ISSN 2250-3153

On Sextic Equation With Five Unknowns

$$2(x^3+y^3)(x-y)=84(z^2-w^2)P^4$$

S.Vidhyalakshmi[1], S. Aarthy Thangam[2], G. Dhanalakshmi[3]

[1]Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamil Nadu, India. email: vidhyasigc@gmail.com

[2]Research Scholar, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamil Nadu, India. email: aarthythangam@gmail.com

[3]M.Phil Scholar, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamil Nadu, India. email: dhanamselvi93@gmail.com

ABSTRACT

The non-homogeneous sextic equation with five unknowns represented by the Diophantine equation is $2(x^3+y^3)(x-y)=84(z^2-w^2)P^4$ analyzed for its patterns of non-zero distinct integral solutions are illustrated. Various interesting relations between the solutions and special numbers namely, polygonal numbers, pyramidal numbers are exhibited.

KEYWORDS

Integral solutions, Non-homogenous equation, Sextic equation.

1. INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems (Dickson, 1952; Carmichael, 1959; Mordell, 1969; Telang, 1996). Particularly, in (Gopalan et al., 2007; Gopalan and Sangeetha, 2010; Gopalan et al., 2010), sextic equations with three unknowns are studied for their integral solutions. (Gopalan and VijayaShankar, 2010; Gopalan et al., 2012; Gopalan et al., 2012; Gopalan et al., 2013; Gopalan et al., 2013; Gopalan et al., 2013; Gopalan et al., 2013) analyze sextic equations with four unknowns for their non-zero integer solutions (Gopalan et al., 2012; Gopalan et al., 2013; Gopalan et al., 2014; Gopalan et al., 2015) analyze sextic equations with five unknowns for their non-zero integer solutions.

This communication concerns with yet another interesting non-homogeneous sextic equation with five unknowns $2(x^3+y^3)(x-y)=84(z^2-w^2)P^4$ given by tuple is analysed for its infinitely many non-zero distinct integer solutions (x, y, z, w, P) satisfying the above equations are obtained. Various interesting properties among the values of x, y, z, w, P are presented.

NOTATIONS

$t_{m,n}$: Polygonal number of rank n with size m.

P_m^n : Pyramidal number of rank n with size m.

2. METHOD OF ANALYSIS

The non-homogeneous sextic equation with five unknowns to be solved for its distinct non-zero integral solutions is

$$2\left(x^3+y^3\right)(x-y)=84\left(z^2-w^2\right)P^4 \tag{1}$$

Introduction of the linear transformations

$$x=u+v,\quad y=u-v,\quad z=2u+v,\quad w=2u-v,\quad u\neq v\neq 0 \tag{2}$$

in (1) leads to

$$u^2+3v^2=84P^4 \tag{3}$$

Different methods of obtaining the patterns of integer solutions to (1) are illustrated below:

2.1 PATTERN: 1

Let

$$P=a^2+3b^2 \tag{4}$$

where a and b are non-zero integers.

Write 84 as

$$84=\left(9+i\sqrt{3}\right)\left(9-i\sqrt{3}\right) \tag{5}$$

Using (4), (5) in (3) and applying the method of factorization, define

$$\left(u+i\sqrt{3}v\right)=\left(9+i\sqrt{3}\right)\left(a+i\sqrt{3}b\right)^4 \tag{6}$$

From which, we have

$$\left.\begin{aligned} u&=9a^4+81b^4-162a^2b^2-12a^3b+36ab^3\\ v&=a^4+9b^4-18a^2b^2+36a^3b-108ab^3\end{aligned}\right\} \tag{7}$$

Using (7) in (2), the values of x, y, z and w are given by

$$\left.\begin{aligned} x(a,b)&=10a^4+90b^4-180a^2b^2+24a^3b-72ab^3\\ y(a,b)&=8a^4+72b^4-144a^2b^2-48a^3b+144ab^3\\ z(a,b)&=19a^4+171b^4-342a^2b^2+12a^3b-36ab^3\\ w(a,b)&=17a^4+153b^4-306a^2b^2-60a^3b+180ab^3\end{aligned}\right\} \tag{8}$$

Thus (4) and (8) represent the non-zero integer solutions to (1).

PROPERTIES

- $x(1,b)+y(1,b)-18P^2(1,b)+24\left[t_{8,b}+t_{10,b}+t_{12,b}+t_{16,b}+t_{18,b}-18P_b^{\,3}\right]=-696b$

- $y(a,1)+z(a,1)-w(a,1)-10P(a^2,1)+12[t_{8,a}+t_{12,a}+t_{14,a}+t_{16,a}-12P_a^{\ 3}]\equiv 0(\bmod 6)$
- $z(a,1)-w(a,1)-x(a,1)+8[P(a^2,1)-36P_a^{\ 3}]\equiv 0(\bmod 48)$
- $z(1,b)-w(1,b)+6[216P_b^{\ 3}-P(1,b^2)-6[t_{10,b}+t_{12,b}+t_{18,b}]]\equiv 0(\bmod 4)$
- $w(a,1)+y(a,1)-x(a,1)-15P(a^2,1)+6[132P_a^{\ 3}-t_{14,a}-t_{16,a}-t_{18,a}]\equiv 0(\bmod 3)$

2.2 PATTERN: 2

Consider (6) as

$$u^2-81P^4=3(P^4-v^2) \tag{9}$$

Write (9) in the form of ratio as

$$\frac{u+9P^2}{P^2+v}=\frac{3(P^2-v)}{u-9P^2}=\frac{\alpha}{\beta}, \quad (\beta\neq 0)$$

which is equivalent to the following two equations

$$\beta u-\alpha v+(9\beta-\alpha)P^2=0$$
$$3\beta v+\alpha u-(9\alpha+3\beta)P^2=0$$

On employing the method of cross multiplication, we get

$$\left.\begin{aligned} u&=9\alpha^2+6\alpha\beta-27\beta^2\\ v&=3\beta^2-\alpha^2+18\alpha\beta\end{aligned}\right\} \tag{10}$$

$$P^2=3\beta^2+\alpha^2 \tag{11}$$

which is satisfied by

$$\alpha=3r^2-s^2$$
$$\beta=2rs$$

Substituting the values of α and β in (10) and (11), we get

$$u=81r^4+9s^4-162r^2s^2+36r^3s-12rs^3$$
$$v=-9r^4-s^4+18r^2s^2+108r^3s-36rs^3$$

Substituting the values of u and v in (2), the non-zero distinct integral values of x and y are given by

ISSN 2250-3153

$$\left.\begin{array}{l} x(r,s)=72r^4+8s^4-144r^2s^2+144r^3s-48rs^3 \\ y(r,s)=90r^4+10s^4-180r^2s^2-72r^3s+24rs^3 \\ z(r,s)=153r^4+17s^4-306r^2s^2+180r^3s-60rs^3 \\ w(r,s)=171r^4+19s^4-342r^2s^2-36r^3s+12rs^3 \\ P(r,s)=3r^2+s^2 \end{array}\right\} \quad (12)$$

Thus (12) represent the non-zero integer solutions to (1).

PROPERTIES

- $w(1,s)-x(1,s)-y(1,s)-P(1,s^2)+36\left[t_{3,s}-6P_s^{\,3}+6t_{3,s-1}\right]\equiv 0(\bmod 6)$
- $x(r,1)+y(r,1)-z(r,1)-3P(r^2,1)+18\left[36P_r^{\,3}-\left(t_{10,r}+t_{14,r}+t_{16,r}\right)\right]\equiv 0(\bmod 2)$
- $z(1,s)-y(1,s)-7P(1,s^2)+6\left[84P_s^{\,3}-\left(t_{14,s}+t_{16,s}+t_{18,s}\right)\right]\equiv 18(\bmod 42)$
- $z(r,1)-x(r,1)-27P(r^2,1)+6\left[t_{12,r}+t_{14,r}+t_{16,r}+t_{18,r}+t_{20,r}+20t_{3,r}-36P_r^{\,3}\right]\equiv 12(\bmod 18)$
- $w(1,s)-z(1,s)-2P(1,s^2)+36\left[t_{16,s}-12P_s^{\,3}\right]\equiv 0(\bmod 12)$

2.3 PATTERN: 3

Write (9) in the form of ratio as

$$\frac{u+9P^2}{3(P^2+v)}=\frac{(P^2-v)}{u-9P^2}=\frac{\alpha}{\beta},(\beta\neq 0)$$

which is equivalent to the following two equations

$$\beta u-3\alpha v+(9\beta-3\alpha)P^2=0$$
$$\beta v+\alpha u-(9\alpha+\beta)P^2=0$$

On employing the method of cross multiplication, we get

$$\left.\begin{array}{l} u=27\alpha^2+6\alpha\beta-9\beta^2 \\ v=-3\alpha^2+\beta^2+18\alpha\beta \end{array}\right\} \quad (13)$$

$$P^2=3\alpha^2+\beta^2 \quad (14)$$

which is satisfied by

$$\alpha=2rs$$
$$\beta=3r^2-s^2$$

Substituting the values of α and β in (13) and (14), we get

$$u=-81r^4-9s^4+162r^2s^2+36r^3s-12rs^3$$
$$v=9r^4+s^4-18r^2s^2+108r^3s-36rs^3$$

Substituting the values of u and v in (2), the non-zero distinct integral values of x and y are given by

$$\left.\begin{aligned}x(r,s)&=-72r^4-8s^4+144r^2s^2+144r^3s-48rs^3\\y(r,s)&=-90r^4-10s^4+180r^2s^2-72r^3s+24rs^3\\z(r,s)&=-153r^4-17s^4+306r^2s^2+180r^3s-60rs^3\\w(r,s)&=-171r^4-19s^4+342r^2s^2-36r^3s+12rs^3\\P(r,s)&=3r^2+s^2\end{aligned}\right\} \quad (15)$$

Thus (15) represent the non-zero integer solutions to (1).

PROPERTIES

- $y(1,s)+10P^2(1,s)=24\left[6P_s^3+t_{8,s}+t_{10,s}\right]$
- $x(r,1)+8\left[P^2(r,1)-108P_r^3+6t_{12,r}\right]=-528r$
- $z(1,s)-y(1,s)+7\left[P(1,s^2)+72p_s^3-6(t_{10,s}+t_{12,s})\right]\equiv 0(\bmod 42)$
- $x(r,1)-z(r,1)-27P(r^2,1)+6\left[36P_r^3+t_{10,r}+t_{12,r}\right]\equiv -6(\bmod 18)$
- $x(1,s)+y(1,s)-w(1,s)-P^2(1,s)+12\left[18P_s^3-t_{8,s}-t_{10,s}\right]=240s$

PATTERN: 4

Write (9) in the form of ratio as

$$\frac{u+9P^2}{3(P^2-v)}=\frac{(P^2+v)}{u-9P^2}=\frac{\alpha}{\beta},(\beta\neq 0)$$

which is equivalent to the following two equations

$$\beta u+3\alpha v+(9\beta-3\alpha)P^2=0$$
$$-\beta v+\alpha u-(9\alpha+\beta)P^2=0$$

On employing the method of cross multiplication, we get

$$\left.\begin{aligned}u&=27\alpha^2+6\alpha\beta-9\beta^2\\v&=3\alpha^2-\beta^2-18\alpha\beta\end{aligned}\right\} \quad (16)$$

$$P^2=3\alpha^2+\beta^2 \quad (17)$$

which is satisfied by

$$\alpha = 2rs$$
$$\beta = 3r^2 - s^2$$

Substituting the values of α and β in (16) and (17), we get

$$u = -81r^4 - 9s^4 + 162r^2s^2 + 36r^3s - 12rs^3$$
$$v = -9r^4 - s^4 + 18r^2s^2 - 108r^3s + 36rs^3$$

Substituting the values of u and v in (2), the non-zero distinct integral values of x and y are given by

$$\left.\begin{aligned} x(r,s) &= -90r^4 - 10s^4 + 180r^2s^2 - 72r^3s + 24rs^3 \\ y(r,s) &= -72r^4 - 8s^4 + 144r^2s^2 + 144r^3s - 48rs^3 \\ z(r,s) &= -171r^4 - 19s^4 + 342r^2s^2 - 36r^3s + 12rs^3 \\ w(r,s) &= -153r^4 - 17s^4 + 306r^2s^2 + 180r^3s - 60rs^3 \\ P(r,s) &= 3r^2 + s^2 \end{aligned}\right\} \quad (18)$$

Thus (18) represent the non-zero integer solutions to (1).

PROPERTIES

- $z(1,s) - x(1,s) - y(1,s) + P(1,s^2) - 18\left[12P_s^3 - 6t_{3,s} - t_{6,s}\right] \equiv 0 \pmod 6$
- $z(r,1) - w(r,1) - 6\left[P(r^2,1) - 36\left[-6P_r^3 + \left(t_{8,r} + t_{10,r} + t_{12,r} + t_{16,r}\right)\right]\right] \equiv 0 \pmod 4$
- $y(1,s) + 8P(1,s) + 4\left[72P_s^3 - 49t_{3,s} - 12t_{10,s}\right] \equiv 0 \pmod{48}$
- $y(r,1) + w(r,1) - x(r,1) + 45P(r^2,1) + 6\left[300t_{3,r} - 396P_r^3 + t_{8,r}\right] \equiv -6 \pmod{30}$
- $z(1,s) - w(1,s) - x(1,s) - y(1,s) + 16\left[108P_s^3 - P(1,s^2) - 6\left(t_{8,s} + t_{6,s} + 2t_{3,s-1}\right)\right] \equiv 0 \pmod{96}$

2.5 PATTERN: 5

Write (9) in the form of ratio as

$$\frac{u + 9P^2}{(P^2 - v)} = \frac{3(P^2 + v)}{u - 9P^2} = \frac{\alpha}{\beta}, \quad (\beta \neq 0)$$

which is equivalent to the following two equations

$$\beta u + \alpha v + (9\beta - \alpha)P^2 = 0$$
$$-3\beta v + \alpha u - (9\alpha + 3\beta)P^2 = 0$$

On employing the method of cross multiplication, we get

$$\left.\begin{aligned} u &= 9\alpha^2 + 6\alpha\beta - 27\beta^2 \\ v &= \alpha^2 - 3\beta^2 - 18\alpha\beta \end{aligned}\right\} \quad (19)$$

$$P^2 = 3\beta^2 + \alpha^2 \quad (20)$$

which is satisfied by

$$\alpha = 3r^2 - s^2$$
$$\beta = 2rs$$

Substituting the values of α and β in (19) and (20), we get

$$u = 81r^4 + 9s^4 - 162r^2s^2 + 36r^3s - 12rs^3$$
$$v = 9r^4 + s^4 - 18r^2s^2 - 108r^3s + 36rs^3$$

Substituting the values of u and v in (2), the non-zero distinct integral values of x and y are given by

$$\left.\begin{aligned} x(r,s) &= 90r^4 + 10s^4 - 180r^2s^2 - 72r^3s + 24rs^3 \\ y(r,s) &= 72r^4 + 8s^4 - 144r^2s^2 + 144r^3s - 48rs^3 \\ z(r,s) &= 171r^4 + 19s^4 - 342r^2s^2 - 36r^3s + 12rs^3 \\ w(r,s) &= 153r^4 + 17s^4 - 306r^2s^2 + 180r^3s - 60rs^3 \\ P(r,s) &= 3r^2 + s^2 \end{aligned}\right\} \quad (21)$$

Thus (21) represent the non-zero integer solutions to (1).

PROPERTIES

- $x(r,1) + y(r,1) - w(r,1) - 3P(r^2,1) + 18\left[36P_r^{\,3} - t_{10,r} - t_{14,r} - t_{16,r}\right] \equiv 0 \pmod{2}$
- $y(1,s) + 8\left[36P_s^{\,3} - P(1,s^2)\right] \equiv 0 \pmod{48}$
- $z(1,s) - x(1,s) - y(1,s) - P(1,s^2) + 18\left[t_{8,s} + t_{10,s} - 12P_s^{\,3}\right] \equiv 0 \pmod{6}$
- $x(r,1) + 6\left[72P_r^{\,3} - 5P(r^2,1) - 4t_{3,r} - 2t_{6,r}\right] \equiv 0 \pmod{4}$
- $z(r,1) + w(r,1) + 12\left[12t_{3,r} - 9P(r^2,1) - 72P_r^{\,3} + 2(t_{8,r} + t_{10,r} + t_{12,r} + t_{14,r} + t_{16,r} + t_{18,r} + t_{20,r})\right] \equiv 48 \pmod{72}$

3. CONCLUSION

In this paper, we have made an attempt to determine different patterns of non-zero distinct integer solutions to the non-homogeneous sextic equations with five unknowns. As the sextic equations are rich in variety, one may search for other forms of sextic equation with variables greater than or equal to five and obtain their corresponding properties.

4. REFERENCES

ISSN 2250-3153

[1] L.E. Dickson, History of theory of numbers, vol.11, Chelsea publishing company, Newyork (1952).

[2] R.D. Carmichael, The theory of numbers and Diophantine analysis, Dover publications, Newyork (1959).

[3] L.J. Mordell, Diophantine equations, Academic press, London (1969).

[4] S.G. Telang, Number Theory, Tata MC Graw Hill Publishing Company, New Delhi (1996).

[5] M.A. Gopalan, Manju Somanath and N. Vanitha, "Parametric Solutions of $x^2 - y^6 = z^2$ ", ActaCiencia Indica XXXIII, vol.3, 2007, 1083-1085.

[6] M.A. Gopalan, G. Sangeetha, "On the Sextic Equations with three unknowns $x^2 - xy + y^2 = (k^2+3)^n z^6$ " , Impact J.Sci.tech, Vol.4, No: 4, (2010), 89-93.

[7] M.A. Gopalan, R. Srikanth and Usha janaki, "Parametric integral solutions of $x^2 - y^2 = 2z^6$", Impact J. Sci. Tech., Vol.4, No:3, 2010, 01-04.

[8] M.A. Gopalan and A. VijayaShankar, " Integral Solutions of the sextic equation $x^4 + y^4 + z^4 = 2w^6$", Indian Journal of Mathematics and Mathematical sciences, Vol.6., No:2, 2010, 241-245.

[9] M.A. Gopalan, S. Vidhyalakshmi and A. VijayaShankar, "Integral Solutions of Non-homogeneous sextic equation $xy + z^6 = w^6$", Impact J. Sci. Tech., Vol.6, No:1, 2012, 47-52.

[10] M.A. Gopalan, S. Vidhyalakshmi and K. Lakshmi, "On the non-homogeneous sextic equation $x^4 + 2(x^2 + w) + x^2y^2 + y^4 = z^4$", IJAMA, 4(2), Dec 2012, 171-173.

[11] M.A. Gopalan, G. Sumathi and S. Vidhyalakshmi, "Gaussian Integer Solutions of sextic equation with four unknowns $x^6 - y^6 = 4z(x^4 + y^4 + w^4)$", Archimedes J. Math., 3(3), 2013, 263-266.

[12] M.A. Gopalan, G. Sumathi and S. Vidhyalakshmi, "Integral Solutions of $x^6 - y^6 = 4z\left(x^4 + y^4 + 4(w^2+2)^2\right)$" interms of Generalised Fibonacci and Lucas Sequences, Diophantus J. Math., 2(2), 2013, 71-75.

[13] M.A. Gopalan, G. Sumathi and S. Vidhyalakshmi, "Integral Solutions of Non-homogeneous sextic equation with four unknowns $x^4 + y^4 + 16z^4 = 32w^6$ ", Antarctica J. Math., 10(6), 2013, 623-629.

[14] M.A. Gopalan, S. Vidhyalakshmi and A. Kavitha, "Observations on the non-homogeneous sextic equation with four unknowns $x^3 + y^3 = 2(K^2+3)z^5w$" IJIRSET, Vol.2, Issue. 5, 2013, May-2013.

[15] M.A. Gopalan, S. Vidhyalakshmi and K. Lakshmi, "Integral Solutions of Non-homogeneous sextic equation with five unknowns $x^3 + y^3 = z^3 + w^3 + 3(x+y)T^5$" IJESRT, 1(10), 2012, 562-564.

[16] M.A. Gopalan, G. Sumathi and S. Vidhyalakshmi, "Integral Solutions of Non-homogeneous sextic equation with five unknowns $x^3 + y^3 = z^3 + w^3 + 6(x+y)t^5$ " Vol.1, issue.2, 2013, 146-150.

[17] M.A. Gopalan, S. Vidhyalakshmi and K. Lakshmi, "Integral Solutions of the Sextic equation with five unknowns $x^6 - 6w^2(xy+z) + y^6 = 2(y^2+w)T^4$" International Journal of Scientific and research Publications, Vol.4, issue.7, July-2014

[18] M.A. Gopalan, S. Vidhyalakshmi and A. Kavitha, "Integral Solutions of the Sextic equation with three unknowns $(4k-1)(x^2+y^2) - (4k-2)xy = 4(4k-1)z^6$", International Journal of Innovation Sciences and Research, Vol.4, No:7, July-2015, 323-328.

AUTHORS :

First Author - S. Vidhyalakshmi, Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamil Nadu, India. email: vidhyasigc@gmail.com

Second Author – S. Aarthy Thangam, Research Scholar, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamil Nadu, India. email: aarthythangam@gmail.com

Third Author – G. Dhanalakshmi, M.Phil Scholar, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, Tamil Nadu, India. email: dhanamselvi93@gmail.com.