ISSN 2250-3153

Applications of Laplace transform Unit step functions and Dirac delta functions

M. Peer Mohamed

Assistant Professor, Department of Mathematics, Marudupandiyar College, Thanjavur-613403

Abstract- Laplace transform plays very important role in the field of science and engineering. It is particularly useful in solving initial value problems involving linear ordinary differential equation with constant coefficient. This paper will discuss the advantage of unit step function in solving initial value problems having discontinuous functions in the area of electric circuit theory, biological modeling and atomic control and servomechanism. Also it will discuss the application of delta function in the area of mechanics.

I. Introduction

The importance of the Laplace transform in the study of initial value problems for linear constant coefficient differential equations is that it replaces the operation of the integrating differential equation in f(t) by much simpler algebraic operations involving F(s) [1]. Unlike previous method, where first a general solution is found, and then the constants in the complementary function are chosen to match the initial conditions, when the Laplace transform method is used the initial condition are incorporated from the start. The task of finding f(t), from its Laplace transform F(s) is called inverting the transform by the Laplace transform table.

1. **Definition:**

Let $f(t),\ t>0$ be given. The Laplace transform of f(t) is defined as $\int_0^\infty f(t)dt, t>0$ ……..(2.1),provided the integral exist, where S is real (or) complex parameter.

[2] Not all function have a laplace transform because the interval (2.1) may fail to exist. For any function f(t);t>0, to be laplace transformable, it must satisfy the exixtance theorem.

>>>> [2] f(t) is piecewise continuous on $0 \le t \le A$, for every a>0 and (ii) f(t) is of exponential order as $t \to \infty$, so that there exist real constant k, c and T such that $\left|f(t)\right| \le ke^{-t}\ for\ all\ t \ge T.$. Then the Laplace transform of f(t), namely F(s) given by (2.1) exist for alls >c. The above condition is also sufficient even though $f(t)=\frac{1}{\sqrt{t}}$, does not satisfy the conditions because it is not piecewise continuous on $0 \le t < \infty$ the laplace of f(t) exist.

Even though there are several properties are there, here we describe some properties which will be used in its applications to be described late.

2.1 Linear property: [3]

The transform of a sum equals, the sum of the transform

$$\left[c_1 f(t)+c_2 g(t)\right]=c_1 L\left[f(t)\right]+c_2 L\left[g(t)\right]$$

where $c_1\ and\ c_2$ ar constants

2.2 Differentiation: [3]

The second important property deals with derivatives

$$L\left[f^n(t)\right]=s^n F(s)-s^{n-1}F(s)-s^{n-2}F(s)-.........-f^{n-1}(0)$$

On the assumption that f(t) and its first (n-1) derivatives are continuous,$f^n(t)$ is piecewise continuous, and all are exponential order so that the laplace transform exist.

2.3 The heaviside step function and dirac delta function

Here we introduce two important function, the Heaviside step function and dirac delta is solving complicated discontinuous function.

Unit step function and laplace and Inverse Laplace.

$$U(t-a)=\begin{cases}1; & t>a\\ 0; & t<a\end{cases} \quad where\ a\ge 0$$

$$L\left[U(t-a)\right]=\int_0^\infty e^{-st}f(t)dt=\frac{e^{-as}}{s};\ s>0$$

$$L\left[f(t-a)U(t-a)\right]=e^{-as}F(s)$$

$$L^{-1}\left[e^{-as}F(s)\right]=f(t-a)u(t-a)$$

Dirac delta function

A very large force acting during very short period of time(ex. Earth Quake). This idea leads to construct dirac delta function.

$$\delta(t-a)=\begin{cases}\infty; t>a\\ 0; t\ne 0\end{cases}$$

$$L\left[\delta(t-a)\right]=e^{-as}\, particular\ at\ a=0$$

$$L\left[\delta(t)\right]=1$$

2. **Applications**

Theory of Automatic Control [4]

A mechanism, whether it involves electrical, mechanical (or) other principles, designed to accomplish such automatic control is called a servomechanism.

Suppose that a missile M is tracking down an enemy aircraft. If at time t the enemy turns through some angle $\phi(t)$, then M must also turn through this angle, if it is to catch with and destroy it. If a mass was aboard M, he could operate

Some steering mechanism to produce the required turns, but since the missile is must be accomplished automatically. To do for a mans eyes, such as a radar beam which will indicate or point to the direction which must be taken by M, we also need something providing a substitute for a man's hands which will turn a shaft through some angle in order to produce the desired turn.

In this application, let us assume that the desired angle of turn as indicated by the radar beam is αt.

Also let $\theta(t)$ denote the angle if turn of the shaft at time t. Because the things are happening so fast we must expect to have a error between two.

ie., $error\ =\theta(t)-\alpha t$

The existence of the error must be **signaled** back to the shaft, so that a compensating turning **effort (or)** torque …. Be produced. If the error is large the torque needed will be large. If the error is small the torque needed will be small. So that requires torque is proportional to the error from mechanics, we know that

$$Torque\ = I\frac{d^2q}{dt^2}$$

i.e., M.I multiplied by angular acceleration.

Since Torque α error

$$I\frac{d^2q}{dt^2}\alpha\left(\theta(t)-\alpha t\right)$$

$$I\frac{d^2q}{dt^2}=k\left(\theta(t)-\alpha t\right)\ where\, k>0..........(1)$$

Is the constant of propertiality.

The minus sign is used because if the error is positive then the torque must opposite it.While the error is negative the torque must be positive.

Assuming that the initial angle and angular velocity are zero as possible conditions.

$$\theta(0)=0,\theta'(0)=0$$

Equation (1) becomes

$$I\,\frac{d^2q}{dt^2} = -k\theta(t) + k\alpha t$$

$$\theta(0) = 0, \theta'(0) = 0$$

$$\frac{d^2q}{dt^2} + \frac{k}{I}\theta(t) = \frac{k}{I}\alpha t$$

$$L\left[\frac{d^2q}{dt^2}\right] + \frac{k}{I}L[\theta] = \frac{k}{I}L[\alpha t]$$

$$s^2L(\theta) - s\theta(0) - \theta'(0) + \frac{k}{I}L[\theta] = \frac{k}{I}\alpha\frac{1}{s^2}$$

$$\left(s^2 + \frac{k}{I}\right)L(\theta) = \frac{k}{I}\alpha\frac{1}{s^2}$$

$$L(\theta) = \frac{k}{I}\frac{\alpha}{s^2\left(s^2 + \frac{k}{I}\right)}$$

$$\theta = \frac{k}{I}L^{-1}\left[\frac{\alpha}{s^2\left(s^2 + \frac{k}{I}\right)}\right]$$

$$= \frac{k\alpha}{I}\left[L^{-1}\left(\frac{1}{s^2}\right)L^{-1}\frac{1}{\left(s^2 + \frac{k}{I}\right)}\right]$$

Use Convolution theorem

$$= \frac{k\alpha}{I}\int_0^t t * \frac{1}{\sqrt{\frac{k}{I}}}\sin\sqrt{\frac{k}{I}}(t-u)\,du$$

$$then\ \ \alpha\sqrt{\frac{k}{I}}\int_0^t t\sin\sqrt{\frac{k}{L}}(t-u)\,du$$

$$\sqrt{\frac{k}{I}}\alpha\left[u\frac{1}{\sqrt{\frac{k}{I}}}\cos\sqrt{\frac{k}{I}}(t-u)\right]_0^t - \int_0^t \frac{1}{\sqrt{\frac{k}{I}}}\cos\sqrt{\frac{k}{I}}(t-u)\,du$$

$$\sqrt{\frac{k}{I}}\alpha\left[\left(\sqrt{\frac{k}{I}}t\right)\right] + \left(\left(\frac{I}{k}\sin\sqrt{\frac{k}{I}}(t-u)\right)_0^t\right)$$

$$\sqrt{\frac{k}{I}}\alpha\left[\sqrt{\frac{I}{K}}t - \left(\frac{I}{k}\sin\sqrt{\frac{k}{I}}t\right)\right]$$

$$\theta(t) = \alpha t - \frac{I}{k}\sin\sqrt{\frac{k}{I}}t$$

Electric circuit [4]

An inductor of 0.1 Henry, a resistor of 10 ohms and an electro magnetic force E(t) volts, where

$$E(t)=\begin{cases}10\,;0<t\le 5\\ 0\;\;;t>5\end{cases}$$ are connected in series. Calculate the current I(t), assuming I (0) = 0.

Solution:

By Kirchoff's law, the differential equation for the above circuit is given by

$$L\frac{di}{dt}+RI=E$$

Here L = 0.1, R = 10, E = E (t)

$$0.1\frac{di}{dt}+10I=E(t)$$

General method:

$$0.1\frac{di}{dt}+10I=10;0<t\le 5$$

$$(i)0<t\le 5\quad \frac{di}{dt}+100I=100$$

$$Ie^{\int 100dt}=\int 100\,e^{\int 100dt}dt+c$$

$$Ie^{\int 100dt}=100\int e^{\int 100dt}\;dt+c$$

$$=e^{100t}+c$$

$$I=1+ce^{-100t}+c$$

$$I(0)=0\Rightarrow c=-1$$

$$So,\;I=1-e^{-100t}\;\;for\;0<t\le 5$$

$$(ii)t>5$$

$$\frac{di}{dt}+100I=0$$

$$\frac{di}{dt}=-100I$$

$$\frac{di}{I}=-100dt$$

$$I=e^{-100t}k.........(1)$$

$$I(5)=1-e^{-100t}$$

$$1-e^{-100t}=e^{-100(5)}k$$

$$k=e^{500}-e^{-100t+500}$$

$$I=\left(e^{500}-e^{-100t+500}\right)e^{-100t}$$

$$I=e^{-100(t-5)}-e^{-100t};t>5$$

Unit Step Function Method;

$$0.1\frac{\mathrm{di}}{\mathrm{dt}}+10\mathrm{I}=\mathrm{E}(t) \quad where\ E(t)=\begin{cases}10 & 0\le t\le 5\\ 0 & t>5\end{cases}$$

$$0.1\frac{\mathrm{di}}{\mathrm{dt}}+10\,\mathrm{I}=10-10H(t-5) \qquad 10+\begin{cases}-10 & t>5\\ 0 & t<5\end{cases}$$

$$\frac{\mathrm{di}}{\mathrm{dt}}+100\,\mathrm{I}=100-100H(t-5) \qquad 10-10H(t-5)$$

Taking Laplace on both sides

$$sL(I)-I(0)+100L(I)=100\left[L(1)-L(H(t-5))\right]$$

$$sL(I)-I(0)+100L(I)=100\left[L(1)-L(H(t-5))\right]$$

$$sL(I)+100L(I)=100\left[\frac{1}{s}-\frac{e^{-5s}}{s}\right]$$

$$L(I)(s+100)=100\left[\frac{1}{s}-\frac{e^{-5s}}{s}\right]$$

$$L(I)=100\left[\frac{1}{s(s+100)}-\frac{e^{-5s}}{s(s+100)}\right]..............(2)$$

Equation (2) solve by partial fraction

$$=1-e^{-100t}-u(t-5)+u(t-5)e^{-100(t-5)}$$

For

$$0<t\le 5 \quad \Rightarrow 1-e^{-100t}$$

$$t>5 \quad \Rightarrow 1-e^{-100t}-1+e^{-100(t-5)}$$

$$=e^{-100(t-5)}-e^{-100t}$$

The results are same in both methods. But the advantage of unit step function is we can solve as one problem.

Biology Problem:

A liquid carries a drug into an organ of volume V $cm^3{}_{/\ sec}$ at ‘a’ rate at a rate of a cm^3/sec and leaves at a rate of ‘b’ $cm^3{}_{/sec}$’, where V, a, b are constants. At time t =o the concentration of the drug is zero and builds up linear function to a maximum of K at t = T, at which time the process is stopped. What is the concentration of the drug in the organ at any time t [4].

The concentration of the drug in the organ any time is , x. The amount of drug in the organ at any time t is given by xV.

The concentration of the drug entering in the organ at time t is a c (t). g / sec.

The concentration of the drug leaving the organ by g / sec.

The rate of change of the amount of drug in the organ is equal to the rate at which the drug enter – the rate at which it leave.

So, $\frac{d}{dt}(cv)=a(t)-bx\,;x(0)=0$

Solve by using unit step function

Here,

$$c(t)=\begin{cases}\frac{kt}{T};0\le t\le T\\ 0 \quad ;t>T\end{cases}$$

$$=\frac{kt}{T}+\begin{cases}\frac{-kt}{T};t>T\\ 0\ ;0\le t\le T\end{cases}$$

$$=\frac{kt}{T}-\frac{kt}{T}\begin{cases}1;t>T\\ 0\ ;t<T\end{cases}$$

So, the differential equation becomes

$$\frac{d}{dt}(xv)=a\left[\frac{kt}{T}-\frac{kt}{T}H(t-T)\right]-bx$$

$$\frac{d}{dt}(xv)=\frac{ak}{T}\left[t-tH(t-T)\right]-bx$$

Taking Laplace transform

$$v\ s\ L(x)-x(0)=\frac{ak}{T}\left[L(t)-L(tH(t-T))-bL(x)\right]$$

$$L(x)[vs+b]=\frac{ak}{T}\left[\frac{1}{s^2}-L\{(t-T+T)H(t-T)\}\right]$$

$$=\frac{ak}{T}\left[\frac{1}{s^2}-L(t-T)H(t-T)-L(TH(t-T))\right]$$

$$=\frac{ak}{T}\left[\frac{1}{s^2}-\frac{e^{-Ts}}{s^2}-\frac{Te^{-Ts}}{s}\right]$$

$$X=\frac{ak}{T}L^{-1}\left[\frac{1}{s^2(vs+b)}-\frac{e^{-Ts}}{s^2(vs+b)}-\frac{Te^{-Ts}}{s(vs+b)}\right]$$

$$\frac{1}{s^2(vs+b)}=\frac{v^2}{b^2(vs+b)}-\frac{v}{b^2s}+\frac{1}{bs^2}$$

$$L^{-1}\left(\frac{1}{s^2(vs+b)}\right)=L^{-1}\left(\frac{v^2}{b^2v\left(s+\frac{b}{v}\right)}-\frac{v}{b^2s}+\frac{1}{bs^2}\right)$$

$$=\frac{v}{b^2}e^{\frac{-b}{v}t}-\frac{v}{b^2}+\frac{t}{b}$$

$$L^{-1}\left(\frac{e^{-sT}}{s^2(vs+b)}\right)=L^{-1}\left[\frac{v}{b^2}\frac{e^{-sT}}{v\left(s+\frac{b}{v}\right)}-\frac{v}{b^2}\frac{e^{-Ts}}{s}-\frac{e^{-Ts}}{bs^2}\right]$$

$$\frac{v}{b^2}H(t-T)e^{\frac{-b(t-T)}{v}}-\frac{v}{b^2}H(t-T)+\frac{1}{b}H(t-T)(t-T)$$

$$L^{-1}\left[\frac{Te^{-sT}}{s(vs+b)}\right]=T\left[L^{-1}\left(\frac{e^{-Ts}}{bs}-\frac{ve^{-Ts}}{b(vs+b)}\right)\right]$$

$$=\left(\frac{T}{b}H(t-T)-\frac{1}{b}H(t-T)e^{\frac{-b}{v}(t-T)}\right)$$

So,

$$X=\frac{ak}{T}\begin{bmatrix}\frac{v}{b^2}e^{\frac{-bt}{v}}-\frac{v}{b^2}+\frac{t}{b}\\ -\frac{v}{b^2}H(t-T)e^{\frac{-b}{v}(t-T)}+\frac{v}{b^2}H(t-T)-\frac{1}{b}H(t-T)(t-T)\\ \frac{T}{b}H(t-T)+\frac{T}{b}H(t-T)e^{\frac{-b}{v}(t-T)}\end{bmatrix}$$

$$for\ 0\le t\le T;\ \frac{akt}{bT}-\frac{akv}{Tb^2}e^{-\frac{b}{v}t}-\frac{vak}{Tb^2}$$

$$\frac{akt}{bT}-\frac{akv}{Tb^2}\left(1-e^{-\frac{b}{v}t}\right)$$

For t > T

$$X=\begin{bmatrix}\frac{akv}{Tb^2}e^{\frac{-bt}{v}}-\frac{akv}{Tb^2}+\frac{akt}{Tb}\\ -\frac{akv}{Tb^2}e^{\frac{-b}{v}(t-T)}+\frac{ak}{Tb}t-\frac{ak}{b}+\frac{T}{b}H(t-T)+\frac{Tak}{Tb}e^{\frac{-b}{v}(t-T)}\end{bmatrix}$$

$$X=\frac{akv}{Tb^2}e^{\frac{-bt}{v}}+\left(\frac{Tak}{Tb}-\frac{vak}{Tb^2}\right)e^{\frac{-b}{v}(t-T)}$$

<u>Application of Dirac delta function</u>[4]

A particle of mass at rest at the origin O on the axis. At t = t_0 it is acted upon by a force for a very short interval of time where the impulse of the force is a constant P_0.

Here we describe at any time t where the particle.

By Newton's law, we can set the differential equation for the problem is

$$m\frac{d^2x}{dt^2}=P_0\delta(t-t_0)\quad ;x(0)=0,\ x'(0)=0$$

Here we have assumed that the impulse of the force applied to the partical of the mass is constant and equal to P_0 . so that the force can be taken as

$$P_0\delta(t-t_0)$$

Taking Laplace on both sides

$$L\left(m\frac{d^2x}{dt^2}\right)=LP_0\delta\left(t-t_0\right)$$

$$m\left(s^2L\left(x\right)-sx\left(0\right)-x'\left(0\right)\right)=P_0e^{-st_0}$$

$$s^2L\left(x\right)=\frac{P_0}{m}e^{-st_0}$$

$$X=L^{-1}\left[\frac{P_0}{m}\frac{e^{-st_0}}{s^2}\right]$$

$$X=\frac{P_0}{m}\left(t-t_0\right)H\left(t-t_0\right)$$

$$=\begin{cases}0 & ;t<t_0\\ \frac{P_0}{m}\left(t-t_0\right) & ;t>t_0\end{cases}$$

II. Conclusion

This paper presented the applications of Laplace transform in various fieds of automatic control, Electric circuit, Biological problem and mechanics. In the first application we have shown the use of Laplace transform in solving linear differential equation,in the second and third application we have shown that the use of unit step function solving the problems involving discontinuous functions. Also we showed that the use of Direc delta function in the last application.

References

[1] Alen Jeffrey;Advanced Engineering Mathematics,HARCOURT ACDEMIC PRESS.

[2] Micheal,D,Greenberg; Advanced Engineering Mathematics,Prentice hall,2nd edition

[3] Dean G,Duffy; Advanced Engineering Mathematics,CRC,Press.

[4] Murrey R, Spigel.Applied Differential Equations ,Springer verlog/

Authors

First Author – Dr. M. Peer Mohamed, Assistant Professor, Department of Mathematics, Marudupandiyar College, Thanjavur-613403, mpinstitutions@gmail.com, 9047247354