}f(z)=\mathcal{J}^{\sigma}\left(z^p-\sum_{n=p+1}^{\infty}|a_n|z^n-\sum_{n=p}^{\infty}|b_n|\overline{(z)}^n\right) \qquad =z^p-\sum_{n=p+1}^{\infty}\left(\frac{p+1}{n+1}\right)^{\sigma}|a_n|z^n-\sum_{n=p}^{\infty}\left(\frac{p+1}{n+1}\right)^{\sigma}\sum_{n=p}^{\infty}|b_n|\overline{(z)}^n,$$

since $f\in AM_p(K,\beta,\gamma)$, then by Theorem (2), we have

$$\sum_{n=p+1}^{\infty}\delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|a_n|$$
$$+\sum_{n=p}^{\infty}\delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|b_n|\le\delta(p,q). \quad (23)$$

We must show

$$\sum_{n=p+1}^{\infty}\delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}\left(\frac{p+1}{n+1}\right)^{\sigma}|a_n|$$
$$+\sum_{n=p}^{\infty}\delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}\left(\frac{p+1}{n+1}\right)^{\sigma}|b_n|$$
$$\le\delta(p,q). \quad (24)$$

But in view of (23) the inequality in (24) holds true if $\left(\frac{p+1}{n+1}\right)^{\sigma}\le 1$, since $\sigma>0$ *and* $p\le n$, therefore (24) holds true and this gives the result.

Theorem 7: Let $f\in AM_p(K,\beta,\gamma)$. Then

$$\left|D_p(\lambda,\beta,\gamma)f(z)\right|\le\left(1+|b_p|\right)z^p+\frac{\delta(p,q)-|b_p|}{\delta(p+1,q)\big[(p-q+1)\big((p-q)K-\beta\big)+K\big]}|z|^{p+1}$$

and

$$\left|D_p(\lambda,\beta,\gamma)f(z)\right|\ge\left(1-|b_p|\right)z^p-\frac{\delta(p,q)-|b_p|}{\delta(p+1,q)\big[(p-q+1)\big((p-q)K-\beta\big)+K\big]}|z|^{p+1}.$$

Proof : Let $f\in AM_p(K,\beta,\gamma)$, then we have

$$\delta(p+1,q)\big[(p-q+1)\big((p-q)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}(|a_n|+|b_n|)$$
$$\le\sum_{n=p+1}^{\infty}\delta(n,q)\big[(n-q)\big((n-q-1)K-\beta\big)+K\big]\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}(|a_n|+|b_n|)\le\delta(p,q)-|b_p|$$

Which implies that

$$\sum_{n=p+}^{\infty}\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}(|a_n|+|b_n|)\le\frac{\delta(p,q)-|b_p|}{\delta(p+1,q)\big[(p-q+1)\big((p-q)K-\beta\big)+K\big]}.$$

Applying this inequality in the following assertion, we obtain

$$\left|D_p(\lambda,\beta,\gamma)f(z)\right|=\left|z^p-\sum_{n=p+1}^{\infty}\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|a_n|z^n-\sum_{n=p}^{\infty}\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}|b_n|\overline{(z)}^n\right|$$

$$\leq \left(1+|b_p|\right)z^p + \sum_{n=p+1}^{\infty}\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}(|a_n|+|b_n|)|z|^n$$

$$\leq \left(1+|b_p|\right)z^p + \sum_{n=p+1}^{\infty}\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}(|a_n|+|b_n|)\,|z|^{p+1}$$

$$\leq \left(1+|b_p|\right)z^p + \frac{\delta(p,q)-|b_p|}{\delta(p+1,q)\left[(p-q+1)\left((p-q)K-\beta\right)+K\right]}|z|^{p+1}.$$

Also, on the other hand we obtain

$$|D_p(\lambda,\beta,\gamma)f(z)| \geq \left(1-|b_p|\right)z^p - \sum_{n=p+1}^{\infty}\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}(|a_n|+|b_n|)|z|^n$$

$$\geq \left(1-|b_p|\right)z^p - \sum_{n=p+1}^{\infty}\left[1+\frac{(n-p)\lambda}{p+\gamma}\right]^{\eta}(|a_n|+|b_n|)|z|^{p+1}$$

$$\geq \left(1-|b_p|\right)z^p - \frac{\delta(p,q)-|b_p|}{\delta(p+1,q)\left[(p-q+1)\left((p-q)K-\beta\right)+K\right]}|z|^{p+1}.$$

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